An alternative to Schwarzschild gravity.

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"The chief attraction of the theory lies in its logical completeness. If a single one of the conclusions drawn from it proves wrong, it must be given up; to modify it without destroying the whole structure seems to be impossible."
Albert Einstein *(1919).*

In the first part we analyze a novel metric for a spherically symmetric mass distribution, here called the *K-gravity metric.* This has a close formal similarity to the Schwarzschild metric, arising from an analytic generalization of the usual factor: $k =_{df} 1/\sqrt{(1-2MG/c^2r)}$, but has quite distinct properties, including infinitely dispersed mass, and lack of an event horizon. In the second part we propose that this may be taken as a general alternative solution for physical gravity within GTR, resulting from an alternative interpretation of the stress-energy tensor, with $T_{\mu\nu} \neq 0$ for empty space, and we consider the empirical differences with standard Schwarzschild gravity. We propose a novel test of GTR is in order, to test this against the usual Schwarzschild solution in the solar system.

Part 1. The K-gravity metric.

1. Introduction.

The *Schwarzschild solution* (Schwarzschild, 1915/16) to Einstein's General Theory of Relativity (GTR) is typically expressed in line metric form, in polar coordinates:

$$ds^2 = c^2d\tau^2 = c^2dt^2/k^2 - k^2dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 \qquad (1)$$

with k ('little k') defined by: $k =_{df} 1/\sqrt{(1-2MG/c^2r)}$. We will analyze an alternative metric, obtained by substituting k with K ('big K'):

$$ds^2 = c^2d\tau^2 = c^2dt^2/K^2 - K^2dr^2 - r^2d\theta^2 - r^2\sin^2\theta\, d\phi^2 \qquad (2)$$

with K defined by:

$$K =_{df} \exp(MG/c^2r) \qquad (3)$$

We refer to (2) as the *K-gravity metric.* K is close to k in weak gravity, i.e. where MG/c^2r <<1, or: $r >> MG/c^2$. This is seen most simply by comparing $1/k^2$ with $1/K^2$.

$$1/k^2 = 1-2MG/c^2r$$

$$1/K^2 = 1 - 2MG/c^2r + (2MG/c^2r)^2(1/2!) - (2MG/c^2r)^3(1/3!) + \ldots$$

They differ in the second-order terms: $(2MG/c^2r)^2$ and higher. $1/K^2$ appears as an *analytic continuation of* $1/k^2$. Comparing k and K directly:

$$k = 1 + MG/c^2r + (3/2)(MG/c^2r)^2 + \ldots$$

$$K = 1 + MG/c^2r + (1/2)(MG/c^2r)^2 + \ldots$$

Hence for large r, $k \approx K + (MG/c^2r)^2$. All subsequent terms of k are larger than corresponding terms of K, so $k > K$ (for all r). Hence Schwarzschild gravity (1) predicts stronger fields than K-gravity (2) for the same source mass, at all $r > 2MG/c^2r$, i.e. everywhere outside the Schwarzschild radius.

We now analyze the K-gravity metric as a solution to Einstein's equation:

$$G_{\mu\nu} = R_{\mu\nu} + ½g_{\mu\nu}R = (8\pi G/c^4)T_{\mu\nu} \qquad (4)$$

$G_{\mu\nu}$ is determined by the K-gravity metric, and we work out the stress-energy tensor, $T_{\mu\nu}$, and corresponding mass-energy distribution ρ, required to produce this metric. Whereas the Schwarzschild solution corresponds to a symmetric mass M at a central region in otherwise empty space, we will see the K-gravity solution corresponds to the same mass M smeared out in space.

2. Generalized line metric functions.

We adopt the usual spherically symmetric metric in polar coordinates: (t,r,θ,ϕ), indexed by: μ,ν = (0,1,2,3), respectively, and we can write both metrics in the form:

$$ds^2 = g_{\mu\nu}\,dx^\mu\,dx^\nu = U(r)dt^2 - V(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta\,d\phi^2 \qquad (5)$$

where U and V are spatial functions of r alone. This means:

$$g_{00} = U, \qquad g_{11} = -V, \qquad g_{22} = -r^2, \qquad g_{33} = -r^2\sin^2\theta \qquad (6)$$

K-gravity and Schwarzschild gravity are defined by alternative choices of U and V. Note K-gravity and Schwarzschild gravity are two different metrics – they cannot be transformed to each other by any coordinate transformation. However they are both spherically symmetric, and because of this, we can use the generalized form (5) to represent the K-gravity metric, and the general solutions for Christoffel symbols, Ricci tensors, etc, in terms of U, V, commonly used to derive the Schwarzschild metric can be used in our derivations for K-gravity.[1] We will compare the two solutions side-by-side as we go for clarity.

[1] (Oas, 2014) is a useful simple source to follow the derivation in this form. Standard texts such as (Landau *etc alia,* 1975), (Misner *et alia,* 1973), (Spivak, 1979), (Wald, 1984) contain all detail needed. (Kay, 1976) is a good reference to the tensor calculus.

3. Useful Identities.

We start with some identities useful in the algebraic calculations.

Identities for k, K, U, V, and their differentials. (7)

K-gravity	***Schwarzschild gravity***
$K = exp(MG/c^2r)$	$k = (1\text{-}2\ MG/c^2r)^{-½}$
Definitions of U and V	
K-gravity:	***Schwarzschild gravity:***
$U = c^2/K^2 = c^2exp(-2MG/c^2r)$	$U = c^2/k^2 = c^2(1\text{-}2MG/c^2r)$
$V = -K^2 = -exp(2MG/c^2r)$	$V = -k^2 = -1/(1\text{-}2MG/c^2r)$
$U = -c^2/V$	$U = -c^2/V$
$V = -c^2/U$	$V = -c^2/U$
$UV = -c^2$	$UV = -c^2$
$U/V = -c^2/K^4$	$U/V = -c^2/k^4$
Derivatives of K and k by r	
$K = exp(MG/c^2r)$	$k = (1\text{-}2\ MG/c^2r)^{-½}$
$K' = -(MG/c^2r^2)K$	$k' = -(MG/c^2r^2)k^3$
$K^2{}' = -(2MG/c^2r^2)K^2$	$k^2{}' = -(2MG/c^2r^2)k^4$
$K^{-1}{}' = (MG/c^2r^2)K^{-1}$	$k^{-1}{}' = (MG/c^2r^2)k$
$K^{-2}{}' = (2MG/c^2r^2)K^{-2}$	$k^{-2}{}' = (2MG/c^2r^2)$
Second derivatives by r	
$K'' = (2MG/c^2r^3)K+(MG/c^2r^2)^2K$	$k'' = (2MG/c^2r^3)k^3+(MG/c^2r^2)^2 3k^5$
Derivatives of U and V in terms of K and k:	
$U' = -2c^2K'/K^3 = (2MG/r^2)/K^2$	$U' = -2c^2k'/k^3 = 2MG/r^2$
$V' = -2KK' = (2MG/c^2r^2)K^2$	$V' = -2kk' = (2MG/c^2r^2)k^4$
$U'' = 4(MG/r^2K)^2 – (4MG/r^3K^2)$	$U'' = -4MG/r^3$
$V'' = -(4MGK^2/c^2r^3)(1 + MG/c^2r)$	$V'' = -(4MGk^4/c^2r^3)(1 + 2MGk^2/c^2r)$
Derivatives of U in terms of U:	
$U' = (2MG/c^2r^2)U$	$U' = 2MG/r^2$
$U'' = –(4MG/c^2r^3)(1–MG/c^2r)U$	$U'' = -4MG/r^3$
$= 4(MG/c^2r^2)^2U – (4MG/c^2r^3)U$	
Derivatives of V in terms of V:	
$V' = -(2MG/c^2r^2)V$	$V' = -(2MG/c^2r^2)V^2$
$V'' = 4(MG/c^2r^2)^2V + (4MG/c^2r^3)V$	$V'' = 8(MG/c^2r^2)^2V^3 + 4(MG/c^2r^3)V^2$

$= (4MG/c^2r^3)(1 + MG/c^2r)V \qquad\qquad = (4MG/c^2r^3)(1 + 2VMG/c^2r)V^2$

4. Christoffel symbols.

Non-vanishing Christoffel symbols in terms of *U* and *V* are for a general spherically symmetric metric:[2]

Christoffel symbols written in U, V *(8)*

$\Gamma^0{}_{01} = \Gamma^0{}_{10} = U'/2U$

$\Gamma^1{}_{00} = U'/2V \qquad \Gamma^1{}_{11} = V'/2V \qquad \Gamma^1{}_{22} = -r/V \qquad \Gamma^1{}_{33} = -r\sin^2\theta/V$

$\Gamma^2{}_{12} = \Gamma^2{}_{21} = 1/r \qquad \Gamma^2{}_{33} = -\cos\theta\sin\theta$

$\Gamma^3{}_{13} = \Gamma^3{}_{31} = 1/r \qquad \Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta$

Others terms are zero.

Substituting the coordinate functions for *U, V* we obtain:

Christoffel symbols in coordinate functions *(9)*

K-gravity:	***Schwarzschild gravity:***
$\Gamma^0{}_{01} = \Gamma^0{}_{10} = MG/c^2r^2$	$\Gamma^0{}_{01} = \Gamma^0{}_{10} = k^2MG/c^2r^2$
$\Gamma^1{}_{00} = -MG/r^2K^4$	$\Gamma^1{}_{00} = -MG/r^2k^2$
$\Gamma^1{}_{11} = MG/c^2r^2$	$\Gamma^1{}_{11} = k^2MG/c^2r^2$
$\Gamma^1{}_{22} = r/K^2$	$\Gamma^1{}_{22} = r/k^2$
$\Gamma^1{}_{33} = r\sin^2\theta/K^2$	$\Gamma^1{}_{33} = r\sin^2\theta/k^2$

Others terms are zero.

