A note on the block restricted isometry property condition*

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Abstract. In this work, the sufficient condition for the recovery of block sparse signals that satisfy $b = \Phi x + \xi$ is investigated. We prove that every block s-sparse signal can be reconstructed by the l_2/l_1-minimization method in the noise-free situation and is stably reconstructed in the noisy measurement situation, if the sensing matrix fulfils the restricted isometry property with $\delta_{ts|\mathcal{I}} < t/(4-t)$ as $0 < t < 4/3$, $ts \geq 2$.

Keywords. block restricted isometry property; block sparse; compressed sensing; _2/l_1 minimization.

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1 Introduction

Compressed sensing is one of novel sampling theory, recently attracting more and more researchers' interest. It plays an critical role in a variety of fields such as signal processing, machine learning, seismology, electrical engineering and statistics. In compressed sensing, we are interested in recovering an unknown signal $x \in \mathbb{R}^N$ that fulfils the undetermined system of linear equations, that is,

$$b = \Phi x + \xi \tag{1.1}$$

where $\Phi \in M \times N$ is a known sensing matrix with $M \ll N$, observed signal $b \in \mathbb{R}^M$ and $\xi \in \mathbb{R}^M$ is an unknown bounded noise. In particular, when the noise vector $\xi = 0$, the linear measurement (1.1) reduces to the noiseless situation, namely,

$$b = \Phi x. \tag{1.2}$$

It is well known that there is not only unique solution to the linear measurement (1.1) or (1.2). However, we assume that the signal x consists of a small number of nonzero coefficients that spread arbitrarily throughout the signal, that is, suppose that x is sparse. Under this assumption, the

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problem has a unique sparse solution. Initially, the way that solves it is to study the l_0-minimization, i.e.,

$$\min_x \|x\|_0, \text{ subject to } b = \Phi x, \tag{1.3}$$

where $\|x\|_0$ counts the number of nonzero elements of the vector x. However, it is nonconvex and NP-hard and accordingly is infeasible. It is now well understood that the l_1-minimization approach offers an effective method for resolving this problem, i.e.,

$$\min_x \|x\|_1, \text{ subject to } b = \Phi x, \tag{1.4}$$

where $\|x\|_1 = \Sigma_{i=1}^N |x_i|$. Of course, the l_1-minimization (1.4) is convex and therefore is computationally tractable. The equivalency[1][2] between the problem (1.3) and the problem (1.4) have been proved by making use of the restricted isometry property (RIP) with a restricted isometry constant (RIC). Let s is a positive integer with $1 \leq s \leq N$, the restricted isometry constant δ_s of order s of a matrix Φ is defined as the smallest nonnegative constant such that

$$1 - \delta_s \leq \|\Phi x\|_2^2 / \|x\|_2^2 \leq 1 + \delta_s \tag{1.5}$$

holds for any s-sparse vectors $x \in \mathbb{R}^N$. Here, we say that $x \in \mathbb{R}^N$ is s-sparse that $\|x\|_0 \leq s \ll N$.

However, in a lot of practical applications, some real-world signals may exhibit some particular sparsity patterns, where the non-zero coefficients arise in some fixed blocks. These non-conventional signals have a number of potential applications in the fields of science and technology, like DNA microarrays[3], equalization of sparse communication channels[4], face recognition[5], source localization[6], reconstruction of multi-band signals[7] and multiple measurement vector model[8]. We think of these signals as block sparse signals. Literature[9] first introduced the concept of block sparsity. Recently, block sparsity recovery has attracted considerable interests; for more details, see [10], [11], [12] and [13].

We assume that a block sparse signal $x \in \mathbb{R}^N$ over block index set $\mathcal{I} = \{d_1, \cdots, d_l\}$ can be represented as:

$$x = [\underbrace{x_1, \cdots, x_{d_1}}_{x[1]}, \underbrace{x_{d_1+1}, \cdots, x_{d_1+d_2}}_{x[2]}, \cdots, \underbrace{x_{N-d_l+1}, \cdots, x_N}_{x[l]}]^T, \tag{1.6}$$

where $x[i]$ stands for the ith block of x associated with the block length d_i and $N = d_1 + d_2 + \cdots + d_l$. We say that a vector $x \in \mathbb{R}^N$ as block s-sparse over index set $\mathcal{I} = \{d_1, \cdots, d_l\}$ when $x[i]$ is nonzero for no more than s indices i. In order to reconstruct a block sparse signal, analogous to the l_0-minimization, we search for the sparsest block sparse vector by employing the l_2/l_0-minimization below proposed by [5]:

$$\min \|x\|_{2,0}, \text{ subject to } b = \Phi x, \tag{1.7}$$

where $\|x\|_{2,0} = \sum_{i=1}^l I(\|x[i]\|_2 > 0)$, and $I(x)$ denotes an indicator function that $I(x) = 1$ or 0 according as $x > 0$ or otherwise. Accordingly, we could define a block s-sparse vector x as $\|x\|_{2,0} \leq s$. However, the l_2/l_0-minimization problem remains NP-hard and computationally intractable. Let $\|x\|_{2,\mathcal{I}} = \sum_{i=1}^l \|x[i]\|_2$. Similar to the case of l_0-minimization, one natural ideal is to substitute the l_2/l_0-minimization with the l_2/l_1-minimization below given by [14], [15]:

$$\min \|x\|_{2,\mathcal{I}}, \text{ subject to } b = \Phi x. \tag{1.8}$$

In order to describe the performance of this approach, the block restricted isometry property (block RIP) was defined by [9].

