Breakdown of Euler Equations – New Approach

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Abstract – The solution for the problem of Breakdown of Euler Equations, like the Millenium Problem for Navier-Stokes equations.

§ 1

Motived by the 6th Millenium Problem, relative to the solution of the Navier-Stokes equations or prove of the inexistence of solutions, obeying certain conditions, I wrote this paper for solve this problem substituting Navier-Stokes by Euler equations, since that these same questions are unsolved for Euler equations, although these last are not on the Clay Institute's list of prize problems.[1] The natural sequence of this paper is the correspondent to Navier-Stokes equations.

In his famous *Méchanique Analitique* (1788), using the notions of total or complete differential and exact differential, and creating the concept of velocity-potential, for an external force with potential (a gradient or conservative external force, which also can be a force equal to zero) Lagrange came to the conclusion that Euler´s equations could be solved only for two specific conditions: (1) for potential (irrotational) flows, and (2) for non-potential (rotational) but steady flows.[2],[3] In Lagrange[3], pp. 536-542, the pressure is represented as λ, the external force components as X, Y, Z, the velocity components as p, q, r, the rectangular coordinates as x, y, z and time as t. The velocity-potential is φ and the force-potential is V.

The solution for pressure obtained by Lagrange in potential flow case was

$$\lambda = V + \frac{d\varphi}{dt} + \frac{1}{2}\left(\frac{d\varphi}{dx}\right)^2 + \frac{1}{2}\left(\frac{d\varphi}{dy}\right)^2 + \frac{1}{2}\left(\frac{d\varphi}{dz}\right)^2,$$

and an arbitrary function of t could be added here because this variable is treated in the integration as a constant, which is nothing more nor less that the Bernouilli's law (the use by Lagrange of d is as our ∂, means partial derivative).

The determination of φ will depend upon equation (continuity equation, the incompressibility condition)

$$\frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dy} = 0,$$

in which after substitution of the expressions $\frac{d\varphi}{dx}, \frac{d\varphi}{dy}, \frac{d\varphi}{dz}$ for p, q, r becomes

$$\frac{d^2\varphi}{dx^2} + \frac{d^2\varphi}{dy^2} + \frac{d^2\varphi}{dz^2} = 0,$$

that is the Laplace's equation.

Thus, conclude Lagrange, all the remaining difficulty will now lie in the integration of this last equation.

Of course that it is possible describe a fluid movement without potential flow and conservative forces, simply by setting the force as

$$\boldsymbol{f} = \nabla p + \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u},$$

given any pressure p and velocity u, both differentiable functions of class C and C^2, respectively, velocity with potential or no, obeying the incompressibility condition or no, but we do not need this kind of force here.

In the present paper we are interested only in conservative external forces, i.e., with potential, including zero, and the validity of incompressible flow condition, which require for the solution of Euler equations a potential velocity for non-steady flows.

I think that the deduction used by Lagrange in Euler's equations can be implemented also in Navier-Stokes equations, and we will come to $\nabla^2 \boldsymbol{u} = 0$. I am hopeful to prove this in next article, concluding this subject.

§ 2

When $\nabla \times \boldsymbol{u} = 0$ then exist a potential function ϕ such that $\boldsymbol{u} = \nabla\phi$. When $\nabla \times \boldsymbol{u} = 0$ and $\nabla \cdot \boldsymbol{u} = 0$ then $\nabla^2\phi = 0$ and $\nabla^2\boldsymbol{u} = 0$, therefore the Navier-Stokes equations are reduced to Euler's equations and the solutions for velocity are given by Laplace's equation, they are harmonic functions, i.e.,

(2.1) $$\nabla^2\boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u}) = (\nabla^2 u, \nabla^2 v, \nabla^2 w) = 0$$

and

(2.2) $$\boldsymbol{u} = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right), \nabla \cdot \boldsymbol{u} = 0 \Longrightarrow \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\right) = 0.$$