5. Ricci tensors.

The Christoffel symbols determine the Ricci tensor, which has four non-zero terms.

Ricci Tensor written in U,V *(10)*

$R_{00} = -U''/2V + U'V'/4V^2 + U'^2/4UV - U'/rV$

$R_{11} = U''/2U - U'^2/4U^2 - U'V'/4UV - V'/Vr$

$R_{22} = rU'/2UV + 1/V - rV'/2V^2 + 1$

$R_{33} = R_{22}\sin^2\theta$

[2] See (Oas 2014), p. 3-4.

Ricci tensor components written in coordinate variables *(11)*

K-gravity:

$R_{00} = 2(MG/cr^2)^2/K^4$

$R_{11} = -4MG/c^2r^3$

$R_{22} = 1-1/K^2$

$R_{33} = R_{22} \sin^2\theta$

$R_{00}/R_{11} = -(MG/2r)/K^4$

Schwarzschild gravity:

$R_{00} = 0$

$R_{11} = 0$

$R_{22} = 0$

$R_{33} = 0$

6. Ricci scalar.

Ricci Scalar written in U and V *(12)*

$R = R^{\mu}{}_{\nu} = g^{\mu\nu}R_{\mu\nu}$

$\quad = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33}$

$\quad = g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}\sin^2\theta\, R_{22}$

$\quad = R_{00}/U - R_{11}/V - R_{22}/r^2 - \sin^2\theta\, R_{22}/(r^2\sin^2\theta)$

$\quad = R_{00}/U - R_{11}/V - 2R_{22}/r^2$

Substituting for $R_{\mu\nu}$ and simplifying:

$R = -U''/UV + U'V'/2UV^2 + U'^2/2VU^2 - 2U'/rUV + 2V'/V^2r - 2/r^2(1+1/V)$ *(13)*

Then substitute for *U, V, U', V', U''* to obtain:

Ricci Scalar in coordinate functions *(14)*

R for K-Gravity

$R = (2/r^2K^2)(K^2 - 1 - (2MG/c^2r))$

$R \approx (4M^2G^2/c^4r^4K^2)$

R is positive and proportional to $1/r^4$ *in its highest term.*

(For Schwarzschild gravity, $R = 0$.)

7. Stress-Energy tensors.

Using the Einstein equation, we can now determine the $T_{\mu\nu}$ components directly. Only diagonal terms can be non-zero, and we obtain three independent equations:

Field Equations written in U, V *(15)*

$(8\pi G/c^4)T_{00} = R_{00} + ½g_{00}R = UV'/rV^2 - (U/r^2)(1+1/V)$

$(8\pi G/c^4)T_{11} = R_{11} + ½g_{11}R = -U'/rU - (V/r^2)(1+1/V)$

$$(8\pi G/c^4)T_{22} = R_{22} + ½g_{22}R$$
$$= (r/2V)(-U'/U + V'/V - rU''/U + rU'V'/2UV + rU'^2/2U^2)$$

The fourth equation, for T_{33}, is equivalent to the third. In the Schwarzschild derivation these are set to zero, and this leads to the solutions: $V = k^2$ and $U = c^2/k^2$. We now use these to solve $T_{\mu\nu}$ for K-gravity. The solutions are given below in a number of forms, including series in $1/r$, and approximations from above and below.

K-gravity Stress-Energy Tensor in coordinate functions

T_{00} for K-Gravity *(16)*

$$T_{00} = Mc^4/4\pi r^3K^4 + c^6/8\pi Gr^2K^4 - c^6/8\pi Gr^2K^2$$
$$= (Mc^4/4\pi r^3K^4) + (c^6/8\pi Gr^2K^4)(1-K^2)$$
$$= (c^6/8\pi Gr^2K^4)(1+(2MG/c^2r)- K^2)$$
$$= -(c^6/8\pi Gr^2K^4)((2MG/c^2r)^2/2!+(2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + \ldots)$$
$$= -(M^2Gc^2/4\pi r^4K^4)(1 + 2(2MG/c^2r)/3! + 2(2MG/c^2r)^2/4! \ldots)$$
$$= -(M^2Gc^2/4\pi r^4K^4) - (c^6/8\pi Gr^2K^4)((2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + \ldots)$$
$$-(M^2Gc^2/4\pi r^4K^2) \approx< T_{00} \approx< -(M^2Gc^2/4\pi r^4K^4) \qquad \text{for large } r$$

T_{11} for K-Gravity *(17)*

$$T_{11} = -T_{00}\,K^4/c^2 = T_{00}\,g_{11}/g_{00}$$
$$= -Mc^2/4\pi r^3 - (1-K^2)(c^4/8\pi Gr^2)$$
$$= (M^2G/4\pi r^4) + (c^4/8\pi Gr^2)((2MG/c^2r)^3/3! + (2MG/c^2r)^4/4! + \ldots)$$
$$(M^2G/4\pi r^4) \approx< T_{11} \approx< (M^2GK^2/4\pi r^4) \qquad \text{for large } r$$

T_{22} for K-Gravity *(18)*

$$T_{22} = T_{11}\,r^2/K^4 = T_{11}\,g_{22}/g_{11}$$

Note that:

$$T_{11} = -T_{00}\,K^4/c^2 = T_{00}V/U = T_{00}\,g_{11}/g_{00}$$
$$T_{22} = -T_{00}\,g_{22}/g_{00},\quad T_{33} = -T_{00}\,g_{33}/g_{00}$$

I.e.: $T_{\mu\mu} = T_{\nu\nu}\,g_{\mu\mu}/g_{\nu\nu}$ *(no summation)*

T_{00} *is negative,* T_{11}, T_{22}, T_{33} *are positive.*

8. Pressure-Density in K-gravity.

In K-gravity the gravitational mass, *M*, is smeared out across space like a fluid. We now determine the distribution of this fluid. We can follow Vojinovic (2010) p.7. for a simple derivation.[3]

> "The stress-energy tensor of a fluid element with density ρ, pressure *p*, and 4-velocity u^μ, is:
>
> $$T_{\mu\nu} = (\rho+p)u_\mu u_\nu + pg_{\mu\nu}$$
>
> We wish to describe the static fluid ($u_1 = u_2 = u_3 = 0$). So the stress-energy obtains the form:
>
> $$T_{00} = \rho u_0 u_0 + p(u_0 u_0 + g_{00}), \qquad T_{11} = pg_{11}, \quad T_{22} = pg_{22}, \quad T_{33} = pg_{33}$$
>
> while other components vanish. Next the 4-velocity vector must be normalized, $u_\mu u_\nu g^{\mu\nu} = -1$, which means that $u_0 u_0 = -g_{00}$."

Applying this to the K-gravity metric gives four equations:

K-Gravity: Pressure-Density Tensor Equations

$$T_{00} = -\rho c^2/K^2, \qquad T_{11} = pK^2, \qquad T_{22} = pr^2, \qquad T_{33} = -pg_{33} \tag{19}$$

Or inversely:

$$\rho = -T_{00} K^2/c^2, \qquad p = T_{11}/K^2, \qquad p = T_{22}/r^2 \tag{20}$$

For the Schwarzschild solution, these are all zero: $T_{\mu\nu} = 0$ so $p=0$ and $\rho = 0$.

We now calculate *p* and ρ for K-gravity. Since from Section 7: $T_{\mu\mu} = T_{\nu\nu} g_{\mu\mu}/g_{\nu\nu}$, there is really only one equation to solve, and: $p = \rho$. We will solve for ρ.

Substituting T_{00} from Equation (16) in the first equation above gives:

$$\rho = -Mc^2/4\pi r^3 K^2 - c^4/8\pi G r^2 K^2 + c^4/8\pi G r^2 \tag{21}$$

$$= (c^4/8\pi G r^2)(1 - 1/K^2 - 2GM/c^2 r K^2)$$

$$= (c^4/8\pi G r^2)(1 - 1/K^2 k^2)$$

Or expanded as a series in $1/r$:

$$\rho = (M^2 G/4\pi r^4) - (c^4/8\pi G r^2)((2MG/c^2 r)^3 (2/3!) - (2MG/c^2 r)^4 (3/4!) + \dots)$$

Approximations from below and above, for large *r*, are:

$$\rho \approx< (M^2 G/4\pi r^4) \qquad \text{for large } r \tag{22}$$

$$\rho \approx> (M^2 G/4\pi r^4 K^2) \qquad \text{for large } r \tag{23}$$

For large *r*, ρ is constrained between these two limits, which are close when *K* is small. Hence ρ varies with M^2/r^4 in the first approximation, for $r >> MG/c^2$. The first-order variation with M^2 may seem odd: when we integrate ρ (next) we find the full integral is proportional to *M*. But this integral is dependant on the behavior at small *r*, i.e. where $r < MG/c^2$, and higher-order terms in $1/r$ and *M* dominate.

Note *K* only becomes substantially larger than 1 in the region of the Schwarzschild (black hole) radius. E.g. at: $r = 2MG/c^2$, $K = exp(1/2) = \sqrt{e} = 1.64872$.

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[3] Vojinovic use the reverse metric signature, so we must reverse signs when we apply this.

When r becomes smaller than this, the value of ρ begins to diverge to infinity. As $r \to 0$, $\rho \to \infty$, and there is a central naked singularity. But we will see when we integrate for the mass that there is no conventional 'black hole' event horizon.

