An approximate non-quantum calculation of the Aharonov-Bohm effect

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In the Aharonov-Bohm effect for a magnetic solenoid a moving charged particle seems to be influenced by the 4-potential in a region where there are no fields in the laboratory frame of reference. The 4-potential should be transformed to the frame of reference of the particle before computing the fields. There is an E field in its frame of reference. The field accelerates a moving charged particle. One of the components of the acceleration vector is in the same direction as the particle's velocity in the first frame of reference. The resulting longitudinal displacement in the path integral, when scaled in units of the de Broglie wavelength for the particle, is approximately the same as the phase of the Aharonov-Bohm solution for long paths. The scalar solution does not require transformation. It follows from the static Coulomb solution and the Newton equations.

I. THE EXTERIOR SOLENOID SOLUTION

The exterior potential solution for a long static magnetic solenoid is

$$\begin{aligned} \boldsymbol{A} &= \mu_0 n I \hat{\boldsymbol{\phi}} r_0^2/(2r) \\ \psi &= 0, \end{aligned}$$

where n is the number of turns per meter, r_0 is the radius, I is the current in each turn, and r is the distance from the center. In order to shorten the expressions, the substitution $k = \mu_0 n I$ is made in the following calculations. After converting from cylindrical to Cartesian coordinates,

$$\begin{aligned} A_x &= -k r_0^2 \sin\phi/[2(x^2+y^2)^{1/2}] \\ A_y &= k r_0^2 \cos\phi/[2(x^2+y^2)^{1/2}] \\ A_z &= 0 \\ \psi &= 0, \end{aligned}$$

with $(x^2+y^2)^{1/2} > r_0$. For $x > 0$, $\phi = \arcsin[y/(x^2+y^2)^{1/2}]$

$$\begin{aligned} A_x &= -k r_0^2 y/[2(x^2+y^2)] && (1) \\ A_y &= k r_0^2 x/[2(x^2+y^2)] && (2) \\ A_z &= 0 \\ \psi &= 0. \end{aligned}$$

The magnetic field, $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, is zero. $\boldsymbol{\nabla}\cdot\boldsymbol{A}$ and the scalar potential are zero, so the Lorentz condition, $\boldsymbol{\nabla}\cdot\boldsymbol{A} + 1/c^2\ \partial\psi/\partial t$, is also zero.

The magnetic Aharonov-Bohm[7] (A-B) effect for this solution is

$$\Delta\phi = \frac{q}{\hbar}\int A_y\ \mathrm{d}y,$$

where q is the charge of the particle and $\hbar$ is Planck's constant divided by 2π. From Eq. (2), the path integral of the y component is

$$\begin{aligned} \int_{-y1}^{+y1} A_y \mathrm{d}y &= \frac{k r_0^2}{2}\arctan\frac{y}{x}\ \Big|_{-y_1}^{+y_1} \\ &= k r_0^2 \arctan\frac{y_1}{x}. \end{aligned} \quad (3)$$

The quantum phase for the path is then

$$\Delta\phi = \frac{q}{\hbar} k r_0^2 \arctan\frac{y_1}{x}. \quad (4)$$

If the solenoid is driven by a low frequency alternating current Eqs. (1) become

$$\begin{aligned} A_x &= -k\sin(\omega t) r_0^2 y/[2(x^2+y^2)] && (5) \\ A_y &= k\sin(\omega t) r_0^2 x/[2(x^2+y^2)] \\ A_z &= 0 \\ \psi &= 0. \end{aligned}$$

A more complete calculation shows that there is still no scalar potential in the solution if the charge of the stationary protons in the wire loop is included. In the SI system of units the $\boldsymbol{E}$ field is

$$\boldsymbol{E} = -\partial\boldsymbol{A}/\partial t - \boldsymbol{\nabla}\psi. \quad (6)$$

The E field for the solution in Eqs. (5) is

$$\begin{aligned} E_x &= k\omega\cos(\omega t) r_0^2 y/[2(x^2+y^2)] \\ E_y &= -k\omega\cos(\omega t) r_0^2 x/[2(x^2+y^2)] \\ E_z &= 0. \end{aligned}$$

