Sedeonic duality-invariant field equations for dyons

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We discuss the theoretical description of dyons having simultaneously both electric and magnetic charges on the basis of space-time algebra of sixteen-component sedeons. We show that the sedeonic equations for electromagnetic field of dyons can be reformulated in equivalent form as the equations for renormalized field potentials, field strengths and single renormalized source. The relations for energy and momentum as well as the relations for Lorentz invariants of renormalized electromagnetic field are derived. Additionally, we discuss the sedeonic second-order Klein-Gordon and first-order Dirac wave equations describing the quantum behavior of dyons in an external electromagnetic field.

1. Introduction

A dyon is a hypothetical point particle, which has simultaneously both electric q_e and magnetic q_m charges was proposed by J. Schwinger in 1969 [1]. In fact, the dyonic concept is a development of the idea of magnetic monopoles proposed previously by P.A.M. Dirac [2, 3]. Magnetic monopoles as well as dions increase the symmetry of the Maxwell equations. Taking into account the magnetic charges and corresponding magnetic currents the Maxwell equations for the electromagnetic field in a vacuum is represented in absolutely symmetric form [1]:

$$\begin{aligned}
&\nabla \cdot \boldsymbol{E} = 4\pi\rho_e, \\
&\frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} - \nabla \times \boldsymbol{H} = -\frac{4\pi}{c}\boldsymbol{j}_e, \\
&\nabla \cdot \boldsymbol{H} = 4\pi\rho_m, \\
&\frac{1}{c}\frac{\partial \boldsymbol{H}}{\partial t} + \nabla \times \boldsymbol{E} = -\frac{4\pi}{c}\boldsymbol{j}_m.
\end{aligned} \tag{1.1}$$

Here ρ_e is a volume density of electric charge; $\boldsymbol{j}_e$ is a volume density of electric current; ρ_m is a volume density of magnetic charge and $\boldsymbol{j}_m$ is a volume density of magnetic current. These equations are invariant under the electromagnetic duality transformations for field strengths and sources [4]:

$$\begin{aligned}
&\boldsymbol{E} \to \boldsymbol{H}, \quad \boldsymbol{H} \to -\boldsymbol{E}, \\
&\boldsymbol{j}_e \to \boldsymbol{j}_m, \quad \boldsymbol{j}_m \to -\boldsymbol{j}_e, \\
&\rho_e \to \rho_m, \quad \rho_m \to -\rho_e.
\end{aligned} \tag{1.2}$$

In recent years, there have been a few publications devoted to the reformulation of equations for electromagnetic field in terms of hypercomplex field potentials. The first approach is based on four-component quaternions, which consist of scalar and vector parts that adequately describe the four-vector concept of special relativity [5-8]. In particular quaternions were applied for the description of dyons [9-11]. However since the system of Maxwell equations consists of four equations for scalar, pseudoscalar, vector and pseudovector values, the application of eight-component algebras is more appropriate. Taking into account this spatial symmetry several approaches have been proposed to describe electromagnetic field on the basis of eight-component octonions [12-16] and octons [17-19]. Particularly these algebras were used for the description of dyonic field [20-22]. However, a consistent relativistic consideration implies equally the space and time symmetries that require using the extended sixteen-component space-time algebras.

Recently we proposed the space-time algebra of sixteen-component sedeons, which takes into account the symmetry of physical values with respect to the space-time inversion and realizes the scalar-vector representation of Poincare group [23]. In particular we considered the equations for massive and massless fields based on sedeonic potentials and space-time operators [24-26]. In the present paper we consider the application of sedeonic algebra to the description of dyonic electromagnetic field and for the reformulation of relativistic quantum equations for dyons in an external electromagnetic field.

2. Algebra of space-time sedeons

To begin with we shortly recall the main properties of sedeons [23]. The algebra of sedeons encloses four groups of values, which are differed with respect to spatial and time inversion.

- Absolute scalars (V) and absolute vectors ($\vec{V}$) are not transformed under spatial and time inversion.
- Time scalars ($V_{\mathbf{t}}$) and time vectors ($\vec{V}_{\mathbf{t}}$) are changed (in sign) under time inversion and are not transformed under spatial inversion.
- Space scalars ($V_{\mathbf{r}}$) and space vectors ($\vec{V}_{\mathbf{r}}$) are changed under spatial inversion and are not transformed under time inversion.
- Space-time scalars ($V_{\mathbf{tr}}$) and space-time vectors ($\vec{V}_{\mathbf{tr}}$) are changed under spatial and time inversion.

Here indexes $\mathbf{t}$ and $\mathbf{r}$ indicate the transformations ($\mathbf{t}$ for time inversion and $\mathbf{r}$ for spatial inversion), which change the corresponding values. All introduced values can be integrated into one space-time sedeon $\tilde{\mathbf{V}}$, which is defined by the following expression:

$$\tilde{\mathbf{V}} = V + \vec{V} + V_{\mathbf{t}} + \vec{V}_{\mathbf{t}} + V_{\mathbf{r}} + \vec{V}_{\mathbf{r}} + V_{\mathbf{tr}} + \vec{V}_{\mathbf{tr}} . \tag{2.1}$$