9. Integrating the mass-energy density.

We now show that the total mass-energy adds up to Mc^2, by integrating ρ over the spatial volume. This is required to match the Newtonian and Schwarzschild solutions in the limit. We first find the indefinite integral:

The mass-energy integral

$$I = \int (\rho)(4\pi r^2 dr) \qquad (24)$$

Substituting from (21):

$$I = \int (-Mc^2/4\pi r^3K^2 - c^4/8\pi Gr^2K^2 + c^4/8\pi Gr^2)(4\pi r^2 dr) \qquad (25)$$
$$= \int (-Mc^2/rK^2 - c^4/2GK^2 + c^4/2G)dr$$

This has the exact solution:

The mass-energy integral solution

$$I = -rc^4/2GK^2 + rc^4/2G + E \qquad (26)$$
$$= (rc^4/2G)(1-1/K^2) + E$$

where E is an arbitrary constant of integration. To verify this calculate:

$$d/dr(rc^4/2GK^2) = c^4/2GK^2 + (-2rc^4/2GK^3)(dK/dr)$$
$$= c^4/2GK^2 + (Mc^2/rK^2)$$

And:

$$d/dr(rc^4/2G) = c^4/2G$$

Next we obtain the limit of I as: $r \to \infty$. We expand the solution in terms of r.

$$I = (rc^4/2G)(1-1/K^2) + E \qquad (27)$$
$$= (rc^4/2G)(1-1+2MG/c^2r -(2MG/c^2r)^2/2! + (2MG/c^2r)^3/3!) - ...) + E$$
$$= Mc^2 - M^2G/r + 2M^3G^2/3c^2r^2 - ... + E$$

As we limit $r \to \infty$ all terms in r disappear and only constant terms remain:

$$I_\infty = Mc^2 + E$$

We will set the constant E equal to 0,[4] so the indefinite integral is Mc^2 at $r = \infty$. Hence the indefinite integral is:

$$I = (rc^4/2G)(1-1/K^2) \qquad (28)$$

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[4] For an empty universe. But in any realistic universe model there is a lot of background mass-energy that has to be included. However it is just the differentials of I that matter for the metric in GTR.

We next obtain the limit of I as $r \rightarrow 0$. To simplify, we can define: $r = \alpha 2MG/c^2$, i.e. r is defined as a multiple α of the fundamental distance: $2MG/c^2$. Thus: $\alpha \rightarrow 0$ as $r \rightarrow 0$, and: $\lim_{r\rightarrow 0} (I) = \lim_{\alpha \rightarrow 0} (I)$. Terms reduce to: $1/K^2 = exp(-2MG/c^2r) = exp(-1/\alpha)$, and: $rc^4/2G = \alpha Mc^2$. Substituting in I:

$$I = (Mc^2)(\alpha(1-exp(-1/\alpha)))$$

We need the value of: $\alpha(1-exp(-1/\alpha))$ as: $\alpha \rightarrow 0$. This goes 0, because:

$exp(-1/\alpha) = 1/exp(1/\alpha)$ and: $exp(1/\alpha)\rightarrow \infty$ as: $\alpha\rightarrow 0$, so: $exp(-1/\alpha) \rightarrow 0$, so:

$\alpha(1-exp(-1/\alpha))\rightarrow \alpha \rightarrow 0$. Hence:

$$I(0) = 0 \qquad \textit{and:} \qquad I(\infty) = Mc^2 \qquad (29)$$

Hence the definite integral over the whole volume of space is:

The total mass-energy integral

$$I_{0 \text{ to } \infty} = {}_{r=0 \text{ to } \infty}\int (\rho)(4\pi r^2 dr) \qquad (30)$$

$$= [(rc^4/2G)(1-1/K^2)]_0^{\infty}$$

$$= Mc^2$$

The total mass-energy of the system is Mc^2.

Note the mass-energy within a radius r is:

$$I_{0 \text{ to } r} = {}_{0 \text{ to } \infty}\int (\rho)(4\pi r^2 dr)$$

$$= [(rc^4/2G)(1-1/K^2)]_0^{r}$$

$$= (rc^4/2G)(1-1/K^2)$$

$$= (rc^4/2G)(2MG/c^2r)(1 - (2MG/c^2r)/2!+(2MG/c^2r)^2/3! - ...)$$

$$= Mc^2(1 - (2MG/c^2r)/2!+(2MG/c^2r)^2/3! - ...) \qquad (31)$$

The factor on the right is larger than $1/K^2$ and smaller than $1/K$ from at least: $r>10(MG/c^2)/3$, hence:

$$Mc^2/K > I_{0 \text{ to } r} > Mc^2/K^2 \qquad \textit{for } r >> MG/c^2 \qquad (32)$$

For $r >> MG/c^2$, the total amount of gravitational mass outside the spherical shell of r is closely approximated by: M^2G/c^2r. Conversely $M(1–MG/c^2r)\approx M/K$ is approximately the gravitational mass within the sphere of radius r. The overall effect on proper acceleration at r is similar to adopting a central mass M/K^2 in the Schwarzschild solution (Section 12). The effect of the reduced mass within the shell combined will weaken the effective mass by M/K^2 (not just M/K). Two further results help confirm the physical consistency of this solution.

Black hole radius is consistent. Although the mass-density increases indefinitely as we approach the center, the Schwarzschild (black hole) radius r_s for the central mass within r is always smaller than r, so there is no conventional 'black hole' event horizon formed inside. The Schwarzschild radius is: $r_s = 2MG/c^2$. The mass within a radius r is: $M = (rc^2/2G)(1-1/K^2)$. Substituting for M we get: $r_s = (2G/c^2)(rc^2/2G)(1-1/K^2) = r(1-1/K^2)$, or: $r_s/r = (1-1/K^2) < 1$. Hence the mass distribution appears consistent, and no problems of singularities arise, except the central (naked) singularity, which appears as in conventional GTR.

Pressure is consistent with a quasi-Newtonian force. Note if we differentiate the mass integral at r by r we get a force term, and this is exactly equal to: $dI/dr = 4\pi r^2 p$. Since $4\pi r^2$ is the surface area at r, this can be interpreted as meaning that the *total internal force of the mass distribution over the surface at r* generates the pressure term, p. Note (differentiating the series (27)) that this is like a gravitational self-attraction: $F = M^2G/r^2 - 4M^3G^2/3c^2r^3 + \dots \approx M^2G/r^2$ for large r, as if the mass M was attracting to itself at a distance of r by a quasi-Newtonian force law. Note this begins to reduce at small r, e.g. at the point where: $r = 4MG/3c^2$, the Newtonian term cancels with higher order terms as: $M^2G/r^2 - 4M^3G^2/3c^2r^3 = 0$.

We have confirmed that a specific mass-energy distribution ρ in GTR will generate the K-gravity metric.

Part 2. K-gravity as a physical theory.

10. K-gravity as an alternative physical theory.

The K-gravity metric has been treated as the solution for a special mass distribution in GTR. This is not for a *central mass,* i.e. within a finite boundary, but for an infinitely extended mass. But we now propose to consider it as *the physical solution for gravity for a central inertial mass,* and test it against the usual Schwarzschild solution.[5]

At first this may seem impossible, because the Schwarzschild solution is known to be the central mass solution in GTR. But there is one assumption in the derivation of the Schwarzschild solution that can be questioned. The two key assumptions are:

(A) *Symmetric distribution assumption:* the inertial mass distribution is radially symmetric and static and of finite extent.

(B) *Stress-energy tensor assumption:* $T_{\mu\nu} = 0$ for empty space, i.e. all space outside the central mass.

These determine the Schwarzschild solution uniquely from GTR. The symmetry assumption (A) defines the type of system being analyzed, and is not questioned. But we now consider the following alternative to (B):

(B*) *K-gravity Assumption:* $T_{\mu\nu} \neq 0$ for empty space. Instead $T_{\mu\nu}$ for an isolated central mass corresponds to the K-gravity metric.[6]

This is now proposed as an alternative physical theory, and this is what we will now refer to as *K-gravity*. This does not contradict GTR: it only contradicts the orthodox interpretation of the stress-energy tensor within GTR. The new interpretation is that a localized inertial mass M produces an *extended gravitational-mass-density field*

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[5] The original motivation for this proposal was that K-gravity is the solution for gravity in a novel unified theory (unpublished except in preprint), which treats gravity as a result of 'stretching space' in a higher dimensional manifold, with K being the strain function. This is not discussed here, and it may be considered as an independent proposal, motivated simply as a mathematical generalization.

[6] $T_{\mu\nu}$ for complex mass distributions requires a superposition principle consistent with the K-gravity metric in the limit of a single mass, as in the following section.

throughout space around it. This does not contradict GTR directly because the assignment of the *stress-energy tensor* is not part of the definition of GTR proper, but something interpreted from other physics. GTR postulates that there is a $g_{\mu\nu}$ characterizing *space-time,* and that there is a $T_{\mu\nu}$ characterizing a *mass-energy distribution,* and a precise connection between these through the Einstein equation. However the specification of $T_{\mu\nu}$ is not determined within GTR: it comes from other specialized branches of physics, i.e. particle and field theories. So what makes us assume that: $T_{\mu\nu} = 0$ for 'empty space' around a mass? It is our essentially classical assumption that mass energy is strictly localized within certain boundaries. It is this assumption that is questioned here, not GTR *per se*. The peculiarity of course is that we are now proposing a non-standard stress-energy tensor, with a contribution from "gravitational mass" not recognized in particle physics. Of course this raises further conceptual questions, but it gives a physically testable theory, and in the remainder if this paper we will examine these empirical consequences without trying to fully resolve the theory.[7]

To summarize, we can view the *K-gravity* metric in two ways. First it is a purely conventional solution for a particular type of mass density distribution. Second, we subsequently propose a novel interpretation, modifying the usual stress-energy tensor law *for a central mass, M*. This is illustrated below.