If two long straight wires are placed parallel to the y axis on either side of the solenoid, along with crossover wires at the ends, the loop integral of the E field, from Eq. (3), is

$$\Delta V = -k\pi r_0^2 \omega\cos(\omega t).$$

The configuration is a limiting case of a common transformer with a one turn secondary winding. The voltage induced in the winding can also be computed from the rate of change of the magnetic flux within the solenoid. The voltage does not depend on the path of integration, so long as it encircles the solenoid. The voltage is zero

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in other cases. These relationships are only valid at low frequencies. The potentials have to be retarded if the solenoid is driven at a high frequency and becomes radiative. The static solution cannot be retarded, because an irreversible information loss has occurred in integrating around the current loops. The nearside and farside conduction electrons would have to be retarded separately with the Liénard–Wiechert[5] retardation equations.

Velocities do not have an absolute significance, so if the $-\partial \boldsymbol{A}/\partial t$ term induces a voltage in a transformer winding a moving charged particle near a static magnetic solenoid should experience an E field.

The infinitesimal Lorentz transform is

$$\begin{aligned} \boldsymbol{r}' &= \boldsymbol{r} - \boldsymbol{v}t \\ t' &= t - \boldsymbol{r}\cdot\boldsymbol{v}/c^2. \end{aligned} \tag{7}$$

The 4-potential transforms in the same way as the coordinates. It is possible to work in a system of units where the scalar potential has the units of time and the vector potential has the units of distance. When the scalar potential is initially known in SI units it is multiplied by ξ to convert it to the units of time. ξ is a constant with the units of s/V. After transformation with the Lorentz transform the vector potential has the units of distance, and it can be converted to SI units by dividing it by ξc^2. The scalar potential is converted back to SI units by dividing it by ξ.

In this system of units Eqs. (7) become

$$\begin{aligned} \boldsymbol{A}' &= \boldsymbol{A} - \boldsymbol{v}\psi \\ \psi' &= \psi - \boldsymbol{A}\cdot\boldsymbol{v}/c^2, \end{aligned} \tag{8}$$

and Eqs. (1) become

$$\begin{aligned} A_x &= -\xi c^2 k r_0^2 y/[2(x^2+y^2)] \\ A_y &= \xi c^2 k r_0^2 x/[2(x^2+y^2)] \\ A_z &= 0 \\ \psi &= 0. \end{aligned}$$

Using Eqs. (8) to transform to the second frame of reference with the velocity v_{0y}, with $v_{0x} = v_{0z} = 0$, then converting back to the SI system,

$$\begin{aligned} A'_x &= -k r_0^2 y/[2(x^2+y^2)] \\ A'_y &= k r_0^2 x/[2(x^2+y^2)] \\ A'_z &= 0 \\ \psi' &= -k r_0^2 v_0 x/[2(x^2+y^2)]. \end{aligned}$$

The initial velocity in the y direction, v_{0y}, has been abbreviated to v_0 in this and the following equations. $\boldsymbol{\nabla}\times\boldsymbol{A}'$ is zero. To first order, $\partial\psi/\partial x' = \partial\psi/\partial x$ and $\partial\psi/\partial y' = \partial\psi/\partial y$. The $\boldsymbol{E}$ field simplifies to $-\boldsymbol{\nabla}\psi'$

$$\begin{aligned} E'_x &= k r_0^2 v_0 \{1/[2(x^2+y^2)] - x^2/(x^2+y^2)^2\} \\ E'_y &= -k r_0^2 v_0 xy/(x^2+y^2)^2 \\ E'_z &= 0. \end{aligned} \tag{9}$$

As discussed in the next section, the transverse component is probably not correctly determined in this solution.