Definition 1.1. *Given a sensing matrix Φ with size $M \times N$, where $M < N$, one says that the measurement matrix Φ obeys the block RIP over $\mathcal{I} = \{d_1, \cdots, d_l\}$ with constants $\delta_{s|\mathcal{I}}$ if for every vector $x \in \mathbb{R}^N$ with block s-sparse over $\mathcal{I}$ such that*

$$1 - \delta_{s|\mathcal{I}} \leq \|\Phi x\|_2^2 / \|x\|_2^2 \leq 1 + \delta_{s|\mathcal{I}} \tag{1.9}$$

holds. We say the smallest constant $\delta_{s|\mathcal{I}}$ that fulfils the above inequality (1.9) as the block RIC corresponding with the matrix Φ.

It is easy to see that the block RIP is an generalization of the standard RIP, but it is a less stringent requirement in comparison with the standard RIP[16]. Eldar et al. [9] proved that any block s-sparse signal could be exactly recovered via the l_2/l_1-minimization as the sensing matrix Φ meets the block RIP with $\delta_{2s|\mathcal{I}} < \sqrt{2} - 1 \approx 0.4142$. One can improve the block RIP, for example, Lin and Li [10] improved the bound to $\delta_{2s|\mathcal{I}} < (77 - \sqrt{1337})/82 \approx 0.4931$, meanwhile obtained another sufficient condition $\delta_{s|\mathcal{I}} < 0.307$. Recently, Gao and ma [13] improved that bound to $\delta_{2s|\mathcal{I}} < 4/\sqrt{41} \approx 0.6246$. Up to now, to the best of our knowledge, there is no work that further concentrates on improvement of the block RIC. Improving the bound concerning block RIC $\delta_{s|\mathcal{I}}$ could bring several advantages. First of all, in compressed sensing, it permits more sensing matrices to be utilized; Then, it permits for reconstructing a block sparse signal with more non-zero coefficients under the condition of the identical matrix Φ; In the end, it provides better error estimation in a general issue to reconstruct signals with noise and mismodeling error; for more information, see [10], [9], [13], [18] and [17]. The purpose of this article is to discuss the improvement for the block RIC, and we will investigate the following minimization for the noisy and mismodeling measurement $b = \Phi x + \xi$ satisfying $\|\xi\|_2 \leq \rho$:

$$\min_x \|x\|_{2,\mathcal{I}}, \text{ subject to } \|\Phi x - b\|_2 \leq \rho. \tag{1.10}$$

First, the following theorem is our main result that gives a sufficient condition of recovery as signal x is not block sparse and the measure is corrupted by the noise. For any $x \in \mathbb{R}^N$, we represent $x_{\max(s)}$ as x with all but the largest s blocks in l_2 norm set to zero and $x_{-\max(s)} = x - x_{\max(s)}$. Set $\tilde{t} = \max\{\sqrt{t}, t\}$.

Theorem 1.1. *We assume that the measurement matrix Φ with size $M \times N (M < N)$ fulfils for $0 < t < 4/3$, $ts \geq 2$*

$$\delta_{ts|\mathcal{I}} < \frac{t}{4-t}. \tag{1.11}$$

If x^* is a solution to problem (1.10), then we have

$$\begin{aligned}\|x^* - x\|_2 \leq& \frac{2\sqrt{2}\rho\sqrt{1+\delta_{ts|\mathcal{I}}}}{t + (t-4)\delta_{ts|\mathcal{I}}}\tilde{t} \\ &+ \frac{1}{2}\sqrt{\frac{2}{s}}\left(\frac{4\left(2\delta_{ts|\mathcal{I}} + \sqrt{(t + (t-4)\delta_{ts|\mathcal{I}})\delta_{ts|\mathcal{I}}}\right)}{t + (t-4)\delta_{ts|\mathcal{I}}} + 1\right)\|x_{-\max(s)}\|_{2,\mathcal{I}}.\end{aligned}$$

Corollary 1.1. *Under the same condition as in Theorem 1.1, suppose that $\xi = 0$ and x is block ts-sparse, then x can be perfectly recovered through the l_2/l_1 minimization (1.8).*

The remainder of this article is organized as follows. In Section 2, we will provide some lemmas. In Section 3, we will offer the proof of the main result. In Section 4, we draw a conclusion for this paper.

2 Auxiliary lemmas

All over this article, we utilize the notations below. x_Π indicates that it holds these blocks indexed by Π of x and otherwise zero. For any block s-sparse vector, $\|x\|_{2,\infty} = \max_{1\le i\le s}\|x[i]\|_2$. $\mathrm{supp}(x) = \{i : \|x[i]\|_2 \neq 0\}$ denotes the block support of x.

The following two lemmas are necessary to the proof of the main result whose proofs are similar to that of Lemmas 1, 2[19]. Denote $C_s^m = \binom{s}{m}$.

Lemma 2.1. *Given vectors $\{x_i : i \in \Pi\}$ in a vector space X with inner product $<\cdot>$, where Π is an index set satisfying $|\Pi| = s$. Suppose that we select all subsets $\Pi_i \in \Pi$ meeting $|\Pi_i| = m$, $i \in \mathcal{J}$ with $|\mathcal{J}| = C_s^m$, then*

$$\sum_{i\in\mathcal{J}}\sum_{j\in\Pi_i} x_j = C_{s-1}^{m-1}\sum_{j\in\Pi} v_j (m \ge 1), \tag{2.1}$$

and

$$\sum_{i\in\mathcal{J}}\sum_{j\neq k\in\Pi_i} < x_j, x_k >= C_{s-2}^{m-2}\sum_{j\neq k\in\Pi} < x_j, x_k > (m \ge 2). \tag{2.2}$$