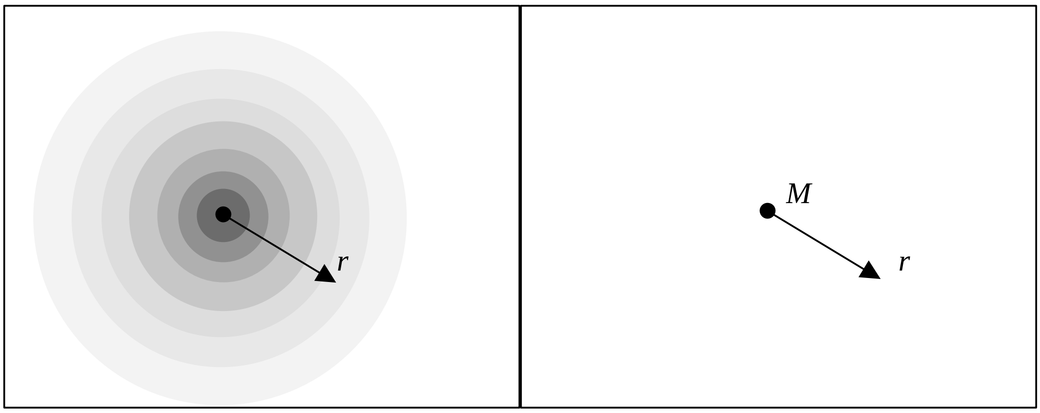

Figure 1(a). A conventional fluid with total mass M, smeared through space, thinning as r increases. Conventional GTR relates $g_{\mu\nu}(r)$ to the mass-energy density-pressure function, $\rho(r)$, via the tensor $T_{\mu\nu}(r)$. The K-gravity solution corresponds to a special case for $\rho(r)$.

Figure 1(b). A single inertial mass, M, produces a metric field $g_{\mu\nu}$. K-gravity takes $g_{\mu\nu}$ to correspond to the conventional solution for the special case of the fluid $\rho(r)$, on the left. The inertial mass M is postulated to generate the 'gravitational mass' field $\rho(r)$.

[7] The relationship between the *metric tensor* and the *stress-energy tensor* and the *mass-energy distribution* is far from clear anyway; see Lehmkuhl (2010) for an interesting recent discussion.

‘Gravitational mass’ thus becomes a *density field in space,* distinguished from the inertial mass at a central point. This is somewhat analogous to the quantisation of classical particles. A classical particle has a mass with a location and trajectory, but is treated in quantum mechanics as being ‘smeared out’ through space, as described by a quantum wave function. This required a radical change in the conception of physical particles. K-gravity proposes the ‘gravitational mass’ of point-like particles is also smeared through space; and this is a similarly radical change in the conventional conception of gravitational mass.[8]

This requires us to distinguish between *inertial mass*, which we conceive as the centralized mass of a localized body, and *gravitational mass*, which now becomes a *mass density function across space.* The conceptual distinction between inertial and gravitational mass was emphasized by Einstein (1919), and played an important role in his thought. ‘Inertial mass’ has a trajectory, and carries energy and momentum, and it is what is accelerated by forces. ‘Gravitational mass’ is the charge for the gravitational field. Einstein recognized the conceptual importance of identifying them (in the Newtonian and GTR theories). But this also introduces the conceptual possibility of separating them, which K-gravity does. There are two key questions:

- Is this proposal coherent as a physical theory of gravity?
- Is it empirically testable and observationally realistic?

It is certainly empirically testable, and we will examine how we may compare predictions of Schwarzschild gravity (1) against K-gravity (2) in the solar system. But we first need to extend the theoretical idea a little. We have defined the K-gravity prediction of $g_{\mu\nu}$ for a single central point mass. But to be claimed as a coherent theory worth testing, we want to show it can be plausibly generalized to deal with distributions of multiple masses. This lets us consider it as a potential general law of nature: the metric (2) by itself appears as a merely *ad hoc* solution.

11. Superposition principle for multiple masses.

We now propose a *superposition principle* for obtaining solutions for $g_{\mu\nu}$ more generally, for multiple masses, with a suitable universal law-like character. The first point is that we cannot do this through simple superposition of the mass-energy density functions ρ (Section 8) for individual masses, because they are not suitably linear. Normally if we define two distinct mass distributions in space, we can simply add their masses together to get a combined mass distribution. This is how we superpose classical inertial mass distributions. But the K-gravity mass-energy density functions ρ, or equivalently the $T_{\nu\mu}$, for individual masses, cannot work like this.[9] We

[8] It should be emphasized that this ‘smearing out’ of the gravitational mass across space does not correspond to the quantum wave function for the mass. Note also that GTR cannot deal consistently with quantum wave functions in the most fundamental respect: viz. quantum distribution of matter is given by *superpositions of position states,* but when these undergo wave function collapse, there is no concept in GTR for the metric tensor to undergo collapse. GTR is deterministic. This is just one sign of fundamental incompleteness of GTR, due to the failure to unify GTR and quantum theory.

[9] Quantum systems do not work like this either: multi-particle systems are superpositions in a Hilbert product space: they do not have the part-whole structure of a classical system. In GTR we normally

cannot take the solution (21) and add $\rho_{(n)}$'s for a collection of N masses together to get a total: $\rho_{total} = \rho_{(n)}$ for the whole system of masses, and then use this to derive $T_{\nu\mu}$ for the whole system, and subsequently derive $g_{\nu\mu}$ from the Einstein equation. The reason is that ρ, or equivalently $T_{\nu\mu}$, for the K-gravity central mass is not linear with mass.

E.g. using the approximation: $T_{11} \approx M^2G/4\pi r^4$, which is accurate for large r, and defining M as a composite mass: $M = M_1+M_2$, we see:

$$\begin{aligned} T_{11}(M_1+M_2) &\approx (M_1+M_2)^2G/4\pi r^4 \\ &= M_1^2G/4\pi r^4 + M_2^2G/4\pi r^4 + 2M_1M_2G/4\pi r^4 \\ &\approx T_{11}(M_1) + T_{11}(M_2) + 2M_1M_2G/4\pi r^4 \end{aligned} \qquad (33)$$

Hence: $T_{11}(M_1+M_2) > T_{11}(M_1) + T_{11}(M_2)$. E.g. when: $M_1 = M_2 = M/2$, we have: $T_{11}(M_1+M_2) \approx 2T_{11}(M_1) + 2T_{11}(M_2)$, not: $T_{11}(M_1+M_2) = T_{11}(M_1) + T_{11}(M_2)$.

We cannot write a superposition principle directly in terms of ρ or $T_{\mu\nu}$. But instead there is a way to define $g_{\mu\nu}$ *directly from the mass distribution without going through the stress-energy tensor at all.* The following treatment is only for static systems: but it shows there is a plausible generalization from the simple metric (2) to a more general law for multiple masses, which must be sufficient for our present purpose.

The most important property of K is the linear separability w.r.t. M. If we define an aggregate mass: $M = M_1+M_2$ then:

$$K(M_1+M_2) = exp((M_1+M_2)G/c^2r) = exp(M_1G/c^2r)exp(M_2G/c^2r) \qquad (34)$$

This corresponds to a basic superposition property: treating a single mass M as a superposition of two component masses, M_1+M_2.

- The effect of imposing an aggregate mass of: (M_1+M_2) on empty space at a given point (by: $K = exp((M_1+M_2)G/c^2r))$ is identical to imposing M_1 on empty space first (by: $K_1 = exp(M_1G/c^2r))$, and then imposing M_2 linearly on the resulting space at the same point (by: $K_{12} = K_1K_2 = exp(M_1G/c^2r)exp(M_2G/c^2r))$.

This symmetry of K is the essential key. k does not have this symmetry. E.g. $k(2M)$ and $k(M)k(M)$ differ by a factor of approximately $1+(2MG/c^2r)^2$ for large r. So the following kind of superposition method is impossible to apply in standard GTR.

The linearity of K means first that there is a well-defined function generalizing K for multiple masses (at multiple positions). This will be called the *K scalar field.* This is defined over all the masses, N, in the space. It is just the product of all the individual $K(M_n,r_{(n)})$*'s* for the individual masses. At any field point K is defined:

$$K \equiv K(M_1,r_{(1)})K(M_2,r_{(2)}) \ldots K(M_N,r_{(N)}) \qquad \textbf{\textit{The K scalar field}} \qquad (35)$$

The $r_{(i)}$*'s* are the distances from the field point O to the masses M_n. We may write this:

$$K \equiv K(M_1/r_{(1)}+ \ldots +M_N/r_{(N)}) = exp((G/c^2)(\textstyle\sum_{n=1 \text{ to } N}(M_n/r_{(n)}))) \qquad (36)$$

This magnitude depends only the masses M_n and their distances: $r_{(n)} = |\mathbf{r}_{(n)}|$ from the field point. I.e. it is independent of the relative directions of the masses. We now

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assume (special-relativistic versions of) classical particle and field theory for the stress-energy tensor. The combination of GTR with quantum theory is fundamentally unresolved anyway.

propose a superposition principle for static mass distributions defined simply in terms of K and its gradient.