The acceleration of a charged particle in the y direction is qE'_y/m

$$a_y = -\frac{q}{m} k r_0^2 v_0 \frac{xy}{(x^2+y^2)^2}.$$

There are no scale changes with an infinitesimal Lorentz transform, so the acceleration in the laboratory system is the same, except for a possible sign inversion. The particle, which has the initial velocity v_0, acquires the additional velocity $\int a_y dt$ in the Newtonian approximation. When the E field is weak the acquired velocity is low, and dt is approximately dy/v_0. There are transverse acceleration terms in the equations, but they will not have a first order effect on the longitudinal solution when $v_y \ll v_0$. The incremental velocity is then

$$\begin{aligned} v_y &= \int_{-y_1}^{y_2} -\frac{qkr_0^2}{m}\frac{xy}{(x^2+y^2)^2}\,\mathrm{d}y \\ &= \frac{qkr_0^2}{2m}\frac{x}{x^2+y^2}\Big|_{y=-y_1}^{y=y_2} \\ &= \frac{qkr_0^2}{2m}\left(-\frac{x}{x^2+y_1^2}+\frac{x}{x^2+y_2^2}\right). \end{aligned}$$

In the second frame of reference the particle is initially at rest at the location $y_2 = -y_1$. The velocity of the particle, relative to an uncharged particle, then steadily increases, reaching a maximum at the closest approach to the solenoid, where y_2 is 0. The relative velocity then steadily diminishes, returning to zero again when the particle is at the location $y_2 = +y_1$ in the laboratory frame of reference. The system would not conserve energy if the particle acquired a net velocity change in its encounter with the solenoid unless there is a way to extract energy from the solenoid, which is not likely with the Maxwell equations.

From the perspective of an observer in the frame of reference of the particle, the retarded location of the moving solenoid is not at the origin when y is zero, but this relationship does not significantly affect the longitudinal acceleration when the particle velocity is low.

The particle's velocity is now $v_0 + v_y$ in the laboratory system. The relative displacement of the particle is $\int v_y\ \mathrm{d}t$. When the E field is weak dt is approximately dy/v_0

$$\begin{aligned} \Delta y &= \int_{y_2=-y_1}^{y_2=+y_1} \frac{qkr_0^2}{2mv_0}\left(-\frac{x}{x^2+y_1^2}+\frac{x}{x^2+y_2^2}\right)\mathrm{d}y_2 \\ &= \frac{qkr_0^2}{2mv_0}\left(-\frac{xy_2}{x^2+y_1^2}+\arctan\frac{y_2}{x}\right)\Big|_{y_2=-y_1}^{y_2=+y_1} \\ &= \frac{qkr_0^2}{mv_0}\left(-\frac{xy_1}{x^2+y_1^2}+\arctan\frac{y_1}{x}\right). \end{aligned}$$

The de Broglie wavelength of the particle in the laboratory frame of reference is

$$\lambda = h/(mv_0).$$

The relative phase in cycles of the de Broglie wavelength is $\Delta y/\lambda$, then the phase in radians is obtained by substituting $\hbar$ for h

$$\Delta\phi = \frac{qkr_0^2}{\hbar}\,(-\frac{xy_1}{x^2+y_1^2} + \arctan\frac{y_1}{x}). \qquad (10)$$

For $y_1 = 10\ x$ this solution is 6.7% less than the value in Eq. (4). It is 0.64% less at $y_1 = 100\ x$. The limiting value for long paths is the same as the A-B solution.

For this particular and unusually simple solution, it is only the scalar potential that matters in Eqs. (9). It is only the vector potential that matters in the first frame of reference. A vector potential transforms into a scalar potential. The solutions represent same problem from the perspectives of two different observers. Since the scalar and vector potentials do not have distinguishable meanings with classical equations, the difference in the parameterization of the equations is of no consequence in this context. Similarly, the Liénard–Wiechert[5] retardation equations represent the transformation of a scalar potential into a vector potential, although in that case the solution contains both vector and scalar terms. The current loop solution would also if the contribution of the stationary protons was not subtracted from the solution.

The electrostatic energy associated with the charge of the conduction electrons is enormous in relation to the energy of the magnetic field within the solenoid, but each conduction electron must be paired with a nearby proton. The electrons and protons must nevertheless be transformed separately if the potentials are retarded or advanced. An electron does not remain nearby to its associated proton as time progresses, and a moving circular loop does not appear to be circular after it is retarded. There are subtle consequences of these relationships.

For these reasons, a zero value for the E and B fields in our frame of reference does not necessarily imply that the fields are zero in other frames of reference. Our frame of reference is as good as any, but solutions representing special cases cannot be generalized if a cancellation of terms occurs in obtaining them. The solution for a special case should be obtained from a more general solution. The inverse calculation does not work unless there is no cancellation of terms for the special case.