Lemma 2.2. *Given a matrix $\Phi \in \mathbb{R}^{M\times N}$, we decompose Φ as a concatenation of column-blocks $\Phi[i]$ with size $M \times d_i$, say,*

$$\Phi = [\underbrace{\phi_1,\cdots,\phi_{d_1}}_{\phi[1]},\underbrace{\phi_{d_1+1},\cdots,\phi_{d_1+d_2}}_{\phi[2]},\cdots,\underbrace{\phi_{N-d_l+1},\cdots,\phi_N}_{\phi[l]}],$$

and a block sparse vector $x \in \mathbb{R}^N (l \ge 2)$ determined by (1.6) and put $\Omega = \{1, 2, \cdots, l\}$. Suppose that we select all subsets $\Pi_i \subset \Omega$ meeting $|\Pi_i| = m$, $i \in \mathcal{J}$ with $|\mathcal{J}| = C_l^m$, and all subsets $\Lambda_j \subset \Omega$ satisfying $|\Lambda_j| = n$, $j \in \mathcal{K}$ with $\mathcal{K} = C_l^n$. Then

$$\sum_{i\in\mathcal{J}}\frac{(l-n)\|\Phi x_{\Pi_i}\|_2^2}{m|\mathcal{J}|} - \sum_{j\in\mathcal{K}}\frac{(l-m)\|\Phi x_{\Lambda_j}\|_2^2}{n|\mathcal{K}|} = \frac{(m-n)\|\Phi x\|_2^2}{l}, \tag{2.3}$$

and as $l \ge m+n$,

$$\sum_{\Pi_i\bigcap\Lambda_j=\emptyset}\frac{l-m-n}{mnl|\mathcal{J}|C_{l-m}^n}\left(\frac{mnl}{l-m-n}\|\Phi(x_{\Pi_i}+x_{\Lambda_j})\|_2^2 - \|\Phi(nx_{\Pi_i}-mx_{\Lambda_j})\|_2^2\right) = \frac{(m+n)^2\|\Phi x\|_2^2}{l^2}. \tag{2.4}$$

The following lemma offers a crucial technical tool to the proof of our main theorem which is from [20]. For any block sparse vector x defined by (1.6), $\|x\|_{2,2} = (\sum_{i=1}^l \|x_i\|_2^2)^{\frac{1}{2}}$.

Lemma 2.3. *For a positive number α and a positive integer s, the block polytope $\tau(\alpha, s) \in \mathbb{R}^{\mathrm{N}}$ is defined by*

$$\tau(\alpha, s) = \{x \in \mathbb{R}^N : \|x\|_{2,\infty} \le \alpha, \|x\|_{2,\mathcal{I}} \le s\alpha\}.$$

For any $x \in \mathbb{R}^N$, the set of block sparse vectors $\mathcal{U}(\alpha, s, x) \in \mathbb{R}^N$ is defined by

$$\mathcal{U}(\alpha, s, x) = \{u \in \mathbb{R}^N : supp(u) \subseteq supp(x), \|u\|_{2,0} \le s, \|u\|_{2,\mathcal{I}} = \|x\|_{2,\mathcal{I}}, \|u\|_{2,\infty} \le \alpha\}.$$

Then we can represent any $x \in \tau(\alpha, s)$ as

$$x = \sum_i \lambda_i u_i,$$

where $u_i \in \mathcal{U}(\alpha, s, x)$, $0 \le \lambda_i \le 1$, $\sum_i \lambda_i = 1$,and $\sum_i \lambda_i \|u_i\|_{2,2}^2 \le s\alpha^2$.

3 Proof of main result

Proof. First, suppose that ts is an integer. Let $x* = x + h$. Similar to the proof of Lemma 3.1 [10] , we have

$$\|h_{-\max(s)}\|_{2,\mathcal{I}} \leq \|h_{\max(s)}\|_{2,\mathcal{I}} + 2\|x_{-\max(s)}\|_{2,\mathcal{I}}. \tag{3.1}$$

Select positive integers m and n satisfying $n \leq m \leq s$ and $m+n = st$. Subsets $\Pi_i, \Lambda_j \subset \{1, 2, \cdots, s\}$ stand for all the possible index set that $|\Pi_i| = m, |\Lambda_j| = n$ with $i \in \mathcal{J}$ and $j \in \mathcal{K}$ that $|\mathcal{J}| = C_s^m$ and $|\mathcal{K}| = C_s^n$.

Denote

$$r = \frac{\|h_{\max(s)}\|_{2,\mathcal{I}} + 2\|x_{-\max(s)}\|_{2,\mathcal{I}}}{s}.$$

Due to

$$\|h_{-\max(s)}\|_{2,\mathcal{I}} \leq sr = n\frac{s}{n}r,$$

and

$$\begin{aligned}\|h_{-\max(s)}\|_{2,\infty} &\leq \frac{\|h_{\max(s)}\|_{2,\mathcal{I}}}{s} \\ &\leq \frac{\|h_{\max(s)}\|_{2,\mathcal{I}} + 2\|x_{-\max(s)}\|_{2,\mathcal{I}}}{s} \\ &\leq r \leq \frac{s}{n}r. \end{aligned} \tag{3.2}$$

Making use of (3.2) and Lemma 2.3, we have $h_{-\max(s)} = \sum_i \lambda_i u_i$, where u_i is block n-sparse, $0 \leq \lambda_i \leq 1$ with $\sum_i \lambda_i = 1$, and $\mathrm{supp}(u_i) \subset \mathrm{supp}\left(h_{-\max(s)}\right)$, $\|u_i\|_{2,\mathcal{I}} = \|h_{-\max(s)}\|_{2,\mathcal{I}}$, $\|u_i\|_{2,\infty} \leq sr/n$, and

$$\sum_i \lambda_i \|u_i\|_{2,2}^2 \leq n\left(\frac{s}{n}r\right)^2 = \frac{s^2 r^2}{n}. \tag{3.3}$$