Let us introduce a scalar-vector basis $\mathbf{a_0}$, $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$, where the element $\mathbf{a_0}$ is an absolute scalar unit ($\mathbf{a_0} \equiv 1$), and the values $\vec{\mathbf{a}}_1$, $\vec{\mathbf{a}}_2$, $\vec{\mathbf{a}}_3$ are absolute unit vectors generating the right Cartesian basis. Further we will indicate the absolute unit vectors by symbols without arrows as $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$. We also introduce the four space-time units $\mathbf{e}_0$, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, where $\mathbf{e_0}$ is an absolute scalar unit ($\mathbf{e_0} \equiv 1$); $\mathbf{e}_1$ is a time scalar unit ($\mathbf{e}_1 \equiv \mathbf{e_t}$); $\mathbf{e}_2$ is a space scalar unit ($\mathbf{e}_2 \equiv \mathbf{e_r}$); $\mathbf{e}_3$ is a space-time scalar unit ($\mathbf{e}_3 \equiv \mathbf{e_{tr}}$). Using space-time basis $\mathbf{e}_{\boldsymbol{\alpha}}$ and scalar-vector basis $\mathbf{a}_{\boldsymbol{\beta}}$ (Greek indexes $\boldsymbol{\alpha}, \boldsymbol{\beta} = 0, 1, 2, 3$), we can introduce unified sedeonic components $V_{\alpha\beta}$ in accordance with following relations:

$$\begin{aligned}
V &= \mathbf{e_0} V_{00} \mathbf{a_0} , \\
\vec{V} &= \mathbf{e_0} \left(V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right) , \\
V_{\mathbf{t}} &= \mathbf{e_1} V_{10} \mathbf{a_0} , \\
\vec{V}_{\mathbf{t}} &= \mathbf{e_1} \left(V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right) , \\
V_{\mathbf{r}} &= \mathbf{e_2} V_{20} \mathbf{a_0} , \\
\vec{V}_{\mathbf{r}} &= \mathbf{e_2} \left(V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right) , \\
V_{\mathbf{tr}} &= \mathbf{e_3} V_{30} \mathbf{a_0} , \\
\vec{V}_{\mathbf{tr}} &= \mathbf{e_3} \left(V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right) .
\end{aligned} \tag{2.2}$$

Then sedeon (2.1) can be written in the following expanded form:

$$\begin{aligned}
\tilde{\mathbf{V}} = &\ \mathbf{e_0} \left(V_{00}\mathbf{a_0} + V_{01}\mathbf{a_1} + V_{02}\mathbf{a_2} + V_{03}\mathbf{a_3} \right) \\
&+ \mathbf{e_1} \left(V_{10}\mathbf{a_0} + V_{11}\mathbf{a_1} + V_{12}\mathbf{a_2} + V_{13}\mathbf{a_3} \right) \\
&+ \mathbf{e_2} \left(V_{20}\mathbf{a_0} + V_{21}\mathbf{a_1} + V_{22}\mathbf{a_2} + V_{23}\mathbf{a_3} \right) \\
&+ \mathbf{e_3} \left(V_{30}\mathbf{a_0} + V_{31}\mathbf{a_1} + V_{32}\mathbf{a_2} + V_{33}\mathbf{a_3} \right) .
\end{aligned} \tag{2.3}$$

The sedeonic components $V_{\alpha\beta}$ are numbers (complex in general). Further we will omit units $\mathbf{a_0}$ and $\mathbf{e_0}$ for the simplicity. The important property of sedeons is that the equality of two sedeons means the equality of all sixteen components $V_{\alpha\beta}$.

Let us consider the multiplication rules for the basis elements $\mathbf{a_n}$ and $\mathbf{e_k}$ (Latin indexes **n**, **k** = 1, 2, 3). The unit vectors $\mathbf{a_n}$ have the following multiplication and commutation rules:

$$\mathbf{a_n}\mathbf{a_n} = \mathbf{a_n}^2 = 1 , \tag{2.4}$$

$$\mathbf{a_n}\mathbf{a_k} = -\,\mathbf{a_k}\mathbf{a_n} \text{ (for } \mathbf{n} \neq \mathbf{k} \text{)}, \tag{2.5}$$

$$\mathbf{a_1}\mathbf{a_2} = i\mathbf{a_3} , \quad \mathbf{a_2}\mathbf{a_3} = i\mathbf{a_1} , \quad \mathbf{a_3}\mathbf{a_1} = i\mathbf{a_2} , \tag{2.6}$$

while the space-time units $\mathbf{e_k}$ satisfy the following rules:

$$\mathbf{e_k e_k} = \mathbf{e_k^2} = 1 , \tag{2.7}$$

$$\mathbf{e_n e_k} = -\mathbf{e_k e_n} \text{ (for } \mathbf{n} \neq \mathbf{k} \text{)}, \tag{2.8}$$

$$\mathbf{e_1 e_2} = i\mathbf{e_3} , \quad \mathbf{e_2 e_3} = i\mathbf{e_1} , \quad \mathbf{e_3 e_1} = i\mathbf{e_2} . \tag{2.9}$$

Here and further the value i is imaginary unit $(i^2 = -1)$. The multiplication and commutation rules for sedeonic absolute unit vectors $\mathbf{a_n}$ and space-time units $\mathbf{e_k}$ can be presented for obviousness as the tables 1 and 2.

Table 1. Multiplication rules for absolute unit vectors $\mathbf{a_n}$.

	$\mathbf{a_1}$	$\mathbf{a_2}$	$\mathbf{a_3}$
$\mathbf{a_1}$	1	$i\mathbf{a_3}$	$-i\mathbf{a_2}$
$\mathbf{a_2}$	$-i\mathbf{a_3}$	1	$i\mathbf{a_1}$
$\mathbf{a_3}$	$i\mathbf{a_2}$	$-i\mathbf{a_1}$	1

Table 2. Multiplication rules for space-time units $\mathbf{e_k}$.

	$\mathbf{e_1}$	$\mathbf{e_2}$	$\mathbf{e_3}$
$\mathbf{e_1}$	1	$i\mathbf{e_3}$	$-i\mathbf{e_2}$
$\mathbf{e_2}$	$-i\mathbf{e_3}$	1	$i\mathbf{e_1}$
$\mathbf{e_3}$	$i\mathbf{e_2}$	$-i\mathbf{e_1}$	1

Note that units $\mathbf{e_k}$ commute with vectors $\mathbf{a_n}$:

$$\mathbf{a_n e_k} = \mathbf{e_k a_n} \tag{2.10}$$

for any $\mathbf{n}$ and $\mathbf{k}$.