The *gradient field of K* is defined in local rectangular coordinates for the empty space at a field point O as usual, with $i = 1,2,3$ and $\underline{\mathbf{x}}_i$ the basis vectors for coordinates: x^i. This is developed only for a static mass distribution here: it can be generalized to a 4-vector form using retarded potentials (by the close analogy with EM theory), and this makes it covariant, but we cannot develop the full theory here. It must be enough for our present purposes to illustrate that a plausible superposition principle exists.

$$\boldsymbol{K} = \boldsymbol{grad}(K) = (\partial K/\partial x^i)\underline{\mathbf{x}}_i \qquad \textbf{\textit{The K gradient field}} \qquad (37)$$

To differentiate note: $\boldsymbol{grad}(r_{(n)}) = \underline{\mathbf{r}}_{(n)}$ and: $\boldsymbol{grad}(M_n/r_{(n)}) = -(M_n/r_{(n)}^2)\underline{\mathbf{r}}_{(n)}$. So (summing over the masses n):

$$\boldsymbol{grad}(K) = \boldsymbol{grad}(\exp((G/c^2)\textstyle\sum(M_n/r_{(n)}))) \qquad (38)$$

$$= -(KG/c^2)\textstyle\sum((M_n/r_{(n)}^2)\underline{\mathbf{r}}_{(n)})$$

This is simply related to the Newtonian acceleration field, as illustrated.

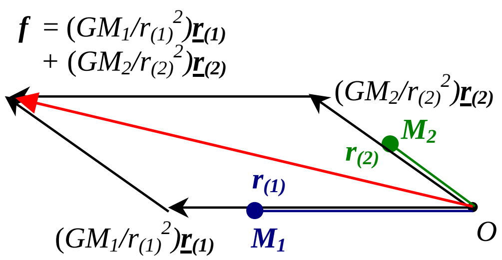

*Figure 2. The vector sum **f** of Newtonian acceleration fields, illustrated for two masses. The K-gradient field is closely related.*

The *K gradient field* is just K/c^2 times the *Newtonian gravitational acceleration vector field.* The latter will be denoted $\boldsymbol{f}$, defined:

$$\boldsymbol{f} = -G\textstyle\sum((M_n/r_{(n)}^2)\underline{\mathbf{r}}_{(n)}) \qquad \text{Newtonian acceleration field} \qquad (39)$$

Thus:

$$\boldsymbol{grad}(K) = K\boldsymbol{f}/c^2 \qquad (40)$$

$\boldsymbol{grad}(K)$ and $\boldsymbol{f}$ are vector fields.

- The magnitude of $\boldsymbol{f}$ may be written: $f = |\boldsymbol{f}| = (\boldsymbol{f}\cdot\boldsymbol{f})^{1/2}$ (a scalar field).
- The direction of $\boldsymbol{f}$ may be written: $\underline{\boldsymbol{f}} = \boldsymbol{f}/f$.
- The magnitude of $\boldsymbol{grad}(K)$ may be written: $grad(K) = |\boldsymbol{grad}(K)|$ (scalar field).

The scalar fields are thus also simply related:

$$grad(K) = Kf/c^2 \qquad (41)$$

We only need to work with one or other of $\boldsymbol{f}$ or $\boldsymbol{grad}(K)$, and we will use $\boldsymbol{f}$. We now state a set of rules to determine $g_{\mu\nu}$ directly for multiple source masses. We state this initially *in the special local rectangular coordinate system, at the field point O, with* x^1 *chosen in the direction of* $\underline{\boldsymbol{f}}$. The $g_{\mu\nu}$ representation is diagonalised in this coordinate system. The first two rules are general for static systems:

- First, the off-diagonal terms with a time component are zero:

$$g_{01} = g_{02} = g_{03} = g_{10} = g_{20} = g_{30} = 0 \qquad (42)$$

- Second, the *t-t* component g_{00} is defined:

$$g_{00} = c^2/K \qquad (43)$$

The third rule for the case where $\underline{\boldsymbol{f}}$ is in the first coordinate direction $\underline{\boldsymbol{x}}_1$ is:

- Third, in our special rectangular coordinates with $x = x^1$ chosen in the direction of $\underline{\boldsymbol{f}}$ the spatial components are:

$$g_{11} = -1-(\boldsymbol{f}.\underline{\boldsymbol{x}}_1/f)^2(K^2-1), \quad \text{all other } g_{ij} = -\delta_{ij}. \qquad (44)$$

Hence the full metric tensor is:

$$\begin{array}{lllll} \textit{Coordinates:} & t=x^0 & x=x^1 & y=x^2 & z=x^3 \\ [g_{\mu\nu}] = & \left(\begin{array}{l} c^2/K^2, \\ 0, \\ 0, \\ 0, \end{array}\right. & \begin{array}{l} 0, \\ -1-(\boldsymbol{f}.\underline{\boldsymbol{x}}_1/f)^2(K^2-1), \\ 0, \\ 0, \end{array} & \begin{array}{l} 0, \\ 0, \\ -1 \\ 0, \end{array} & \left.\begin{array}{l} 0 \\ 0 \\ 0 \\ -1 \end{array}\right) \end{array} \qquad (45)$$

Note first that the 'time dilation' component, viz. $g_{00}/c^2 = (\partial\tau/\partial t)^2$, is always given by $1/K^2$. All masses M_n contribute to this by the factor: $exp(GM_n/c^2r_{(n)})$, independent of their direction from the field point. Thus a field point at the center of mass of a galaxy with a first moment of inertia *Mr* will have the same time dilation effect as a field point at distance *r* from a single mass *M.* This is in conformity with ordinary GTR, within a factor of: K^2/k^2.

Now for the space components, which determine the accelerations, in this special case: $(\boldsymbol{f}.\underline{\boldsymbol{x}}_1/f)^2 = 1$, because $\boldsymbol{f}$ is chosen in the direction of $\underline{\boldsymbol{x}}_1$ and $\boldsymbol{f}.\underline{\boldsymbol{x}}_1 = f$. So we can just write: $g_{11} = -K^2$. It is written in the functional form above to compare with the form of the more general case, which is:

$$g_{ij} = -\delta_{ij} - (\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_j/f)(K^2-1) \qquad (46)$$

This is the third rule generalized for rectangular coordinates (x^i) rotated with respect to $\underline{\boldsymbol{f}}$ in a plane of $\underline{\boldsymbol{f}}$ by an angle θ. The dot product: $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i$ gives the magnitude of $\boldsymbol{f}$ in the $\underline{\boldsymbol{x}}_i$ direction, and we may write this as: $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i = f_i$.

For consistency as a tensor relation, this metric (46) in rotated spatial coordinates in the *x-y* plane of $\boldsymbol{f}$, by an angle θ, must match that obtained through the usual coordinate transformation rule: $g_{\mu\nu}' = g_{kl}\,(\partial x^k/\partial x'^{\mu})(\partial x^l/\partial x'^{\nu})$, with the Jacobian $(\partial x^k/\partial x'^{\mu})$ for rotation defined:

$$\begin{pmatrix} 1, & 0, & 0, & 0 \\ 0, & \cos\theta, & -\sin\theta, & 0 \\ 0, & \sin\theta, & \cos\theta, & 0 \\ 0, & 0, & 0, & 1 \end{pmatrix} \qquad (47)$$

This transforms the simple K-gravity metric (2) or (45) to:

K-Gravity metric tensor for M in rotated Cartesian coordinates *(48)*

$$[g_{(1)\mu\nu}]' = \begin{pmatrix} c^2/K^2, & 0, & 0, & 0 \\ 0, & -1-\cos^2\theta(K^2-1), & -\cos\theta\sin\theta(K^2-1), & 0 \\ 0, & -\cos\theta\sin\theta(K^2-1), & -1-\sin^2\theta(K^2-1), & 0 \\ 0, & 0, & 0, & -1 \end{pmatrix}$$

We need to confirm we get the same result by using our general rule (46) to assign the components in a rotated frame. First consider the simple central mass case, where: $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i = f$, and use (46) to assign components in a rotated frame. In the simple frame, $x = x^1$ is chosen in the direction $\boldsymbol{f}$, so: $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1 = f$, and $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2 = \boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3 = 0$. Now suppose we rotate in the x-y plane by θ, as in the transformation (35). We find that:

$$\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1 = f\cos\theta \qquad \boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2 = f\sin\theta \qquad (49)$$

This is simply the vector geometry of rotating $\boldsymbol{f}$. Thus we find the components directly from (46) as:

$$g_{11} = -\delta_{11} - (f\cos\theta/f)(f\cos\theta/f)(K^2-1) = -1 - \cos^2\theta(K^2-1) \qquad (50)$$

$$g_{22} = -\delta_{22} - (f\sin\theta/f)(f\sin\theta/f)(K^2-1) = -1 - \sin^2\theta(K^2-1)$$

$$g_{12} = -\delta_{12} - (f\cos\theta/f)(f\sin\theta/f)(K^2-1) = -\cos\theta\sin\theta(K^2-1)$$

$$g_{21} = -\delta_{21} - (f\sin\theta/f)(f\cos\theta/f)(K^2-1) = -\cos\theta\sin\theta(K^2-1)$$

This confirms we get the same result using (46) directly as we get by transforming the diagonalised metric to the rotated coordinate system (48). This holds generally when $\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_i = f_i < f$, because f_i/f acts as a constant when we differentiate the g_{ij}. We may write the rule (46) in a generalized matrix form:

K-metric for a static system in rectangular coordinates *(51)*

$$\begin{pmatrix} c^2/K^2, & 0, & 0, & 0 \\ 0, & -1-(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)^2(K^2-1), & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)(K^2-1), & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)(K^2-1) \\ 0, & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)(K^2-1), & -1-(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)^2(K^2-1), & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)(K^2-1) \\ 0, & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_1/f)(K^2-1), & -(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_2/f)(K^2-1), & -1-(\boldsymbol{f}\bullet\underline{\boldsymbol{x}}_3/f)^2(K^2-1) \end{pmatrix}$$

and this is consistent with general coordinate rotations. This is the metric represented in *orthogonal rectangular coordinates* (in any direction). To get general coordinates we apply tensor transformations as usual.