Analogously, we can decompose $h_{-\max(s)}$ as

$$h_{-\max(s)} = \sum_i \gamma_i v_i,$$
$$h_{-\max(s)} = \sum_i \nu_i w_i,$$

where v_i is block m-sparse, w_i is block $(t-1)s$-sparse with

$$\sum_i \gamma_i \|v_i\|_{2,2}^2 \leq \frac{s^2 r^2}{m} \tag{3.4}$$

$$\sum_i \nu_i \|w_i\|_{2,2}^2 \leq \frac{sr^2}{t-1}. \tag{3.5}$$

Notice that $h_{-\max(s)}$ is block s-sparse, and utilizing Cauchy-Schwarz inequality to any block s-sparse vector x, $\|x\|_{2,\mathcal{I}}^2 = (\sum_i \|x[i]\|_2 \cdot 1)^2 \leq s \sum_i \|x[i]\|_2^2 = s\|x\|_{2,2}^2$, we have

$$r^2 = s^{-2}\left(\|h_{\max(s)}\|_{2,\mathcal{I}} + 2\|x_{-\max(s)}\|_{2,\mathcal{I}}\right)^2$$

$$
\begin{aligned}
&= s^{-2}\left(\|h_{\max(s)}\|_{2,\mathcal{I}}^2 + 4\|h_{\max(s)}\|_{2,\mathcal{I}}\|x_{-\max(s)}\|_{2,\mathcal{I}} + 4\|x_{-\max(s)}\|_{2,\mathcal{I}}^2\right)\\
&\le s^{-2}\left(s\|h_{\max(s)}\|_{2,2}^2 + 4\sqrt{s}\|h_{\max(s)}\|_{2,2}\|x_{-\max(s)}\|_{2,\mathcal{I}} + 4\|x_{-\max(s)}\|_{2,\mathcal{I}}^2\right)\\
&= s^{-1}\|h_{\max(s)}\|_{2,2}^2 + 4s^{-\frac{3}{2}}\|h_{\max(s)}\|_{2,2}\|x_{-\max(s)}\|_{2,\mathcal{I}} + 4s^{-2}\|x_{-\max(s)}\|_{2,\mathcal{I}}^2. \qquad (3.6)
\end{aligned}
$$

For $1 \le t < 4/3$, exploiting the notion and monotonicity (Page 1404 [10]) of $\delta_{s|\mathcal{I}}$, we have

$$
\begin{aligned}
\left\langle \Phi h_{\max(s)}, \Phi h\right\rangle &\le \|\Phi h_{\max(s)}\|_2\|\Phi h\|_2\\
&\le \sqrt{1+\delta_{s|\mathcal{I}}}\|h_{\max(s)}\|_2\|\Phi h\|_2\\
&\le \sqrt{1+\delta_{ts|\mathcal{I}}}\|h_{\max(s)}\|_2\|\Phi h\|_2. \qquad (3.7)
\end{aligned}
$$

Since x^* is the feasible solve to (1.10), we have

$$
\|\Phi h\|_2 \le \|\Phi(x-x^*)\|_2 \le \|\Phi x - b\|_2 + \|\Phi x^* - b\|_2 \le 2\rho. \qquad (3.8)
$$

Putting (3.8) into (3.7), we have

$$
\left\langle \Phi h_{\max(s)}, \Phi h\right\rangle \le 2\rho\sqrt{1+\delta_{ts|\mathcal{I}}}\|h_{\max(s)}\|_2. \qquad (3.9)
$$

For simplicity, we use $G_{m,n}$ for

$$
\begin{aligned}
G_{m,n} :=& \frac{s-n}{mC_s^m}\sum_{i\in\mathcal{J},k}\lambda_k\left(m^2\|\Phi(h_{\Pi_i}+\frac{n}{s}u_k)\|_2^2 - n^2\|\Phi(h_{\Pi_i}-\frac{m}{s}u_k)\|_2^2\right)\\
&+\frac{s-m}{nC_s^n}\sum_{j\in\mathcal{K},k}\gamma_k\left(n^2\|\Phi(h_{\Lambda_j}+\frac{m}{s}v_k)\|_2^2 - m^2\|\Phi(h_{\Lambda_j}-\frac{n}{s}v_k)\|_2^2\right). \qquad (3.10)
\end{aligned}
$$

Let $\theta(m,n,t) = 2mn(t-2)+(m-n)^2$. The following two equalities both hold, whose proof that may use Lemma 2.2 are similar to that of identity (14) and (15) [19]. The detail process is omitted. The equality

$$
\begin{aligned}
&\frac{\theta(m,n,t)(t-1)}{mnC_s^mC_{s-m}^n}\sum_{\Pi_i\bigcap\Lambda_j=\phi}\left(\frac{mn}{t-1}\|\Phi(h_{\Pi_i}+h_{\Lambda_j})\|_2^2 + \|\Phi(nh_{\Pi_i}-mh_{\Lambda_j})\|_2^2\right)\\
&\qquad = tG_{m,n} + 2mn(t-2)t^2\left\langle \Phi h_{\max(s)}, \Phi h\right\rangle \qquad (3.11)
\end{aligned}
$$

holds for $0 < t < 1$. The equality

$$
\begin{aligned}
&\theta(m,n,t)\sum_k \nu_k\left(\|\Phi(h_{\max(s)}+(t-1)w_k)\|_2^2 - \|(t-1)\Phi(h_{\max(s)}-w_k)\|_2^2\right)\\
&\qquad = -(3t-4)G_{m,n} + 2((t-1)s^2-mn)t^3\left\langle \Phi h_{\max(s)}, \Phi h\right\rangle \qquad (3.12)
\end{aligned}
$$

holds for $1 \le t < 4/3$. As to $\theta(m,n,t)$, as ts is even, we can set $m=n=ts/2$; as ts is odd, we can put $m=n+1=(ts+1)/2$. It is no difficult to check that $\theta(m,n,t)<0$ for the both situation.