In sedeonic algebra we assume the Clifford multiplication of vectors. The sedeonic product of two vectors $\vec{A}$ and $\vec{B}$ can be presented in the following form:

$$\vec{A}\vec{B} = \left(\vec{A} \cdot \vec{B}\right) + \left[\vec{A} \times \vec{B}\right] . \tag{2.11}$$

Here we denote the sedeonic scalar multiplication of two vectors (internal product) by symbol "$\cdot$" and round brackets

$$\left(\vec{A} \cdot \vec{B}\right) = A_1 B_1 + A_2 B_2 + A_3 B_3 , \tag{2.12}$$

and sedeonic vector multiplication (external product) by symbol "$\times$" and square brackets

$$\left[\vec{A} \times \vec{B}\right] = i\left(A_2 B_3 - A_3 B_2\right)\mathbf{a_1} + i\left(A_3 B_1 - A_1 B_3\right)\mathbf{a_2} + i\left(A_1 B_2 - A_2 B_1\right)\mathbf{a_3} . \tag{2.13}$$

Note that in sedeonic algebra the expression for the vector product differs from analogous expression in common used Gibbs-Heaviside vector algebra. For the transition from sedeons to the vector algebra the replacement

$$i\left[\vec{A} \times \vec{B}\right] \Rightarrow -\boldsymbol{A} \times \boldsymbol{B} \tag{2.14}$$

should be made in all vector expressions. Here $\boldsymbol{A} \times \boldsymbol{B}$ is the vector product of $\boldsymbol{A}$ and $\boldsymbol{B}$ vectors in Gibbs-Heaviside algebra.

3. The equations for electromagnetic field of dyons

The sedeonic wave equation for electromagnetic field of electric and magnetic charges can be written in the following form [25,26]:

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\tilde{\mathbf{W}} = \tilde{\mathbf{J}} . \tag{3.1}$$

Here $\tilde{\mathbf{W}}$ is the sedeon of electromagnetic field potential

$$\tilde{\mathbf{W}} = i\mathbf{e}_1\varphi_e - i\mathbf{e}_2\varphi_m + \mathbf{e}_1\vec{A}_m + \mathbf{e}_2\vec{A}_e , \tag{3.2}$$

where φ_e is electric scalar potential, φ_m is magnetic scalar potential, $\vec{A}_e$ is electric vector potential, $\vec{A}_m$ is magnetic vector potential. We use the following sedeonic definition of Hamilton nabla operator and vectors $\vec{A}_e$ and $\vec{A}_m$:

$$\begin{aligned} \vec{\nabla} &= \frac{\partial}{\partial x}\mathbf{a}_1 + \frac{\partial}{\partial y}\mathbf{a}_2 + \frac{\partial}{\partial z}\mathbf{a}_3 , \\ \vec{A}_e &= A_{e1}\mathbf{a}_1 + A_{e2}\mathbf{a}_2 + A_{e3}\mathbf{a}_3 , \quad . \\ \vec{A}_m &= A_{m1}\mathbf{a}_1 + A_{m2}\mathbf{a}_2 + A_{m3}\mathbf{a}_3 . \end{aligned} \tag{3.4}$$

The sedeonic source is

$$\tilde{\mathbf{J}} = -i\mathbf{e}_1 4\pi\rho_e - \mathbf{e}_2 \frac{4\pi}{c}\vec{j}_e + i\mathbf{e}_2 4\pi\rho_m - \mathbf{e}_1 \frac{4\pi}{c}\vec{j}_m , \tag{3.3}$$

where ρ_e is a volume density of electric charge, $\vec{j}_e$ is a density of electric current, ρ_m is a volume density of magnetic charge and $\vec{j}_m$ is a density of magnetic current.
The electric and magnetic field strengths are defined as

$$\begin{aligned} \vec{E} &= -\frac{1}{c}\frac{\partial \vec{A}_e}{\partial t} - \vec{\nabla}\varphi_e + i\left[\vec{\nabla}\times\vec{A}_m\right], \\ \vec{H} &= -\frac{1}{c}\frac{\partial \vec{A}_m}{\partial t} - \vec{\nabla}\varphi_m - i\left[\vec{\nabla}\times\vec{A}_e\right]. \end{aligned} \tag{3.5}$$

The potentials satisfy the Lorentz gauge conditions

$$\frac{1}{c}\frac{\partial \varphi_e}{\partial t} + \left(\vec{\nabla}\cdot\vec{A}_e\right) = 0 , \tag{3.6}$$

$$\frac{1}{c}\frac{\partial \varphi_m}{\partial t} + \left(\vec{\nabla}\cdot\vec{A}_m\right) = 0 . \tag{3.6}$$

Then we have

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\varphi_e - i\mathbf{e}_2\varphi_m + \mathbf{e}_1\vec{A}_m + \mathbf{e}_2\vec{A}_e\right) = \mathbf{e}_3\vec{E} - i\vec{H}, \tag{3.7}$$

and the wave equation (3.1) is rewritten as

$$\left(i\mathbf{e}_1 \frac{1}{c}\frac{\partial}{\partial t} - \mathbf{e}_2\vec{\nabla}\right)\left(\mathbf{e}_3\vec{E} - i\vec{H}\ \right) = -i\mathbf{e}_1 4\pi\rho_e - \mathbf{e}_2 \frac{4\pi}{c}\vec{j}_e + i\mathbf{e}_2 4\pi\rho_m - \mathbf{e}_1 \frac{4\pi}{c}\vec{j}_m . \tag{3.8}$$