To verify this is physically realistic we examine accelerations next. These are also needed to discuss empirical tests, in the subsequent section.

12. Acceleration of a stationary test particle.

We will use: U^μ for the velocity 4-vector and A^μ for the acceleration 4-vector. These are the differentials of the x^μ w.r.t. proper time, $d\tau$. Thus for a *stationary test particle* at the field point: $U^0 = dt/d\tau = c/\surd g_{00} = K$, and $U^i = dU^i/d\tau = 0$ for the spatial velocities. The general tensor relationship for acceleration is:

$$A^\kappa = U^\lambda \nabla_\lambda U^\kappa$$
$$= U^\lambda(\partial U^\kappa/\partial x^\lambda + \Gamma^\kappa{}_{\lambda\mu}U^\mu) \qquad (52)$$

For the stationary particle, only $U^0 \neq 0$, and this simplifies to:

$$A^\kappa = U^0(\partial U^\kappa/\partial x^0 + \Gamma^\kappa{}_{00}U^0)$$
$$= (U^0)^2\Gamma^\kappa{}_{00} \qquad (53)$$

For a Schwarzschild-type metric, the only non-vanishing Christoffel symbol is $\Gamma^1{}_{00}$. So the proper-time acceleration: $d^2x/d\tau^2$ of a stationary particle at a field point ***O*** is:

$$A^1 = (c^2/g_{00})\Gamma^1{}_{00} = (c^2/g_{00})(\partial g_{00}/\partial x)(1/2g_{11}) \qquad (54)$$

This is then equal to:

$$A^1 = (c^2/g_{00})\Gamma^1{}_{00} = -(c^2K^2/c^2)(c^2\partial K^{-2}/\partial x)(1/2K^2)$$
$$= -\tfrac{1}{2}c^2(\partial K^{-2}/\partial x) \qquad (55)$$

In the case of the simple single-mass K, the differential is simply: $\partial K^{-2}/\partial x = 2MG/c^2r^2K^2$, and the result is: $A^1 = MG/r^2K^2$. (*c.f.* the Schwarzschild result is: $A^1 = MG/r^2$. Thus we see that the Schwarzschild acceleration is greater by a factor of K^2.)

This is the acceleration with respect to proper time. The acceleration in real time t *for a stationary test particle with:* $dx/dt = 0$ is then: $a = d^2x/dt^2 = A^1(d\tau/dt)^2 = MG/r^2K^4$. (c.f. the Schwarzschild result is: $a = MG/r^2k^2$.)

In the more general case with multiple masses, we do not have a spherically symmetric metric, the Christoffel symbols $\Gamma^\kappa{}_{00}$ other than $\Gamma^1{}_{00}$ are not generally vanishing, and we have to go back to the more general equation (53). However we can use the special assumption at the field point ***O***, that we have chosen $x = x^1$ in the direction of ***f***. The differentials of K tangent directions are then zero *at this point*, and *for this point* the Christoffel symbols $\Gamma^\kappa{}_{00}$ do vanish except for $\Gamma^1{}_{00}$.

For the generalized *K scalar field,* the differential: $\partial K^{-2}/\partial x$ is of course no longer simply: $2MG/c^2r^2K^2$. It is given through the gradient function: $\partial K/\partial x = \mathbf{grad}(K).\underline{\mathbf{x}} = K\mathbf{f}.\underline{\mathbf{x}}/c^2$. We have: $\partial K^{-2}/\partial x = -2\mathbf{f}.\underline{\mathbf{x}}/c^2K^2$. Thus the result of calculating A^1 is more generally:

$$A^1 = (c^2/g_{00})\Gamma^1{}_{00} = -\tfrac{1}{2}c^2(\partial K^{-2}/\partial x) = \mathbf{f}.\underline{\mathbf{x}}/K^2 \qquad (56)$$

This conforms to the Newtonian acceleration (within the factor $1/K^2$) because $\mathbf{f}.\underline{\mathbf{x}}$ is just the Newtonian acceleration.

We can give a simple example to illustrate. Take a field-point ***O*** half-way between two masses of magnitude M and $2M$ respectively, at a distance r_0 from each.

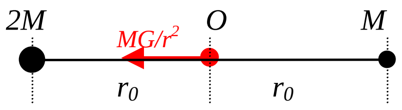

Figure 3. Field point halfway between two unequal masses.

The resultant Newtonian acceleration is towards the larger mass with: $f = MG/r^2$. The function K in x is:

$$K = exp((G/c^2)(2M/(r_0\text{-}x)+M/(r_0+x)) \qquad (57)$$

The magnitude at the field point, where $x = 0$, is simply: $K = exp((G/c^2)(3M/r_0)$. However notice the signs of the variable x are different in the two denominators in K, so when we differentiate we get:

$$\partial K/\partial x = ((G/c^2)(2M/(r_0\text{-}x)^2)\text{-} (G/c^2)(M/(r_0 +x)^2))K \qquad (58)$$

The value at the field point where $x = 0$ is: $\partial K/\partial x_{|x=0} = (GM/c^2r_0)^2K = Kf/c^2$. And this is what gives the correct acceleration.

(56) is not a general expression for K either of course, because it does not show the general dependence on the other two coordinates, y and z. The general expression is rather given by defining radial distance variables:

$$r_{(1)} = \sqrt{((r_0+x)^2+y^2+z^2)}, \qquad r_{(2)} = \sqrt{((r_0\text{-}x)^2+y^2+z^2)} \qquad (59)$$

and then writing:

$$K = exp((G/c^2)(M/r_{(1)} +2M/r_{(2)})) \qquad (60)$$

When we differentiate this w.r.t. x, y and z we get the same result.

The result is that *because we have chosen* ***x*** *in the direction* ***f*** *at the field point* ***O****, only the differential w.r.t. x is non-zero at that point.* This is why the matrix is diagonal *at the point* ***O*** in this coordinate system. However when we have multiple masses, the differentials in all directions are involved. And we are no longer writing general functions over the entire space: only special cases at the point ***O***.

We conclude this discussion of a superposition principle for K-gravity here. Of course it is a further problem to generalize this for *dynamic systems.* Source masses in motion may be treated as retarded sources, like moving electric charges in electrodynamics. But this is beyond the scope here. This development is only meant to justify raising the question of the empirical accuracy of (2), by showing that it has a plausible generalization to a more general theory for multiple masses, with the general character of a natural law. There are various choices to fully generalize it, and a theoretical development may drag on indefinitely. But there is a clear path to empirically testing it. We do not have to establish a full theory for this. This was also the situation when GTR was first developed: it was subject to initial tests against Newtonian gravity without fully understanding its theoretical implications. Similarly, K-gravity may be immediately tested against Schwarzschild gravity.

13. Empirical tests.

We now consider how K-gravity compares empirically against Schwarzschild gravity. The only available testing domain is solar system gravity, with the sun acting as an approximately spherical central mass for the major gravitational effect. This is a weak gravity domain, and the two metrics (1) and (2) give very similar predictions for this, because (a) the functional form of the metrics are very similar, predicting very similar qualitative effects in weak gravity, and (b) k and K are very close in these weak fields, giving very similar quantitative effects. The critical term: MG/c^2r is about 10^{-8} for the gravity of the sun at roughly 1 AU.[10] Hence the terms k and K from the sun for inner planetary orbits typically differ by about: $(k - K) \approx (MG/c^2r)^2 \approx 10^{-16}$. This is not directly detectible in itself. Rather, the key difference is for *accelerations* of slow-moving bodies, which differ by the factor: $K^2 \approx 1+2MG/c^2r$ (Section 12). Accelerations by the sun at orbits around 1 AU calculated with the Schwarzschild solution will be about $1+10^{-8}$ times larger than those calculated using K-gravity using the same M_{sun}/r. The accuracy to which we can measure $M_{sun}G$ is a critical limiting factor for testing this.[11] The relative uncertainty in $M_{sun}G$ is currently claimed to be around 10^{-11}. This precision would make predicted differences well-measurable in principle. However this accuracy is obtained from averaging over hundreds of thousands of measurements of planets and space probes taken over decades (Pitjeva 2015). But to test K-gravity directly through accelerations we would need very precise measurements *taken at single orbits*. The relative error of 10^{-11} in $M_{sun}G$ claimed for averaged results is not relevant to this: error in single experiments (with space probes) is much poorer than this. And it is not a simple matter of measuring acceleration at a single orbit: we have to compare measurements of acceleration *at 2 different orbits.* We need to investigate whether gravitational experiments at single orbits can be made sufficiently accurately to decide between the two theories.

But before we go on to look at this, the first thing to emphasize is that the classic tests of GTR against Newtonian gravity do not distinguish the Schwarzschild solution from K-gravity. The relativistic phenomena of K-gravity are *qualitatively* identical to those of Schwarzschild gravity (in weak gravity).[12] Bending of light, gravitational red shift, time dilation and orbital precession all work in K-gravity almost exactly as in Schwarzschild gravity. These phenomenon represent distinctive *mechanisms* in GTR that are absent from Newtonian theory, and hence they provide the primary observations to compare those two theories. But there are no such qualitative differences (i.e. no distinct causal mechanisms) between Schwarzschild gravity and

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[10] While MG/c^2r is only about 7×10^{-10} for the gravity of the Earth at the surface.