By exploiting the definition of tk order block RIC and observing that h_{Π_i}, v_i are block m-sparse and h_{Λ_j}, u_i are block n-sparse obeying $m+n=ts$, we have

$$
G_{m,n} \ge \frac{s-n}{mC_s^m}\sum_{i\in\mathcal{J},k}\lambda_k\left(m^2(1-\delta_{ts|\mathcal{I}})\|h_{\Pi_i}+\frac{n}{s}u_k\|_2^2 - n^2(1+\delta_{ts|\mathcal{I}})\|h_{\Pi_i}-\frac{m}{s}u_k\|_2^2\right)
$$

$$+\frac{s-m}{nC_s^n}\sum_{j\in\mathcal{K},k}\gamma_k\bigg(n^2(1-\delta_{ts|\mathcal{I}})\|h_{\Lambda_j}+\frac{m}{s}v_k\|_2^2-m^2(1+\delta_{ts|\mathcal{I}})\|h_{\Lambda_j}-\frac{n}{s}v_k\|_2^2\bigg).$$

Note that $< h_{\Pi_i}, u_k >=< h_{\Lambda_j}, v_k >= 0$, because of the support of $h_{\Pi_i}(h_{\Lambda_j})$ does not intersect with the support of $u_k(v_k)$. Therefore,

$$\begin{aligned}
G_{m,n}\geq&\frac{s-n}{mC_s^m}\sum_{i\in\mathcal{J},k}\lambda_k\bigg(m^2(1-\delta_{ts|\mathcal{I}})\|h_{\Pi_i}\|_2^2+\frac{m^2n^2}{s^2}(1-\delta_{ts|\mathcal{I}})\|u_k\|_2^2\\
&-n^2(1+\delta_{ts|\mathcal{I}})\|h_{\Pi_i}\|_2^2-\frac{m^2n^2}{s^2}(1+\delta_{ts|\mathcal{I}})\|u_k\|_2^2\bigg)\\
&+\frac{s-m}{nC_s^n}\sum_{j\in\mathcal{K},k}\gamma_k\bigg(n^2(1-\delta_{ts|\mathcal{I}})\|h_{\Lambda_j}\|_2^2+\frac{m^2n^2}{s^2}(1-\delta_{ts|\mathcal{I}})\|v_k\|_2^2\\
&-m^2(1+\delta_{ts|\mathcal{I}})\|h_{\Lambda_j}\|_2^2-\frac{m^2n^2}{s^2}(1+\delta_{ts|\mathcal{I}})\|v_k\|_2^2\bigg)\\
=&\frac{s-n}{mC_s^m}((m^2-n^2)-(m^2+n^2)\delta_{ts|\mathcal{I}})\sum_{i\in\mathcal{J}}\|h_{\Pi_i}\|_2^2-\frac{s-n}{mC_s^m}2\frac{m^2n^2}{s^2}C_s^m\delta_{ts|\mathcal{I}}\sum_k\lambda_k\|u_k\|_2^2\\
&+\frac{s-m}{nC_s^n}(-(m^2-n^2)-(m^2+n^2)\delta_{ts|\mathcal{I}})\sum_{j\in\mathcal{K}}\|h_{\gamma_j}\|_2^2-\frac{s-m}{nC_s^n}2\frac{m^2n^2}{s^2}C_s^n\delta_{ts|\mathcal{I}}\sum_k\gamma_k\|v_k\|_2^2.
\end{aligned}$$

By making use of (2.1), we have

$$\begin{aligned}
G_{m,n}\geq&((m^2-n^2)-(m^2+n^2)\delta_{ts|\mathcal{I}})\frac{s-n}{mC_s^m}C_{s-1}^{m-1}\|h_{\max(s)}\|_2^2-\frac{2(s-n)mn^2}{s^2}\delta_{ts|\mathcal{I}}\sum_k\lambda_k\|u_k\|_2^2\\
&+(-(m^2-n^2)-(m^2+n^2)\delta_{ts|\mathcal{I}})\frac{s-m}{nC_s^n}C_{s-1}^{n-1}\|h_{\max(s)}\|_2^2-\frac{2(s-m)m^2n}{s^2}\delta_{ts|\mathcal{I}}\sum_k\gamma_k\|v_k\|_2^2.
\end{aligned}$$

Obviously, for any vector $x\in\mathbb{R}^N$ determined by (1.6), we could rewrite l_2-norm $\|x\|_2$ as

$$\|x\|_2=\left(\sum_{i=1}^l\|x[i]\|_2^2\right)^{\frac{1}{2}}=\|x\|_{2,2}.$$

Due to (3.3) and (3.4), we have

$$\begin{aligned}
G_{m,n}\geq&((m^2-n^2)-(m^2+n^2)\delta_{ts|\mathcal{I}})\frac{s-n}{s}\|h_{\max(s)}\|_2^2-\frac{2(s-n)mn^2}{s^2}\delta_{ts|\mathcal{I}}\frac{s^2r^2}{n}\\
&+(-(m^2-n^2)-(m^2+n^2)\delta_{ts|\mathcal{I}})\frac{s-m}{s}\|h_{\max(s)}\|_2^2-\frac{2(s-m)m^2n}{s^2}s^2\delta_{ts|\mathcal{I}}\frac{s^2r^2}{m}\\
=&\left(\frac{(m^2-n^2)(m-n)}{s}-\frac{(m^2+n^2)(2s-(m+n))\delta_{ts|\mathcal{I}}}{s}\right)\|h_{\max(s)}\|_2^2\\
&-2mn\delta_{ts|\mathcal{I}}r^2(2s-(m+n))\\
=&\big(t(m-n)^2+(m^2+n^2)(t-2)\delta_{ts|\mathcal{I}}\big)\|h_{\max(s)}\|_2^2+2mn\delta_{ts|\mathcal{I}}r^2s(t-2).
\end{aligned}\tag{3.13}$$

First, we consider the case of $1\leq t<4/3$.