Producing action of the operator on the left side of this equation and separating the values with different space-time properties, we obtain the system of Maxwell's equations for electric and magnetic charges

$$\begin{aligned}
&\left(\vec{\nabla}\cdot\vec{E}\right)=4\pi\rho_e,\\
&\frac{1}{c}\frac{\partial\vec{E}}{\partial t}+i\left[\vec{\nabla}\times\vec{H}\right]=-\frac{4\pi}{c}\vec{j}_e,\\
&\left(\vec{\nabla}\cdot\vec{H}\right)=4\pi\rho_m,\\
&\frac{1}{c}\frac{\partial\vec{H}}{\partial t}-i\left[\vec{\nabla}\times\vec{E}\right]=-\frac{4\pi}{c}\vec{j}_m.
\end{aligned} \tag{3.9}$$

However, the electric and magnetic properties of dyons are not independent, since their electric q_e and magnetic q_m charges belong to the same point particle. Therefore, the following relations hold [27]:

$$\begin{aligned}
&\frac{1}{q_e}\varphi_e=\frac{1}{q_m}\varphi_m,\\
&\frac{1}{q_e}\vec{A}_e=\frac{1}{q_m}\vec{A}_m,\\
&\frac{1}{q_e}\rho_e=\frac{1}{q_m}\rho_m,\\
&\frac{1}{q_e}\vec{j}_e=\frac{1}{q_m}\vec{j}_m.
\end{aligned} \tag{3.10}$$

To describe the dyons it is convenient to introduce the renormalized sources:

$$\begin{aligned}
&\rho=\frac{\rho_e}{q_e}=\frac{\rho_m}{q_m},\\
&\vec{j}=\frac{\vec{j}_e}{q_e}=\frac{\vec{j}_m}{q_m},
\end{aligned} \tag{3.11}$$

and renormalized field potentials:

$$\begin{aligned}
&\varphi=\frac{1}{q_e}\varphi_e=\frac{1}{q_m}\varphi_m,\\
&\vec{A}=\frac{1}{q_e}\vec{A}_e=\frac{1}{q_m}\vec{A}_m.
\end{aligned} \tag{3.12}$$

Taking into account (3.11) and (3.12) the field potential (3.2) and source (3.3) can be rewritten as

$$\tilde{\mathbf{W}}=\left(i\mathbf{e}_1q_e-i\mathbf{e}_2q_m\right)\varphi+\left(\mathbf{e}_1q_m+\mathbf{e}_2q_e\right)\vec{A}=\left(i\mathbf{e}_1\varphi+\mathbf{e}_2\vec{A}\right)\left(q_e-i\mathbf{e}_3q_m\right), \tag{3.13}$$

$$\tilde{\mathbf{J}}=-4\pi\left(i\mathbf{e}_1q_e-i\mathbf{e}_2q_m\right)\rho-\frac{4\pi}{c}\left(\mathbf{e}_2q_e+\mathbf{e}_1q_m\right)\vec{j}=-4\pi\left(i\mathbf{e}_1\rho+\mathbf{e}_2\frac{1}{c}\vec{j}\right)\left(q_e-i\mathbf{e}_3q_m\right). \tag{3.14}$$

Substituting (3.13) and (3.14) in the wave equation (3.1) and multiplying on $\left(q_e+i\mathbf{e}_3q_m\right)$ from the right we obtain following renormalized wave equation:

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\varphi+\mathbf{e}_2\vec{A}\right)=-i4\pi\mathbf{e}_1\rho-\mathbf{e}_2\frac{4\pi}{c}\vec{j}\,. \tag{3.15}$$

Let us introduce new renormalized electric and magnetic field strengths of dyons

$$\vec{\mathcal{E}}=-\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-\vec{\nabla}\varphi\,, \tag{3.16}$$

$$\vec{\mathcal{H}}=-i\left[\vec{\nabla}\times\vec{A}\right]. \tag{3.17}$$

Since renormalized potentials satisfy the Lorentz gauge condition

$$\frac{1}{c}\frac{\partial\varphi}{\partial t}+\left(\vec{\nabla}\cdot\vec{A}\right)=0\,,$$

we have

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\left(i\mathbf{e}_1\varphi+\mathbf{e}_2\vec{A}_e\right)=\mathbf{e}_3\vec{\mathcal{E}}-i\vec{\mathcal{H}}, \tag{3.18}$$

and the wave equation (3.15) is rewritten as

$$\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\left(\mathbf{e}_3\vec{\mathcal{E}}-i\vec{\mathcal{H}}\right)=-i4\pi\mathbf{e}_1\rho-\mathbf{e}_2\frac{4\pi}{c}\vec{j}. \tag{3.19}$$

Producing action of the operator on the left side of this equation and separating the values with different space-time properties, we obtain the system of Maxwell's equations for dyons in the following form:

$$\begin{aligned}&\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)=4\pi\rho,\\&\frac{1}{c}\frac{\partial\vec{\mathcal{E}}}{\partial t}+i\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]=-\frac{4\pi}{c}\vec{j},\\&\left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right)=0,\\&\frac{1}{c}\frac{\partial\vec{\mathcal{H}}}{\partial t}-i\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]=0.\end{aligned} \tag{3.20}$$

There are the simple relations between electric $\vec{E}$ and magnetic $\vec{H}$ fields and renormalized $\vec{\mathcal{E}}$ and $\vec{\mathcal{H}}$. Taking into account (3.12) the definitions (3.5) can be rewritten as

$$\begin{aligned}\vec{E}&=-q_e\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-q_e\vec{\nabla}\varphi+iq_m\left[\vec{\nabla}\times\vec{A}\right],\\\vec{H}&=-q_m\frac{1}{c}\frac{\partial\vec{A}}{\partial t}-q_m\vec{\nabla}\varphi-iq_e\left[\vec{\nabla}\times\vec{A}\right].\end{aligned} \tag{3.21}$$