[11] The CODATA (2014) recommended value of the gravitational constant G alone has a relative uncertainty of 4.7×10^{-5}, which is poor. This is the uncertainty provided by laboratory-scale experiments, which cannot provide a test of K-gravity. Hence there is about the same error in estimates of solar or planetary masses. The accuracy of $M_{sun}G$ is much better; see Pitjeva (2015).

[12] In strong gravitational fields, close to the Schwarzschild radius, the theories diverge sharply – e.g. there is no event horizon in K-gravity - but we cannot yet observe such fields in any detail, and there is no experimental confirmation for the existence of the Schwarzschild black hole event horizon yet.

K-gravity, only fine quantitative differences. The classic tests of Schwarzschild gravity against Newtonian gravity are not sensitive enough to distinguish Schwarzschild gravity against K-gravity.

However there is one set of observations which is potentially precise enough, viz. the Pioneer spacecraft trajectories. This initially promised to give a sensitive quantitative measurement of gravitational acceleration over a large radial trajectory. This appears to be the only direct measurement to date of sufficient precision to directly test between the two metrics. If the Pioneer data had unambiguously confirmed Schwarzschild gravity, this would have contradicted K-gravity. But instead anomalies famously appeared in the data, inconsistent with Schwarzschild gravity. In an earlier study (unpublished pre-print; 2004), it was found that these anomalies are close to the predictions of K-gravity. But this evidence is now weak, because, after many years searching for a conventional explanation, it has now been claimed (Turyshev *et alia*, 2012) that the anomalies are due to anisotropic radiation from the spacecraft, and many physicists now accept this. But this proposed explanation is by no means certain, and there is no experimental replication of the phenomenon. This is discussed briefly below, and it is proposed a new experiment is the only way to decisively test the matter. First however we look briefly at the basic concept of testing the theories through measurement of accelerations. Although this is not the most practical method, it reveals essential concepts.

The conceptual starting point is that the predicted difference in accelerations, for slowly moving test particles in weak gravity, is K^2 (Section 12). For Earth orbit (1 AU), K^2 is about $1+10^{-8}$. For Saturn orbit (10 AU), it is reduced to about $1+10^{-9}$.

Now for a stationary observer *at a fixed orbit*, adopting K-gravity instead of Schwarzschild gravity is essentially the same as recalibrating the estimated magnitude of $M_{sun}G$ for the sun by the factor K^2 at that orbit. Testability at first sight seems to depend upon whether acceleration measurements are made accurately enough to detect the difference between $M_{sun}G$ and $M_{sun}GK^2$. But measuring absolute accelerations at one orbit is no good: for these are what we use to *determine* $M_{sun}G$ in the first place (under the assumption of the Schwarzschild metric). Any such observation at a single orbit is equally consistent with K-gravity: we would just recalibrate $M_{sun}G$ by the factor K^2. Instead we must compare *accelerations across different orbits.*[13] This point is critical and needs a brief analysis.

$M_{sun}G$ may be measured quite accurately: to a relative uncertainty of around 10^{-9} - 10^{-10}, using space probes, *at a single orbit* (over a period of several orbits, i.e. several years for 1 AU). So it might seem a difference of K^2 could be immediately detected in absolute accelerations. But to repeat the point above, this is wrong. $M_{sun}G$ at a single orbit may be calculated from measuring *acceleration* (of orbiting bodies or space probes), and then using the assumption of Schwarzschild gravity to infer $M_{sun}G$. If we assume K-gravity instead, we would just infer that $M_{sun}G$ is larger by K^2, using K for the orbit where we measured the acceleration. Note because K-gravity is weaker, we infer a *larger* $M_{sun}G$ from the same observed acceleration. $M_{sun}G$ inferred from Schwarzschild gravity from a single orbit will be consistent by definition with $M_{sun}GK^2$ inferred from K-gravity.

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[13] The same applies to time dilation or red shift effects, but red shift effects are measured to relative error only about 10^{-6} (Will 2014 p.13-15) and are not sensitive enough. Measurement of the precession of the perihelion of Mercury is less accurate again.

To use acceleration measurements, we can to measure accelerations at *two different orbits*, and compare their values. (The inference to a value of $M_{sun}G$ is really a proxy for acceleration.) Suppose we first determine (proper) *accelerations:* $M_{sun}G/r_1^2$ and $M_{sun}G/r_2^2$ at two different orbits, using the assumption of Schwarzschild gravity to infer $M_{sun}G$. Their difference is:

$$\Delta_N = (M_{sun}G/r_1^2 - M_{sun}G/r_2^2) = (M_{sun}G)(1/r_1^2 - 1/r_2^2) \qquad (61)$$

We can measure this accurately to the sum of relative uncertainties in the terms. This uncertainty involves the r terms as well as $M_{sun}G$. Let us define this uncertainty (to one standard error) as: $\pm\varepsilon M_{sun}G/r_1^2$. Now this depends on the measurements we have done at both orbits. If we do a very careful measurement at a primary orbit r_1 we may get a small error, but for a good comparison we need a similarly careful measurement at r_2 and as we will now see, we need r_2 to be in a suitable range to maximize the predicted acceleration differences.

To see the predictions of K-gravity, we can *recalibrate* $M_{sun}G$ to the value: $M_{sun}GK_1^2$, at the primary orbit r_1, and then use this value for $M_{sun}G$ at r_2. The accelerations predicted by K-gravity will then be very close to: $M_{sun}GK_1^2/r_1^2K_1^2 = M_{sun}G/r_1^2$ and $M_{sun}GK_1^2/K_2^2r_2^2$. Their difference will then be predicted as:

$$\Delta_K = (M_{sun}G/r_1^2 - M_{sun}GK_1^2/K_2^2r_2^2) = (M_{sun}G)(1/r_1^2 - K_1^2/K_2^2r_2^2) \qquad (62)$$

Expanding the K term, this is approximately:

$$\Delta_K \approx (M_{sun}G)(1/r_1^2 - 1/r_2^2 - (1/r_2^2)(2M_{sun}G/c^2)(1/r_1 - 1/r_2)) \qquad (63)$$

Then the absolute difference: $\Delta_K - \Delta_N$ is:

$$\Delta_K - \Delta_N = -(M_{sun}G)(1/r_2^2)(2M_{sun}G/c^2)(1/r_1 - 1/r_2) \qquad (64)$$

This is the difference between the two theories for the accelerations predicted at r_2.

Define $\sigma = r_1/r_2$, so this is:

$$\Delta_K - \Delta_N = -(M_{sun}G\sigma^2/r_1^2)(2M_{sun}G/c^2r_1)(1-\sigma) \qquad (65)$$

Now we need to compare this magnitude to the error term: $\pm\varepsilon M_{sun}G/r_1^2$.

Dividing gives:

$$(\Delta_K - \Delta_N)/(\varepsilon M_{sun}G/r_1^2) = -2\sigma^2(1-\sigma)(M_{sun}G/c^2r_1)(1/\varepsilon) \qquad (66)$$

Effects become conclusively detectible when this is substantially greater than 1, say on the scale of 10.[14] Let us set this to *±20* to define a *conclusively detectible limit,* so:

$$\sigma^2(1-\sigma)(M_{sun}G/c^2r_1) = \pm 10\varepsilon \qquad \textit{Conclusively detectible limit for } \varepsilon$$

Now we can put in approximate numbers, for $r_1 = 1\ AU$ as: $M_{sun}G/c^2r_1 \approx 10^{-8}$, so:

$$\varepsilon \approx \pm\sigma^2(1-\sigma)10^{-9}$$

This tells us the maximum limit of ε required at different choices of $\sigma = r_1/r_2$ to achieve a clear detection of the K-gravity effect.

[14] Theoretically 4-5 standard errors is sufficient; but the likelihood of systematic error through miscalculation of small effects, like radiation pressure, means we want a better precision to conclusively confirm or disconfirm an effect.

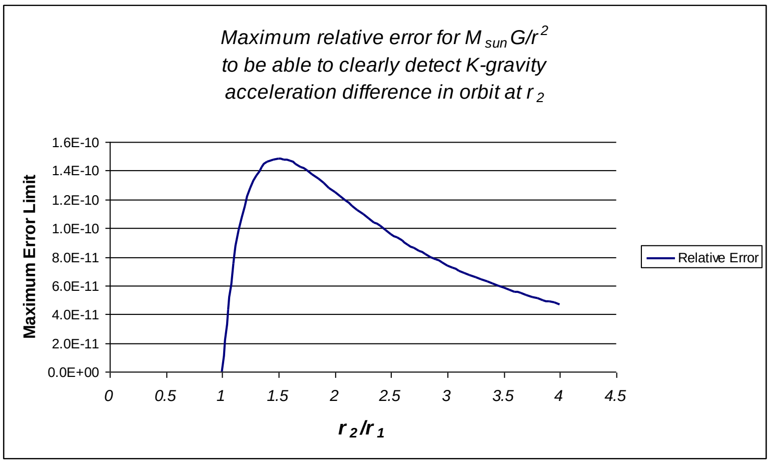

Figure 4. Graph of ε against r_2/r_1

This illustrates minimal precisions required in acceleration measurements to achieve experimental precision of the z-score = *±20*, for a range of the second orbit radius. The best precision is found when r_2/r_1 is 1.5. (Or inversely, *0.66*). I.e. assuming the closest orbit is at r_1 = 1 AU, the second orbit would need to be around 1.5 AU for the most robust experiment. Orbits from about 1.3 – 2.5 AU will be good enough if we can achieve relative measurement error better than about 10^{-10} for the acceleration measurements at both orbits. Note this error must include compensation terms for forces other than the solar gravity component, e.g. solar radiation, solar wind, dust or small particle collisions, planetary tugs, possible EM forces, oblateness of the sun, and velocity of the probe. (The term *r* can be measured with sufficient accuracy).