Since $\theta(m,n,t)$ is not lager than 0, $h_{max(s)}$ is block s-sparse and w_k is block $(t-1)s$-sparse combining with the definition of ts order block RIC $\delta_{ts|\mathcal{I}}$, then

the left side hand (LSH) of Eq.(3.12)

$$\leq \theta(m,n,t)\sum_k \nu_k\bigg((1-\delta_{ts|\mathcal{I}})\|h_{\max(s)}+(t-1)w_k\|_2^2-(1+\delta_{ts|\mathcal{I}})\|(t-1)(h_{\max(s)}-w_k)\|_2^2\bigg).$$

Observe that the support of $h_{\max(s)}$ does not intersect with the support of w_k, thus

$$\begin{aligned}\text{LSH of Eq.(3.12)} \leq{}& \theta(m,n,t)\sum_k \nu_k\bigg((1-\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2+(1-\delta_{ts|\mathcal{I}})(t-1)^2\|w_k)\|_2^2\\ &-(t-1)^2(1+\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2-(t-1)^2(1+\delta_{ts|\mathcal{I}})\|w_k\|_2^2\bigg)\\ ={}& \theta(m,n,t)\sum_k \nu_k\bigg(((1-(t-1)^2)-(1+(t-1)^2)\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2\\ &-2(t-1)^2\delta_{ts|\mathcal{I}}\|w_k\|_2^2\bigg).\end{aligned}$$

By applying (3.5) to the above inequality, we have

$$\begin{aligned}\text{LSH of Eq.(3.12)} \leq{}& \theta(m,n,t)\bigg(((1-(t-1)^2)-(1+(t-1)^2)\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2\\ &-2\delta_{ts|\mathcal{I}}sr^2(t-1)\bigg).\end{aligned}\tag{3.14}$$

It follows from the assumption of Theorem 1.1 that $s\geq 3/2$, so it is no hard to see that

$$mn\geq \frac{t^2s^2-1}{4}=\frac{(2-t)^2s^2-1}{4}+(t-1)s^2>(t-1)s^2,\tag{3.15}$$

for $1\leq t<4/3$. Combining with (3.9), (3.13) and (3.15), we have

the right side hand (RSH) of Eq. (3.12)

$$\begin{aligned}&\geq -(3t-4)\bigg(\left(t(m-n)^2+(m^2+n^2)(t-2)\delta_{ts|\mathcal{I}}\right)\|h_{\max(s)}\|_2^2+2mn\delta_{ts|\mathcal{I}}r^2s(t-2)\bigg)\\ &+4\rho\sqrt{1+\delta_{ts|\mathcal{I}}}((t-1)s^2-mn)t^3\|h_{\max(s)}\|_2.\end{aligned}\tag{3.16}$$

Let the LSH minus the RSH of the eq. (3.12), then

$$\begin{aligned}0\leq{}&\theta(m,n,t)\bigg(((1-(t-1)^2)-(1+(t-1)^2)\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2-2\delta_{ts|\mathcal{I}}sr^2(t-1)\bigg)\\ &+(3t-4)\bigg(\left(t(m-n)^2+(m^2+n^2)(t-2)\delta_{ts|\mathcal{I}}\right)\|h_{-\max(s)}\|_2^2+2mn\delta_{ts|\mathcal{I}}r^2s(t-2)\bigg)\\ &-4\rho\sqrt{1+\delta_{ts|\mathcal{I}}}((t-1)s^2-mn)t^3\|h_{\max(s)}\|_2.\end{aligned}$$

By using (3.6) to the above inequality, the fact that for any vector x, $\|x\|_{2,2}=(\sum_{i=1}^{l}\|x[i]\|_2^2)^{\frac{1}{2}}=\|x\|_2$ and some elementary calculations, then

$$0\leq 2((t-1)s^2-mn)t^2\bigg((t+(t-4)\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2$$

$$-\left(\frac{4\delta_{ts|\mathcal{I}}\|x_{-\max(s)}\|_{2,\mathcal{I}}}{\sqrt{s}}+2\rho t\sqrt{1+\delta_{ts|\mathcal{I}}}\right)\|h_{\max(s)}\|_2-\frac{4\delta_{ts|\mathcal{I}}\|x_{-\max(s)}\|_{2,\mathcal{I}}^2}{s}\Bigg). \tag{3.17}$$

Next, we take into account the case of $0<t<1$.

Utilizing Lemma 4.1[21], we have

$$\begin{aligned}\|\Phi h_{\max(s)}\|_2^2 &\le (1+\delta_{s|\mathcal{I}})\|h_{\max(s)}\|_2^2\\ &=\left(1+\delta_{\frac{1}{t}ts|\mathcal{I}}\right)\|h_{\max(s)}\|_2^2\\ &\le\left(1+\left(\frac{2}{t}-1\right)\delta_{ts|\mathcal{I}}\right)\|h_{\max(s)}\|_2^2\\ &\le\frac{(1+\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2}{t}.\end{aligned} \tag{3.18}$$

By (3.8) and (3.18), we have

$$\begin{aligned}\left\langle\Phi h_{\max(s)},\Phi h\right\rangle &\le \|\Phi h_{\max(s)}\|_2\|\Phi h\|_2\\ &\le 2\rho\frac{\sqrt{(1+\delta_{ts|\mathcal{I}})t}}{t}\|h_{\max(s)}\|_2.\end{aligned} \tag{3.19}$$