II. THE INTERIOR SOLENOID SOLUTION

The interior potential solution in the Lorentz gauge is

$$\boldsymbol{A} = kr\hat{\boldsymbol{\phi}}/2$$
$$\psi = 0.$$

In Cartesian coordinates

$$A_x = -ky/2$$
$$A_y = kx/2$$
$$A_z = 0$$
$$\psi = 0.$$

$\boldsymbol{\nabla}\cdot\boldsymbol{A}$ is zero. $\boldsymbol{\nabla}\times\boldsymbol{A}$ is $B_x = 0$, $B_y = 0$, $B_z = k$. In a system of units where $\boldsymbol{A}$ has the units of distance,

$$A_x = -\xi c^2 ky/2$$
$$A_y = \xi c^2 kx/2$$
$$A_z = 0$$
$$\psi = 0.$$

Transforming to the frame of reference of the particle with Eqs. (8) and the velocity v_y, then converting back to SI units,

$$A'_x = -ky/2$$
$$A'_y = kx/2$$
$$A'_z = 0$$
$$\psi' = -kv_y x/2.$$

$-\boldsymbol{\nabla}\psi'$ is

$$E'_x = kv_y/2$$
$$E'_y = 0$$
$$E'_z = 0.$$

There is no longitudinal component of the E field in the interior region. To first order only, the magnetic field is the same in the second frame of reference as it was in the first. The magnetic force on the particle is $\boldsymbol{F} = q\boldsymbol{v}\times\boldsymbol{B}$. The transverse force due to the E field is half of that value. $\boldsymbol{v}$ has the other sign in the second frame of reference, so the total transverse force on the particle is half of what the solution is known to be.

The Thomas precession[1,3] was originally used to explain a factor of two discrepancy in the spin-orbit coupling of the hydrogen atom. The velocity of conduction electrons in the solenoid windings is so low that the precession would not be expected to affect this solution, but it might. A factor of two discrepancy for the magnetic field also occurs in the Einstein - de Haas effect[8], which is attributed to the spin of the electron. Another possibility is that the potentials of each moving conduction electron need to be advanced to the coordinates of the moving test particle. From this perspective, the test particle is in the third frame of reference. The Lorentz transform is for the second frame of reference. Is there any evidence that we know how to transform to the third frame of reference?

The inconsistency in the calculation for the interior region casts doubt on the validity of the transverse terms in the exterior region. These relationships could be investigated with a miniature cathode ray tube near a static magnetic toroid. There will be no deflection of the electron beam if the transverse component of the E field is zero. A static magnetic toroid cannot supply power to the electron beam, but it could influence the amount of current drawn from the power supply. Miniature cathode ray tubes have not been manufactured anywhere in the world for many years, but limited supplies and data sheets are still available from internet sources. They were produced in various sizes.

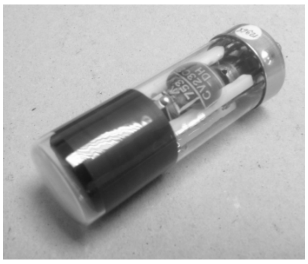

FIG. 1. A CV2302 28 mm cathode ray tube. Photo credit: langrex.co.uk.

If an electron beam is projected through the center of a magnetic toroid and modulated sinusoidally, the configuration is not essentially different from applying an alternating current to a wire passing through the center of the toroid. The electron beam could be in a glass tube. The next step would be to pulse the electron beam so that a single bunch of electrons passes through the toroid. Does the electron bunch induce a voltage in the toroid windings? If it does, there must be a back-reaction that affects the velocity of any electron passing through the toroid when the toroid windings are carrying a direct current. Consequently, the voltage drop along a wire carrying a direct current would not be uniformly distributed along the wire. Due to the low velocity of conduction electrons in metal, the effect would be extremely small. There should also be a current induced in the wires leading to a moving parallel plate capacitor, since there are no electrons between the capacitor plates to respond to the $-\partial \boldsymbol{A}/\partial t$ term. The velocity of a moving capacitor can be far higher than the velocity of conduction electrons, but the capacitance will be small, typically less than a picofarad, so the current required to charge the capacitor will also be small.