Multiplying these relations respectively on q_e and q_m we have

$$q_e\vec{E}+q_m\vec{H}=-\left(q_e^2+q_m^2\right)\left(\frac{1}{c}\frac{\partial\vec{A}}{\partial t}+\vec{\nabla}\varphi\right)=\left(q_e^2+q_m^2\right)\vec{\mathcal{E}} \tag{3.22}$$

and

$$-q_m\vec{E}+q_e\vec{H}=-\left(q_e^2+q_m^2\right)i\left[\vec{\nabla}\times\vec{A}\right]=\left(q_e^2+q_m^2\right)\vec{\mathcal{H}}. \tag{3.23}$$

From these expressions we obtain the following relations between the field strengths:

$$\vec{\mathcal{E}}=\frac{1}{\left(q_e^2+q_m^2\right)}\left(q_e\vec{E}+q_m\vec{H}\right), \tag{3.24}$$

$$\vec{\mathcal{H}}=\frac{1}{\left(q_e^2+q_m^2\right)}\left(-q_m\vec{E}+q_e\vec{H}\right). \tag{3.25}$$

4. The relations for energy and momentum of dyon electromagnetic field

Multiplying (3.19) on $\left(\mathbf{e}_3\vec{\mathcal{E}}-i\vec{\mathcal{H}}\right)$ from the left we have

$$\left(\mathbf{e}_3\vec{\mathcal{E}}-i\vec{\mathcal{H}}\right)\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\left(\mathbf{e}_3\vec{\mathcal{E}}-i\vec{\mathcal{H}}\right)=-\left(\mathbf{e}_3\vec{\mathcal{E}}-i\vec{\mathcal{H}}\right)\left(i4\pi\mathbf{e}_1\rho+\mathbf{e}_2\frac{4\pi}{c}\vec{j}\right). \tag{4.1}$$

Equating in (4.1) the components with different space-time properties we get

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left(\vec{\mathcal{E}}^2+\vec{\mathcal{H}}^2\right)-i\frac{c}{4\pi}\left(\vec{\nabla}\cdot\left[\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right]\right)+\left(\vec{\mathcal{E}}\cdot\vec{j}\right)=0, \tag{4.2}$$

$$\begin{aligned}&\frac{1}{8\pi}\vec{\nabla}\left(\vec{\mathcal{E}}^2+\vec{\mathcal{H}}^2\right)-i\frac{1}{4\pi c}\frac{\partial}{\partial t}\left[\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right]\\&-\frac{1}{4\pi}\left\{\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)\vec{\mathcal{E}}+\left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right)\vec{\mathcal{H}}\right\}+\rho\vec{\mathcal{E}}+i\left[\vec{\mathcal{H}}\times\vec{j}\right]=0,\end{aligned} \tag{4.3}$$

$$\frac{1}{4\pi}\left\{\left(\vec{\mathcal{E}}\cdot\frac{\partial\vec{\mathcal{H}}}{\partial t}\right)-\left(\vec{\mathcal{H}}\cdot\frac{\partial\vec{\mathcal{E}}}{\partial t}\right)\right\}-i\frac{c}{4\pi}\left\{\left(\vec{\mathcal{E}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]\right)+\left(\vec{\mathcal{H}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]\right)\right\}-\left(\vec{\mathcal{H}}\cdot\vec{j}\right)=0, \tag{4.4}$$

$$\begin{gathered}-i\frac{1}{4\pi}\left\{\left[\vec{\mathcal{E}}\times\frac{\partial\vec{\mathcal{E}}}{\partial t}\right]+\left[\vec{\mathcal{H}}\times\frac{\partial\vec{\mathcal{H}}}{\partial t}\right]\right\}+\frac{c}{4\pi}\left\{\vec{\mathcal{E}}\left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right)-\vec{\mathcal{H}}\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)\right\}\\ +\frac{c}{4\pi}\left\{\left[\vec{\mathcal{E}}\times\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]\right]-\left[\vec{\mathcal{H}}\times\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]\right]\right\}+c\vec{\mathcal{H}}\rho-i\left[\vec{\mathcal{E}}\times\vec{j}\right]=0.\end{gathered} \tag{4.5}$$

The expression (4.2) is the analog of Pointing theorem for dyons. The value

$$W=\frac{\vec{\mathcal{E}}^2+\vec{\mathcal{H}}^2}{8\pi} \tag{4.6}$$

is the renormalized volume density of field energy, while vector

$$\vec{P}=-i\frac{c}{4\pi}\left[\vec{\mathcal{E}}\times\vec{\mathcal{H}}\right] \tag{4.7}$$

is the vector of renormalized energy flux density (Pointing's vector). Using (3.24) and (3.25) one can see that

$$W=\frac{1}{8\pi(q_e^2+q_m^2)}\left(\vec{E}^2+\vec{H}^2\right), \tag{4.8}$$

while

$$\vec{P}=-i\frac{c}{4\pi(q_e^2+q_m^2)}\left[\vec{E}\times\vec{H}\right]. \tag{4.9}$$

5. Relations for Lorentz invariants of dyon electromagnetic field

Using sedeonic algebra it is easy to derive the relations for the values

$$\begin{aligned}I_1&=\vec{\mathcal{E}}^2-\vec{\mathcal{H}}^2,\\ I_2&=\left(\vec{\mathcal{E}}\cdot\vec{\mathcal{H}}\right),\end{aligned} \tag{5.1}$$

which are the Lorentz invariants of dyon electromagnetic field. Indeed substituting (3.24) and (3.25) we have

$$I_1=\frac{4q_e^2q_m^2}{(q_e^2+q_m^2)^2}\left(\vec{E}\cdot\vec{H}\right), \tag{5.2}$$

and

$$I_2=\frac{q_eq_m}{(q_e^2+q_m^2)^2}\left(\vec{H}^2-\vec{E}^2\right). \tag{5.3}$$

Thus one can see that I_1 and I_2 are renormalized Lorentz invariants of real electromagnetic field.