Since relative measurement error of 10^{-10} is at the present limit for measuring accelerations of space probes, it is possible for this experiment to be done, but not easy, and it would take many years. Data for such a test is not available from previous experiments. It requires high-precision measurements at two appropriate distances from the sun, but these have not been done. Without a theory here is no way to guess the appropriate distances. And no analysis of data from gravitational experiments has been undertaken to test this. Any inconsistencies present in current gravitational data that might confirm K-gravity have gone unexplained. Current analyses do not envisage a possibility like K-gravity, where the gravitational field *changes shape significantly compared to Schwarzschild gravity with radial distance.*[15]

But there is a much better way to do the experiment than by measuring accelerations separately at two orbits; viz. by carefully observing the trajectory of a probe in radial free-fall. The optimal experiment is a probe launched at an optimal speed, traveling radially from Earth to about Jupiter orbit, over several years. Such an experiment has not been done; but the tracking of the Pioneer spacecraft at more distant orbits (traveling from around 10 AU to 80 AU) provides a similar experiment in principle. As an experiment to test K-gravity this is far from ideal: it is in weaker gravity, and the speed is not optimal. But as far I know this is the only data of sufficient precision available to potentially test the hypothesis of K-gravity.

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[15] Variations conceived in the framework of the Paramistised Post-Newtonian formalism do not allow for such differences, and neither do popular viable alternative theories of gravity.

There are two reasons it provides a more accurate test of acceleration than experiments at closer orbits. First it was taken over a long period of time, so small differences in acceleration accumulate. Second, it involves a free-fall trajectory over a substantial range of r, from about 10 AU (when the spacecraft left Saturn's orbit) to around 80 AU. This second point is most important. The functions K and k which determine accelerations *change shape over changes of r.* It is easier to detect the predicted anomalies in radial trajectories observed over an appropriate range of r than to detect differences in accelerations at two orbits directly.

This also leads to a critical realization when analyzing the effects on radial trajectories. Because K gravity is *weaker* than Schwarzschild gravity, for the same assumed $M_{sun}G/r$, we intuitively expect it to predict that probes traveling in free-fall outwards to large r will travel *faster* under the K-gravity metric. "*The Pioneers have been slowing down faster than predicted … some tiny extra force … must be acting on the probes, braking their outward motion."* (Musser 1998). This is true if the cause of slowing is a non-gravitational force. But if the cause is a modified form of gravity, the opposite is the case: a weaker rather than stronger form of gravity is required.

Because $M_{sun}G/r$ is initially *calibrated* from the inner solar system, on the assumption of Schwarzschild gravity, from the point of view of K-gravity this leads us to *underestimate* the magnitude of $M_{sun}G/r$ (by $1/K^2$). If K-gravity is correct, then we should *increase* the conventional magnitude of $M_{sun}G/r$ by this factor, i.e. K^2. In weak gravity, the differences between K and k are very small, and we will get almost the right acceleration predictions for K-gravity from the conventional Schwarzschild analysis - but by applying it with the larger value: $M_{sun}GK^2/r$ instead of $M_{sun}G/r$. This is what we saw in the analysis above.

If K-gravity is correct, we should notice the spacecraft slowing down faster than expected on the basis of the Schwarzschild solution. There should be an increasing delay in the expected position. This is exactly what was observed with the Pioneer spacecraft. Anomalies of about a 16 seconds delay in the expected journey to around 80 AU appeared, and I have found a similar magnitude of difference (predicting about 12 - 18 seconds delay, sensitive to uncertainties in initial parameters).

However as noted above, the Pioneer situation has subsequently become unclear. Turyshev *et alia* (2012) identify a faint source of anisotropic heat radiation from the Pioneer spacecraft as the cause of slowing. But if such a tiny factor is able to be overlooked for 20 years, who knows if there are further tiny factors also overlooked? Tiny effects are amplified over a long period of time, and there are multiple possible effects to calculate, e.g. radiation and particle pressure from the sun, small planetary pulls, 'dark matter', heat anisotropy, dust collisions, so-called 'frame-dragging' effects, possible tiny EM forces, and even the Hubble expansion of the universe is on roughly the same scale. The upshot is that the analysis of the Pioneer trajectories is vulnerable to too many possible uncertainties *precisely in the magnitude of anomalies predicted by K-gravity* to provide a conclusive test of K-gravity.

Experimental replication is the best way to resolve the question of the Pioneer anomalies. But replication of the original experiment is not feasible. However we do not have to replicate experiments exactly*:* rather, we replicate them to test for possible alternative *causes* of phenomenon. This is where having an alternative theory to test against is critical. It lets us design variations of the original experiment, calculated to enhance anomalous effects on the hypothesis of a specific alternative cause.

A decisive experiment to test K-gravity, and simultaneously try to replicate the Pioneer phenomenon on the hypothesis that K-gravity, may be done in a time-frame of around 3-4 years, by precisely tracking a probe in free-fall traveling from roughly Earth to Jupiter orbit. The speed is optimized to amplify the anomaly predicted by K-gravity. This is a more efficient and robust experiment than trying to measure accelerations of probes at orbits of 1 AU and 1.5 AU to high precision. Calculation of optimal initial trajectory speeds and predicted effects will be given in a subsequent paper.

In summary, K-gravity is on the cusp of current testability of GTR. It represents a domain of *the largest plausible undetected error* that conventional GTR might still have in weak gravity. The test would set a new limit to the tested accuracy of GTR. This falls outside the main program for testing GTR, viz. through the *parameterized post-Newtonian formalism.* There are a number of alternative theories to GTR (including Brans–Dicke theory, string theory, loop gravity, etc) but these are practically untestable, while K-gravity is readily testable. And it may reveal something quite unexpected instead.[16]

[16] There is another reason to test it from the authors' perspective. K-gravity was originally developed as a consequence of a powerful unified theory that makes a number of strong predictions (published only in pre-prints 2004/2014), of a type so far not considered in physics. If K-gravity is disconfirmed, this type of unified theory is probably wrong, despite making other strong and accurate novel predictions. If K-gravity is confirmed it has a very strong case, and indeed, we can probably claim to know the form of the fundamental unified theory, and certainly dismiss all other present candidates, such as string theory. This connection will be given in subsequent papers.

References.

Einstein, Albert. 1919. "What Is The Theory Of Relativity?" *German History in Documents and Images*. http://germanhistorydocs.ghi-dc.org/pdf/eng/EDU_Einstein_ENGLISH.pdf

Kay, David C. 1976. *Theory and Problems of Tensor Calculus.* MsGraw-Hill.

Landau, Lev D. and Lifshitz, Evgeny M. 1975. *Course on Theoretical Physics Vol. 2: The Classical Theory of Fields*. Butterworth-Heinemann.

Misner, C.W., Thorne, K., Wheeler, J.A. 1973. *Gravitation.* W.H. Freeman and Co.

Musser, George. 1998. “Pioneering Gas Leak?”, *Scientific American*, December 1998, pp.12-13.

Pitjeva, E.V. 2015. “Determination of the Value of the Heliocentric Gravitational Constant from Modern Observations of Planets and Spacecraft.” *Journal of Physical and Chemical Reference Data* **44**, 031210 (2015)

Schwarzschild, K. (1916). "Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie". Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften. 7: 189–196. English translation: Antoci, S., Loinger, A. 1999. "On the gravitational field of a mass point according to Einstein's theory". arXiv:physics/9905030.

Spivak, Michael. 1979. *A Comprehensible Introduction to Differential Geometry.* Publish or Perish.

Turyshev, S. G.; Toth, V. T.; Kinsella, G.; Lee, S.-C.; Lok, S. M.; Ellis, J. 2012. "Support for the Thermal Origin of the Pioneer Anomaly". *Physical Review Letters*. ***108** (24): 241101. arXiv:1204.2507*.

Wald, Robert M. 1984. *General Relativity*. University of Chicago Press.

Will, Clifford M. 2014. “The Confrontation between General Relativity and Experiment.” (Living Review series). arXiv:1403.7377v1 [gr-qc] 28 Mar 2014

Web References.

CODATA 2014 https://en.wikipedia.org/wiki/Committee_on_Data_for_Science_and_Technology

Lehmkuhl, Dennis. 2010. “Mass-Energy-Momentum in General Relativity. Only there because of Spacetime?” http://philsci-archive.pitt.edu/5137/1/Lehmkuhl_MassEnergyMomentum_and_spacetime_structure.pdf

NASA Website Pioneer homepage: http://spaceprojects.arc.nasa.gov/Space_projects/pioneer/Pnhome.html

Oas, Gary. 2014 “Full derivation of the Schwarzchild solution”. https://web.stanford.edu/~oas/SI/SRGR/notes/SchwarzschildSolution.pdf

Vojinovic, Marko, 2010. “Schwarzschild Solution in General Relativity.” http://gfm.cii.fc.ul.pt/events/lecture_series/general_relativity/gfm-general_relativity-lecture4.pdf