Such experiments do not represent a measurement of the vector potential. It is the integral of the E field that is measured. There is no significant E field for moving particles near the center of the toroid, where the vector potential is approximately constant. The velocity of a free charged particle moving through the toroid is nevertheless at a maximum at that point on the trajectory.

Some experimental results for a parallel plate capacitor driven by alternating current near a magnetic toroid that is also driven by alternating current are shown in Ref. 4. Sadly, the authors concluded that the configuration does not conserve energy, but the coupling exhibited by the data is probably valid. From Eq. (6), the energy density of the E field is

$$\frac{1}{2}\epsilon \boldsymbol{E}\boldsymbol{\cdot}\boldsymbol{E} = \frac{1}{2}\epsilon[|\partial \boldsymbol{A}/\partial t|^2 + |\boldsymbol{\nabla}\psi|^2 + 2(\partial \boldsymbol{A}/\partial t)\boldsymbol{\cdot}(\boldsymbol{\nabla}\psi)].$$

The energy density of the scalar and vector potentials cannot be computed separately because there is no known way of telling them apart in dynamic solutions.

The measurable penetrating power of the $-\partial \boldsymbol{A}/\partial t$ term cannot be any better than is allowed by the Maxwell equations. It will cause a redistribution of the conduction electrons on the surface of a metal object, causing the E field to disappear, even though the vector potential is still present. The dipole moment caused by the charge redistribution could not be sustained in a conductor without the $-\partial \boldsymbol{A}/\partial t$ term. The $-\partial \boldsymbol{A}/\partial t$ and $-\boldsymbol{\nabla}\psi$ terms are separately present inside the metal object, but the energy density is zero. (This is just a model to serve as a design tool. The actual governing equations are in terms of the second derivatives, as constrained by the Maxwell equations.)

The inseparability of the scalar and vector potentials can be viewed in another way. In the Lorentz gauge the far field potential solution for a current loop antenna contains only vector potential terms. The solution for a dipole antenna contains both scalar and vector terms. The potential solutions are completely different, but they do not represent two different kinds of radiation.

From yet another perspective, there is no magnetic field associated with a non-spinning charge that is at rest in our frame of reference, yet the dial on a moving magnetometer would register a magnetic field. That is because there would be a magnetic field if the magnetometer was at rest near a moving charge, and we have no way of knowing which frame of reference we are in. The magnetometer dial can be read from any frame of reference, so it does not matter what frame of reference we are in.

It is possible that the $-\partial \boldsymbol{A}/\partial t$ and $-\boldsymbol{\nabla}\psi$ terms actually are distinguishable in a physical sense, in which case the energy density within a metal object immersed in a high frequency electromagnetic field is minuscule but not zero. Such solutions, if they exist, are not representable with the Maxwell equations.

III. SCALAR SOLUTIONS

If a charged particle is moving at the center of a metal sphere, then a voltage is suddenly applied to the surface of the sphere, the particle will no longer be at the center when the potentials propagate to its location. That will result in an asymmetry in the potentials between the forward and reverse directions.

Alternatively, in the low frequency quasi-static approximation, the scalar potential within a Faraday cage is the same as the voltage on the external surface.

$$\begin{aligned} \boldsymbol{A} &= 0 \\ \psi &= \xi\psi_0 \sin(\omega t). \end{aligned}$$

Transforming to the frame of reference of the particle with Eqs. (8)

$$\begin{aligned} \boldsymbol{A} &= -\xi\psi_0 \boldsymbol{v} \sin(\omega t) \\ \psi &= \xi\psi_0 \sin(\omega t). \end{aligned}$$

In SI units

$$\begin{aligned} -\partial \boldsymbol{A}/\partial t &= \psi_0 \boldsymbol{v}\omega \cos(\omega t)/c^2 \\ -\boldsymbol{\nabla}\psi &= 0. \end{aligned} \tag{11}$$

The solution does not conserve charge, but it would if the sphere is placed inside a larger sphere and the two enclosures connected by a wire with a sinusoidal voltage source in series with it. For a charged sphere ψ_0 is $q_0/(4\pi\epsilon_0 r_0)$. The solution for two concentric spheres becomes

$$\boldsymbol{E} = \frac{q_0 \boldsymbol{v} \omega \cos(\omega t)}{4\pi\epsilon_0 c^2}(\frac{1}{r_0} - \frac{1}{r_1}).$$

There is obviously no requirement that the enclosures be spherical. It has experimentally verified to a high level of precision that stationary charged particles inside a Faraday cage do not sense an E field[3].