Multiplying both parts of equation (3.19) on sedeon ($\mathbf{e}_{\mathbf{tr}}\vec{\mathcal{E}}+i\vec{\mathcal{H}}$) from the left we have:

$$\left(\mathbf{e}_3\vec{\mathcal{E}}+i\vec{\mathcal{H}}\right)\left(i\mathbf{e}_1\frac{1}{c}\frac{\partial}{\partial t}-\mathbf{e}_2\vec{\nabla}\right)\left(\mathbf{e}_3\vec{\mathcal{E}}-i\vec{\mathcal{H}}\right)=-\left(\mathbf{e}_3\vec{\mathcal{E}}+i\vec{\mathcal{H}}\right)\left(i4\pi\mathbf{e}_1\rho+\mathbf{e}_2\frac{4\pi}{c}\vec{j}\right). \tag{5.4}$$

Equating in (5.2) the components with different space-time properties we obtain the following relations for the Lorentz invariants of dyon electromagnetic field:

$$\frac{1}{8\pi}\frac{\partial}{\partial t}\left\{\vec{\mathcal{E}}^2-\vec{\mathcal{H}}^2\right\}=-\left(\vec{\mathcal{E}}\cdot\vec{j}\right)-i\frac{c}{4\pi}\left\{\left(\vec{\mathcal{E}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]\right)+\left(\vec{\mathcal{H}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]\right)\right\}, \tag{5.5}$$

$$\begin{gathered}\frac{c}{8\pi}\vec{\nabla}\left\{\vec{\mathcal{E}}^2-\vec{\mathcal{H}}^2\right\}=\frac{c}{4\pi}\left\{\vec{\mathcal{E}}\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)+\left(\vec{\mathcal{E}}\cdot\vec{\nabla}\right)\vec{\mathcal{E}}-\vec{\mathcal{H}}\left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right)-\left(\vec{\mathcal{H}}\cdot\vec{\nabla}\right)\vec{\mathcal{H}}\right\}\\ +i\frac{1}{4\pi}\left\{\left[\vec{\mathcal{E}}\times\frac{\partial\vec{\mathcal{H}}}{\partial t}\right]+\left[\vec{\mathcal{H}}\times\frac{\partial\vec{\mathcal{E}}}{\partial t}\right]\right\}-c\rho\vec{\mathcal{E}}+i\left[\vec{\mathcal{H}}\times\vec{j}\right],\end{gathered} \tag{5.6}$$

$$\frac{1}{4\pi}\frac{\partial}{\partial t}\left\{\left(\vec{\mathcal{E}}\cdot\vec{\mathcal{H}}\right)\right\}=i\frac{c}{4\pi}\left\{\left(\vec{\mathcal{E}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{E}}\right]\right)-\left(\vec{\mathcal{H}}\cdot\left[\vec{\nabla}\times\vec{\mathcal{H}}\right]\right)\right\}-\left(\vec{\mathcal{H}}\cdot\vec{j}\right), \tag{5.7}$$

$$\begin{aligned}&\frac{c}{4\pi}\vec{\nabla}\left\{\left(\vec{\mathcal{E}}\cdot\vec{\mathcal{H}}\right)\right\}=\frac{c}{4\pi}\left\{\vec{\mathcal{E}}\left(\vec{\nabla}\cdot\vec{\mathcal{H}}\right)+\vec{\mathcal{H}}\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right)-\left(\vec{\mathcal{E}}\cdot\vec{\nabla}\right)\vec{\mathcal{H}}+\left(\vec{\mathcal{H}}\cdot\vec{\nabla}\right)\vec{\mathcal{E}}\right\}\\&-i\frac{1}{4\pi}\left\{\left[\vec{\mathcal{E}}\times\frac{\partial\vec{\mathcal{E}}}{\partial t}\right]-\left[\vec{\mathcal{H}}\times\frac{\partial\vec{\mathcal{H}}}{\partial t}\right]\right\}-c\vec{\mathcal{H}}\rho+i\left[\vec{\mathcal{E}}\times\vec{j}\right].\end{aligned} \tag{5.8}$$

6. The sedeonic Klein-Gordon equation for dyons

The sedeonic wave equation for the quantum particle with electric charge q_e and magnetic charge q_m in an external electromagnetic field described by electric φ_e, $\vec{A}_e$ and magnetic φ_m, $\vec{A}_m$ potentials is obtained from the equation for free particle [28,29] by the following replacements:

$$\begin{aligned}&\frac{\partial}{\partial t}\ \rightarrow\ \frac{\partial}{\partial t}+\frac{i}{\hbar}q_e\varphi_e+\frac{i}{\hbar}q_m\varphi_m,\\&\vec{\nabla}\ \rightarrow\ \vec{\nabla}-\frac{i}{\hbar c}q_e\vec{A}_e-\frac{i}{\hbar c}q_m\vec{A}_m,\end{aligned} \tag{6.1}$$

and can be written as

$$\left\{i\mathbf{e}_1\frac{1}{c}\left(\frac{\partial}{\partial t}+\frac{i}{\hbar}q_e\varphi_e+\frac{i}{\hbar}q_m\varphi_m\right)-\mathbf{e}_2\left(\vec{\nabla}-\frac{i}{c\hbar}q_e\vec{A}_e-\frac{i}{c\hbar}q_m\vec{A}_m\right)-i\mathbf{e}_3\frac{mc}{\hbar}\right\}^2\tilde{\boldsymbol{\psi}}=0. \tag{6.2}$$