By taking advantage of the concept of ts order block RIC and (2.1), we have

$$\begin{aligned}&\text{the LSH of the Eq. 3.11}\\ &=\frac{\theta(m,n,t)(t-1)}{mnC_s^mC_{s-m}^n}\sum_{\Pi_i\bigcap\Lambda_j=\phi}\left(\frac{mn}{t-1}\|\Phi(h_{\Pi_i}+h_{\Lambda_j})\|_2^2+\|\Phi(nh_{\Pi_i}-mh_{\Lambda_j})\|_2^2\right)\\ &\le\frac{\theta(m,n,t)(t-1)}{mnC_s^mC_{s-m}^n}\sum_{\Pi_i\bigcap\Lambda_j=\phi}\left(\frac{mn}{t-1}(1-\delta_{ts|\mathcal{I}})\|(h_{\Pi_i}+h_{\Lambda_j})\|_2^2+(1+\delta_{ts|\mathcal{I}})\|(nh_{\Pi_i}-mh_{\Lambda_j})\|_2^2\right)\\ &=\frac{\theta(m,n,t)(t-1)}{mnC_s^mC_{s-m}^n}\Bigg(\frac{mn}{t-1}(1-\delta_{ts|\mathcal{I}})\Bigg(C_{s-m}^n\sum_{i\in\mathcal{J}}\|h_{\Pi_i}\|_2^2+C_{s-n}^m\sum_{j\in\mathcal{K}}\|h_{\Lambda_j}\|_2^2\Bigg)\\ &+(1+\delta_{ts|\mathcal{I}})\Bigg(n^2C_{s-m}^n\sum_{i\in\mathcal{J}}\|h_{\Pi_i}\|_2^2+m^2C_{s-n}^m\sum_{j\in\mathcal{K}}\|h_{\Lambda_j}\|_2^2\Bigg)\Bigg)\\ &=\frac{\theta(m,n,t)(t-1)}{mnC_s^mC_{s-m}^n}\Bigg(\frac{mn}{t-1}(1-\delta_{ts|\mathcal{I}})\Bigg(C_{s-m}^nC_{s-1}^{m-1}\|h_{\max(s)}\|_2^2+C_{s-n}^mC_{s-1}^{n-1}\|h_{\max(s)}\|_2^2\Bigg)\\ &+(1+\delta_{ts|\mathcal{I}})\Bigg(n^2C_{s-m}^nC_{s-1}^{m-1}\|h_{\max(s)}\|_2^2+m^2C_{s-n}^mC_{s-1}^{n-1}\|h_{\max(s)}\|_2^2\Bigg)\Bigg)\\ &=\frac{\theta(m,n,t)(t-1)}{mnC_s^mC_{s-m}^n}\Bigg(\frac{mn}{t-1}(1-\delta_{ts|\mathcal{I}})\left(C_{s-m}^nC_{s-1}^{m-1}+C_{s-n}^mC_{s-1}^{n-1}\right)\|h_{\max(s)}\|_2^2\\ &+(1+\delta_{ts|\mathcal{I}})\left(n^2C_{s-m}^nC_{s-1}^{m-1}+m^2C_{s-n}^mC_{s-1}^{n-1}\right)\|h_{\max(s)}\|_2^2\Bigg)\\ &=\theta(m,n,t)(t+(t-2)\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2.\end{aligned} \tag{3.20}$$

By combining (3.13) with (3.19), we have

$$\text{the RSH of the Eq. (3.11)}$$

$$\geq t\bigg(\left(t(m-n)^2+(m^2+n^2)(t-2)\delta_{ts|\mathcal{I}}\right)\|h_{\max(s)}\|_2^2+2mn\delta_{ts|\mathcal{I}}r^2s(t-2)\bigg)$$
$$+4mn\rho\sqrt{1+\delta_{ts|\mathcal{I}}}(t-2)t\sqrt{t}\|h_{\max(s)}\|_2.$$

Let the LSH minus the RSH of Eq. (3.11), then

$$\begin{aligned}
0 &\leq \theta(m,n,t)(t+(t-2)\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2 \\
&- t\bigg(\left(t(m-n)^2+(m^2+n^2)(t-2)\delta_{ts|\mathcal{I}}\right)\|h_{\max(s)}\|_2^2+2mn\delta_{ts|\mathcal{I}}r^2s(t-2)\bigg) \\
&- 4mn\rho\sqrt{1+\delta_{ts|\mathcal{I}}}(t-2)t\sqrt{t}\|h_{\max(s)}\|_2 \\
&\leq 2t(t-2)mn\bigg((t+(t-4)\delta_{ts|\mathcal{I}})\|h_{\max(s)}\|_2^2 \\
&- \left(\frac{4\delta_{ts|\mathcal{I}}\|x_{-\max(s)}\|_{2,\mathcal{I}}}{\sqrt{s}}+2\rho\sqrt{(1+\delta_{ts|\mathcal{I}})t}\right)\|h_{\max(s)}\|_2-\frac{4\delta_{ts|\mathcal{I}}\|x_{-\max(s)}\|_{2,\mathcal{I}}^2}{s}\bigg).
\end{aligned} \tag{3.21}$$

By (3.15) and the assumption of $\delta_{ts|\mathcal{I}}<\frac{t}{4-t}$, it is easy to see that the above two inequalities given by (3.17) and (3.21) are second-order inequalities about $\|h_{\max(s)}\|_2$, where the quadratic coefficients are negative.