Quasi-static approximations do not provide non-trivial solutions to the Maxwell equations. The calculations have to be complete enough to represent the second derivatives of the potentials to obtain a full solution. The second derivatives are not always needed in low order approximations.

At 1 KV an electron has a velocity of $0.06c$. For $\psi_0 =$ 10 V and a frequency of 1 MHz Eq. (11) evaluates to 0.013 V/m peak. The current in the wire leading to the sphere will vary as ω^1 in charging and discharging the capacitance of the sphere. $-\partial \boldsymbol{A}/\partial t$ will vary as ω^2, so the vector potential in the first frame of reference can be set aside at low frequencies, although it would be required for obtaining a solution to the Maxwell equations.

The calculation does not transform nothing into something. The fields are there all along, but the contribution of individual charged particles is canceled by other charged particles when the test particle is stationary. The fields propagate at the speed of light from each charged particle to the detector. If a cancellation of terms occurs it does not happen until after the fields have propagated to the detector. (It is not possible to determine if the fields propagate at c in static solutions, but solutions that are static in our frame of reference are not static in any other frame of reference.)

Transforming the integral over the surface of a sphere is not the same as transforming and then integrating. The integral in our frame of reference has no meaning for an observer in the frame of reference of a moving test particle. At low velocities the retarded integral over the surface of a sphere that is moving in our frame of reference is equivalent to evaluating the integral in the frame of reference of a moving test particle when the sphere is at rest. There will be relativistic corrections if the v^2 terms are carried, and they are not necessarily symmetrical between the two perspectives.

To first order only, and except for a sign inversion, a more complete solution for a sphere that is at rest in our frame of reference could be obtained by applying the Liénard–Wiechert retardation equations to a moving and radially pulsating charged sphere. The solutions to the Liénard–Wiechert equations are always solutions to the Maxwell equations. As with the current loop solution, there are two velocities and three frames of reference in the pulsating sphere solution, so there may be some inconsistencies with the equations for the second frame of reference. Experiments only accurate to within a factor of about two would therefore not constitute a confirmation of these calculations – the discrepancy would not necessarily be due to calibration errors.

In being optimized for other purposes, the high frequency response of conventional cathode ray tubes is very poor. The limitation could be worked around by modulating the grid with a nearby frequency to produce a beat frequency. Nonlinear interactions are required for there to be a beat frequency, so the grid modulation should be of large amplitude. A custom design would not have this limitation. Other styles of vacuum tubes may be usable, but a short beam length will reduce the sensitivity, and the anode might shield the signal.

There are other considerations, such as the skin effect, which prevents a high frequency transverse E field from penetrating metal. The transverse component is the only component in the far field, but this is not a far field solution. It is probably not yet possible to reliably determine if there is a signal, but it is not a difficult experiment. With battery powered microprocessor based instrumentation it is not even necessary to have a connection to the outside world, minimizing signal leakage problems. A similar configuration has been proposed for a quantum experiment[2].

To first order, there are no fields in the frame of reference of the particle when ω is zero, indicating that it is not necessary to transform to its frame of reference for obtaining the 4-potential version of the scalar A-B effect. The difference in the behavior of the scalar and vector solutions occurs because, until magnetic monopoles are discovered, the magnetic field is a transformed E field.

In the following calculations there is a charged metal sphere at the origin with two small holes on opposite sides. From a great distance a charged particle is projected toward the nearby hole with the initial velocity v_0. The particle is accelerated by the E field, and it acquires the velocity $v_0 + v$. It then drifts across the diameter of the sphere without further acceleration and exits the hole on the other side. The voltage on the surface of the sphere has to be arbitrarily low in order for these calculations to be valid. It is an infinitesimal voltage.