Here m is the mass of particle. Taking into account the relations for dyonic potential (3.12) this equation can be represented in the following form:

$$\left\{i\mathbf{e}_1\frac{1}{c}\left(\frac{\partial}{\partial t}+\frac{i}{\hbar}q\varphi\right)-\mathbf{e}_2\left(\vec{\nabla}-\frac{i}{c\hbar}q\vec{A}\right)-i\mathbf{e}_3\frac{mc}{\hbar}\right\}\left\{i\mathbf{e}_1\frac{1}{c}\left(\frac{\partial}{\partial t}+\frac{i}{\hbar}q\varphi\right)-\mathbf{e}_2\left(\vec{\nabla}-\frac{i}{c\hbar}q\vec{A}\right)-i\mathbf{e}_3\frac{mc}{\hbar}\right\}\tilde{\boldsymbol{\psi}}=0 \tag{6.3}$$

where

$$q=q_e^2+q_m^2.$$

Producing the action of operators on the left side of equation we obtain

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta+\frac{m^2c^2}{\hbar^2}+\frac{2iq}{c\hbar}\left(\frac{\varphi}{c}\frac{\partial}{\partial t}+\left(\vec{A}\cdot\vec{\nabla}\right)\right)+\frac{q}{c^2\hbar^2}\left(A^2-\varphi^2\right)\right]\tilde{\boldsymbol{\psi}}+\mathbf{e}_3\frac{iq}{c\hbar}\vec{\mathcal{E}}\,\tilde{\boldsymbol{\psi}}-\frac{q}{c\hbar}\vec{\mathcal{H}}\,\tilde{\boldsymbol{\psi}}=0. \tag{6.4}$$

In this equation the term

$$\mathbf{e}_3\frac{iq}{c\hbar}\vec{\mathcal{E}}\,\tilde{\boldsymbol{\psi}} \tag{6.5}$$

describes the interaction of dyon with renormalized electric field, while the term

$$\frac{q}{c\hbar}\vec{\mathcal{H}}\,\tilde{\boldsymbol{\psi}} \tag{6.6}$$

describes the interaction of dyon with renormalized magnetic field.

On the other hand, the sedeonic wave equation (6.3) can be rewritten as the system of Maxwell-like equations for some quantum field [23]. Let us introduce following new operator and values:

$$\begin{aligned}&\partial=\frac{1}{c}\frac{\partial}{\partial t},\\&\varphi_0=\frac{q}{c\hbar}\varphi,\\&\vec{A}_0=\frac{q}{c\hbar}\vec{A},\\&m_0=\frac{c}{\hbar}m.\end{aligned} \tag{6.7}$$

Then wave equation (6.3) is rewritten in the following compact form:

$$\left(i\mathbf{e}_1(\partial+i\varphi_0)-\mathbf{e}_2\left(\vec{\nabla}-i\vec{A}_0\right)-i\mathbf{e}_3 m_0\right)\left(i\mathbf{e}_1(\partial+i\varphi_0)-\mathbf{e}_2\left(\vec{\nabla}-i\vec{A}_0\right)-i\mathbf{e}_3 m_0\right)\tilde{\boldsymbol{\psi}}=0, \tag{6.8}$$

or introducing operator

$$\hat{\nabla}=i\mathbf{e}_1(\partial+i\varphi_0)-\mathbf{e}_2\left(\vec{\nabla}-i\vec{A}_0\right)-i\mathbf{e}_3 m, \tag{6.9}$$

in the very compact form

$$\hat{\nabla}\hat{\nabla}\tilde{\boldsymbol{\psi}}=0. \tag{6.10}$$

On the other hand, let us defined a field $\tilde{\mathbf{G}}$ according to

$$\tilde{\mathbf{G}}=\hat{\nabla}\tilde{\boldsymbol{\psi}}. \tag{6.11}$$

Then wave equation (6.10) takes the following equivalent form:

$$\hat{\nabla}\tilde{\mathbf{G}}=0. \tag{6.12}$$

Let us choose [25] the wave function as

$$\tilde{\boldsymbol{\psi}}=i\psi_1\mathbf{e}_1-i\psi_2\mathbf{e}_2+\psi_3-i\psi_4\mathbf{e}_3+\vec{\psi}_1\mathbf{e}_2+\vec{\psi}_2\mathbf{e}_1-\vec{\psi}_3\mathbf{e}_3+i\vec{\psi}_4, \tag{6.13}$$

and field $\tilde{\mathbf{G}}$ as

$$\tilde{\mathbf{G}}=-g_1+ig_2\mathbf{e}_3+ig_3\mathbf{e}_1-ig_4\mathbf{e}_2+\vec{G}_1\mathbf{e}_3-i\vec{G}_2+\vec{G}_3\mathbf{e}_2+\vec{G}_4\mathbf{e}_1, \tag{6.14}$$

then taking into account (6.11) we have the following definitions of $\tilde{\mathbf{G}}$ components through the components of wave function:

$$\begin{aligned}
g_1&=(\partial+i\varphi_0)\psi_1+\left(\left(\vec{\nabla}-i\vec{A}_0\right)\cdot\vec{\psi}_1\right)+m_0\psi_4,\\
g_2&=(\partial+i\varphi_0)\psi_2+\left(\left(\vec{\nabla}-i\vec{A}_0\right)\cdot\vec{\psi}_2\right)-m_0\psi_3,\\
g_3&=(\partial+i\varphi_0)\psi_3+\left(\left(\vec{\nabla}-i\vec{A}_0\right)\cdot\vec{\psi}_3\right)+m_0\psi_2,\\
g_4&=(\partial+i\varphi_0)\psi_4+\left(\left(\vec{\nabla}-i\vec{A}_0\right)\cdot\vec{\psi}_4\right)-m_0\psi_1,\\
\vec{G}_1&=-(\partial+i\varphi_0)\vec{\psi}_1-\left(\vec{\nabla}-i\vec{A}_0\right)\psi_1+i\left[\left(\vec{\nabla}-i\vec{A}_0\right)\times\vec{\psi}_2\right]+m_0\vec{\psi}_4,\\
\vec{G}_2&=-(\partial+i\varphi_0)\vec{\psi}_2-\left(\vec{\nabla}-i\vec{A}_0\right)\psi_2-i\left[\left(\vec{\nabla}-i\vec{A}_0\right)\times\vec{\psi}_1\right]-m_0\vec{\psi}_3,\\
\vec{G}_3&=-(\partial+i\varphi_0)\vec{\psi}_3-\left(\vec{\nabla}-i\vec{A}_0\right)\psi_3-i\left[\left(\vec{\nabla}-i\vec{A}_0\right)\times\vec{\psi}_4\right]+m_0\vec{\psi}_2,\\
\vec{G}_4&=-(\partial+i\varphi_0)\vec{\psi}_4-\left(\vec{\nabla}-i\vec{A}_0\right)\psi_4+i\left[\left(\vec{\nabla}-i\vec{A}_0\right)\times\vec{\psi}_3\right]-m_0\vec{\psi}_1.
\end{aligned} \tag{6.15}$$