Consequently, through a straightforward calculation, we have

$$\begin{aligned}
\|h_{\max(s)}\|_2 \leq & \frac{\frac{4\delta_{ts|\mathcal{I}}\|x_{-\max(s)}\|_{2,\mathcal{I}}}{\sqrt{s}}+2\rho\sqrt{1+\delta_{ts|\mathcal{I}}}\tilde{t}}{2(t+(t-4)\delta_{ts|\mathcal{I}})} \\
&+ (2(t+(t-4)\delta_{ts|\mathcal{I}}))^{-1}\Bigg(\left(\frac{4\delta_{ts|\mathcal{I}}\|x_{-\max(s)}\|_{2,\mathcal{I}}}{\sqrt{s}}+2\rho\sqrt{1+\delta_{ts|\mathcal{I}}}\tilde{t}\right)^2 \\
&+ 16(t+(t-4)\delta_{ts|\mathcal{I}})\frac{\delta_{ts|\mathcal{I}}}{s}\|x_{-\max(s)}\|_{2,\mathcal{I}}\Bigg)^{\frac{1}{2}},
\end{aligned}$$

where $\tilde{t}=\max\{t,\sqrt{t}\}$. Note the fact that for fixed $0<q\leq 1$, any non-negative x,y, $(x+y)^q\leq x^q+y^q$. Hence,

$$\|h_{\max(s)}\|_2 \leq \frac{2\rho\sqrt{1+\delta_{ts|\mathcal{I}}}\tilde{t}+2\left(\delta_{ts|\mathcal{I}}+\sqrt{(t+(t-4)\delta_{ts|\mathcal{I}})\delta_{ts|\mathcal{I}}}\right)\|x_{-\max(s)}\|_{2,\mathcal{I}}/\sqrt{s}}{t+(t-4)\delta_{ts|\mathcal{I}}}. \tag{3.22}$$

Easily verify that for any block s-sparse vector x, $\|x\|_{2,2}=\left(\sum_i\|x[i]\|_2^2\right)^{1/2}\leq\sqrt{\|x\|_{2,\infty}}\sqrt{\|x\|_{2,\mathcal{I}}}$. And employing Cauchy-Schwarz to any block s-sparse vector x, $\|x\|_{2,\mathcal{I}}=\sum_i\|x[i]\|_2\cdot 1\leq s^{\frac{1}{2}}(\sum_i\|x[i]\|_2^2)^{\frac{1}{2}}=s^{\frac{1}{2}}\|x\|_{2,2}$ and combining with (3.1), we have

$$\begin{aligned}
\|h_{-\max(s)}\|_{2,2} &\leq \sqrt{s^{-1}\|h_{\max(s)}\|_{2,\mathcal{I}}}\sqrt{\|h_{\max(s)}\|_{2,\mathcal{I}}+2\|x_{-\max(s)}\|_{2,\mathcal{I}}} \\
&\leq \sqrt{s^{-1}\|h_{\max(s)}\|_{2,\mathcal{I}}^2+2s^{-1}\|h_{\max(s)}\|_{2,\mathcal{I}}\|x_{-\max(s)}\|_{2,\mathcal{I}}} \\
&\leq \sqrt{\|h_{\max(s)}\|_{2,2}^2+2s^{-\frac{1}{2}}\|h_{\max(s)}\|_{2,2}\|x_{-\max(s)}\|_{2,\mathcal{I}}}.
\end{aligned} \tag{3.23}$$

By utilizing (3.22) and (3.23), we obtain

$$\|h\|_2=(\|h_{\max(s)}\|_2^2+\|h_{-\max(s)}\|_2^2)^{\frac{1}{2}}$$

$$
\begin{aligned}
&\leq \left(\|h_{\max(s)}\|_2^2 + \|h_{\max(s)}\|_{2,2}^2 + 2s^{-\frac{1}{2}}\|h_{\max(s)}\|_{2,2}\|x_{-\max(s)}\|_{2,\mathcal{I}}\right)^{\frac{1}{2}} \\
&= \left(2\|h_{\max(s)}\|_2^2 + 2s^{-\frac{1}{2}}\|h_{\max(s)}\|_{2,2}\|x_{-\max(s)}\|_{2,\mathcal{I}}\right)^{\frac{1}{2}} \\
&\leq \left((\sqrt{2}\|h_{\max(s)}\|_2)^2 + 2\sqrt{2}\|h_{\max(s)}\|_{2,2}(2s)^{-\frac{1}{2}}\|x_{-\max(s)}\|_{2,\mathcal{I}} + \left((2s)^{-\frac{1}{2}}\|x_{-\max(s)}\|_{2,\mathcal{I}}\right)^2\right)^{\frac{1}{2}} \\
&= \sqrt{2}\|h_{\max(s)}\|_2 + (2s)^{-\frac{1}{2}}\|x_{-\max(s)}\|_{2,\mathcal{I}} \\
&\leq \frac{2\sqrt{2}\rho\sqrt{1+\delta_{ts|\mathcal{I}}}\tilde{t} + 2\sqrt{2}(\delta_{ts|\mathcal{I}} + \sqrt{(t+(t-4)\delta_{ts|\mathcal{I}})\delta_{ts|\mathcal{I}}})\|x_{-\max(s)}\|_{2,\mathcal{I}}/\sqrt{s}}{t+(t-4)\delta_{ts|\mathcal{I}}} \\
&+ \|x_{-\max(s)}\|_{2,\mathcal{I}}/\sqrt{2s} \\
&\leq \frac{2\sqrt{2}\rho\sqrt{1+\delta_{ts|\mathcal{I}}}\tilde{t}}{t+(t-4)\delta_{ts|\mathcal{I}}} \\
&+ \frac{1}{2}\sqrt{\frac{2}{s}}\left(\frac{4\left(2\delta_{ts|\mathcal{I}} + \sqrt{(t+(t-4)\delta_{ts|\mathcal{I}})\delta_{ts|\mathcal{I}}}\right)}{t+(t-4)\delta_{ts|\mathcal{I}}} + 1\right)\|x_{-\max(s)}\|_{2,\mathcal{I}}.
\end{aligned}
$$

If ts is not an integer, we set $t's = [ts]$, then $t's$ is an integer with $t' > t$. For $t' < 4/3$, we have $\delta_{t's|\mathcal{I}} = \delta_{ts|\mathcal{I}} < t/(4-t) < t'/(4-t')$. Analogous to the proof above, we can prove the result under the condition of $\delta_{ts|\mathcal{I}} < t/(4-t)$ with $ts \notin \mathbb{Z}$.

□

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