The E field in the exterior region of the sphere is $E = q/(4\pi\epsilon_0 r^2)$. The potential at its surface, relative to a point at infinity, is $\psi_0 = q/(4\pi\epsilon_0 r_0)$, where r_0 is the radius. Solving the second equation for q and substituting it into the first provides the equation $E = \psi_0 r_0/r^2$. The acceleration of the particle is qE/m, and it acquires the velocity $qE\ dt/m$, with $dt = dr/(v_0 + v)$. v is small when the voltage on the sphere is low, so the equation simplifies to $dt \approx dr/v_0$. v_0 is assumed to be positive.

During the inbound traverse the particle acquires the incremental radial velocity

$$v_i = \int_{r=r_2}^{r=r_1} \frac{q}{m}\frac{\psi_0 r_0}{r^2}\frac{\mathrm{d}r}{v_0} = -\frac{q\psi_0 r_0}{m v_0}(\frac{1}{r_1} - \frac{1}{r_2}). \quad (12)$$

The particle is slowed if it has the same sign as the charge on the sphere. The radial displacement that occurs during the inbound traverse is $\int v_i \mathrm{d}t \approx \int v_i \mathrm{d}r/v_0$.

$$\Delta r_i = \int_{r_1=r_2}^{r_1=r_0} -\frac{q\psi_0 r_0}{m v_0^2}(\frac{1}{r_1} - \frac{1}{r_2})\ \mathrm{d}r_1 = \frac{q\psi_0 r_0}{m v_0^2}(\frac{r_1}{r_2} - \ln r_1)\Big|_{r_1=r_2}^{r_1=r_0} = \frac{q\psi_0 r_0}{m v_0^2}(\frac{r_0}{r_2} - 1 + \ln\frac{r_2}{r_0}).$$

After exiting the hole on the other side of the sphere the outbound incremental radial velocity is

$$v_o = \int_{r=r_1}^{r=r_2} \frac{q}{m}\frac{\psi_0 r_0}{r^2}\frac{\mathrm{d}r}{v_0} = \frac{q\psi_0 r_0}{m v_0}(\frac{1}{r_1} - \frac{1}{r_2}).$$

The radial displacement that occurs during the outbound portion of the trajectory is

$$\Delta r_o = \int_{r_1=r_0}^{r_1=r_2} \frac{q\psi_0 r_0}{m v_0^2}(\frac{1}{r_1} - \frac{1}{r_2})\ \mathrm{d}r_1 = \frac{q\psi_0 r_0}{m v_0^2}(-\frac{r_1}{r_2} + \ln r_1)\Big|_{r_1=r_0}^{r_1=r_2} = \frac{q\psi_0 r_0}{m v_0^2}(\frac{r_0}{r_2} - 1 + \ln\frac{r_2}{r_0}).$$

When the particle source and target are symmetrically located on either side of the sphere there is no net change in the velocity or position during the traverses. In the full symmetrical path integral the particle behaves as though there were no external E field. However, it drifts across the diameter of the sphere with the velocity $v_0 + v$ rather than v_0, resulting in a permanent position shift, relative to an uncharged particle. For $r_2 = \infty$ and $r_1 = r_0$ Eq. (12) becomes

$$v = -\frac{q\psi_0}{m v_0}.$$

This equation is not valid unless v is small in relation to v_0. The relative displacement that occurs during the drift is $vt = 2vr_0/v_0$

$$\Delta r = -2\frac{q\psi_0 r_0}{m v_0^2}.$$

Substituting $r_0 = tv_0/2$

$$\Delta r = -\frac{q\psi_0 t}{m v_0}.$$

Rescaling by the de Broglie wavelength as in Eq. (10),

$$\Delta\phi = -q\psi_0 t/\hbar.$$

The solution is the same as the A-B scalar solution[7].

This effect is large. For an electron energy of 1 KeV and a sphere radius of 1 cm a voltage change of only 3.8 μV would account for one fringe in the interference pattern. A voltage this low qualifies as an infinitesimal voltage in most cases.

It is concluded in Ref. 6 that surface charges in a metal tube cancel the scalar A-B effect. The acceleration of the electrons in the external E field will still be present, implying that the phase shift for the full path is about the same as the A-B solution. Quantum solutions cannot be localized[7], making it difficult to attribute the phase shift to any one portion of the trajectory.

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