Separating in wave equation (6.12) the values with different space-time properties we obtain the system of Maxwell-like equations for quantum field $\tilde{\mathbf{G}}$:

$$\begin{aligned}
&(\partial+i\varphi_0)g_1+\left(\left(\vec{\nabla}-i\vec{A}_0\right)\cdot\vec{G}_1\right)-m_0g_4=0,\\
&(\partial+i\varphi_0)g_2+\left(\left(\vec{\nabla}-i\vec{A}_0\right)\cdot\vec{G}_2\right)+m_0g_3=0,\\
&(\partial+i\varphi_0)g_3+\left(\left(\vec{\nabla}-i\vec{A}_0\right)\cdot\vec{G}_3\right)-m_0g_2=0,\\
&(\partial+i\varphi_0)g_4+\left(\left(\vec{\nabla}-i\vec{A}_0\right)\cdot\vec{G}_4\right)+m_0g_1=0,\\
&(\partial+i\varphi_0)\vec{G}_1+\left(\vec{\nabla}-i\vec{A}_0\right)g_1+i\left[\left(\vec{\nabla}-i\vec{A}_0\right)\times\vec{G}_2\right]+m_0\vec{G}_4=0,\\
&(\partial+i\varphi_0)\vec{G}_2+\left(\vec{\nabla}-i\vec{A}_0\right)g_2-i\left[\left(\vec{\nabla}-i\vec{A}_0\right)\times\vec{G}_1\right]-m_0\vec{G}_3=0,\\
&(\partial+i\varphi_0)\vec{G}_3+\left(\vec{\nabla}-i\vec{A}_0\right)g_3-i\left[\left(\vec{\nabla}-i\vec{A}_0\right)\times\vec{G}_4\right]+m_0\vec{G}_2=0,\\
&(\partial+i\varphi_0)\vec{G}_4+\left(\vec{\nabla}-i\vec{A}_0\right)g_4+i\left[\left(\vec{\nabla}-i\vec{A}_0\right)\times\vec{G}_3\right]-m_0\vec{G}_1=0.
\end{aligned} \tag{6.16}$$

Multiplying each of the equations (6.16) to the corresponding field strength and adding these equations to each other, we obtain that

$$\left(\partial + i\varphi_0\right)W + \left(\left(\vec{\nabla} - i\vec{A}_0\right)\cdot\vec{P}\right) = 0\,. \tag{6.17}$$

Here

$$W = g_1^2 + g_2^2 + g_3^2 + g_4^2 + \vec{G}_1^2 + \vec{G}_2^2 + \vec{G}_3^2 + \vec{G}_4^2 \tag{6.18}$$

and

$$\vec{P} = g_1\vec{G}_1 + g_2\vec{G}_2 + g_3\vec{G}_3 + g_4\vec{G}_4 - i\left[\vec{G}_1\times\vec{G}_2\right] + i\left[\vec{G}_3\times\vec{G}_4\right] \tag{6.19}$$

are analogues of volume density of energy and flux energy respectively [24]. The expression (6.17) is an analog of Pointing theorem for quantum field $\tilde{\mathbf{G}}$ describing the kinematics of quantum dyon in an external electromagnetic fieled.

7. Note on sedeonic Dirac equation for dyons

The equation (6.8) admits a special class of solutions that are described by the sedeonic first-order wave equation [24]:

$$\left(i\mathbf{e}_1\left(\partial + i\varphi_0\right) - \mathbf{e}_2\left(\vec{\nabla} - i\vec{A}_0\right) - i\mathbf{e}_3 m_0\right)\tilde{\boldsymbol{\psi}} = 0\,. \tag{7.1}$$

In fact, the dyons described by this equation do not create a quantum field $\tilde{\mathbf{G}}$, since equation (7.1) means that

$$\tilde{\mathbf{G}} = \hat{\nabla}\tilde{\boldsymbol{\psi}} = 0\,. \tag{7.2}$$

It is clear that increasing the order of the equation (7.1), namely acting on it by the operator $\left(i\mathbf{e}_1\left(\partial + i\varphi_0\right) - \mathbf{e}_2\left(\vec{\nabla} - i\vec{A}_0\right) - i\mathbf{e}_3 m_0\right)$ from the left we arrive at the second-order wave equation (6.8).

8. Conclusion

Thus, we have shown that in the frames of sedeonic algebra the sedeonic wave equation for electromagnetic field of dyons can be reformulated in equivalent form as the system of Maxwell equations for renormalized field strengths $\vec{\mathcal{E}}$, $\vec{\mathcal{H}}$ and single type of renormalized source ρ , $\vec{j}$. The relations for energy and momentum of renormalized electromagnetic field (Pointing theorem) as well as the relations for Lorentz invariants of renormalized electromagnetic field have been derived. Also we have shown that the sedeonic second-order Klein-Gordon equation describing the quantum kinematics of dyons in an external electromagnetic field can be reformulated in the form of Maxwell-like equations for the quantum field $\tilde{\mathbf{G}}$ and the analogue of Pointing theorem for this field have been derived. Additionally we shown that the sedeonic first-order Dirac wave equation describes the dyons, which do not create the quantum field $\tilde{\mathbf{G}}$.

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