It is clear that there is no uniqueness solution in all cases, in special when the velocity is both irrotational and incompressible, even if the velocity vanishes at infinity. Defining $\phi(x, y, z, t) = \phi^0(x, y, z)T(t)$, $T(0) = 1$, $T(t) \not\equiv 1$, then we have $\boldsymbol{u} = \nabla\phi = T(t)\nabla\phi^0 = T(t)\boldsymbol{u}^{\boldsymbol{0}}(x, y, z)$ and so there are endless possibilities for constructing $\boldsymbol{u}$ given $\boldsymbol{u}^{\boldsymbol{0}}$, because there are endless possibilities for constructing $T(t)$ with $T(0) = 1$, even if $\lim_{r\to\infty} \boldsymbol{u} = T(t) \lim_{r\to\infty} \boldsymbol{u}^{\boldsymbol{0}} = 0$, where $r =$

$\sqrt{x^2+y^2+z^2}$. Exception if the initial velocity is identically null, when for the previous reasoning the velocity is $\boldsymbol{u}=0$ unique.

A more long way to see this is for example as follow. If $\nabla\cdot\boldsymbol{u}=0$ and $\nabla\times\boldsymbol{u}=0$ then $\nabla^2\boldsymbol{u}=0$. For $\boldsymbol{u}=(u_1,u_2,u_3)$ and $\boldsymbol{w}=(w_1,w_2,w_3)$, defining $w_i=A(t)u_i+B_i(t), 1\le i\le 3$, we will have $\nabla\cdot\boldsymbol{w}=0,\ \nabla\times\boldsymbol{w}=0$ and $\nabla^2\boldsymbol{w}=0$.

If $\boldsymbol{u}=\nabla\phi$ solves the Navier-Stokes equations then, from

(2.3.1) $$\nabla p+\frac{\partial\boldsymbol{u}}{\partial t}+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}=\nu\nabla^2\boldsymbol{u}$$

(2.3.2) $$\nabla p+\nabla\left(\frac{\partial\phi}{\partial t}\right)+(\nabla\times\boldsymbol{u})\times\boldsymbol{u}+\frac{1}{2}\nabla|\boldsymbol{u}|^2= \\ =\nu\big(\nabla(\nabla\cdot\boldsymbol{u})-\nabla\times(\nabla\times\boldsymbol{u})\big)$$

(2.3.3) $$\nabla p+\nabla\left(\frac{\partial\phi}{\partial t}\right)+\nabla\left(\frac{1}{2}|\boldsymbol{u}|^2\right)=0$$

(2.3.4) $$\nabla\left(p+\frac{\partial\phi}{\partial t}+\frac{1}{2}|\boldsymbol{u}|^2\right)=0,$$

we obtain

(2.4) $$p+\frac{\partial\phi}{\partial t}+\frac{1}{2}|\boldsymbol{u}|^2=\theta(t),$$

which is the Bernouilli's law without external force.

With a gradient external force $\boldsymbol{f}=\nabla U$ we will have

(2.5) $$p+\frac{\partial\phi}{\partial t}+\frac{1}{2}|\boldsymbol{u}|^2=U+\theta(t).$$

For $\boldsymbol{w}$ defined as above, substituting $\boldsymbol{u}\mapsto\boldsymbol{w}$ in the Navier-Stokes equations comes

(2.6) $$p+\frac{\partial\varphi}{\partial t}+\frac{1}{2}|\boldsymbol{w}|^2=U+\theta(t),$$

where $\varphi=A(t)\phi$, and p is the new pressure for the velocity $\boldsymbol{w}$.

If $A(0)=1$ and $B_i(0)=0,\ 1\le i\le 3$, then $\boldsymbol{u}$ and $\boldsymbol{w}$ obey the same initial condition and both solve the Navier-Stokes (and Euler) equations and they are incompressible and potential flows. Thus, in this case, there is no uniqueness solution, for $A(t)\not\equiv 1$ and $B(t)\not\equiv 0$, i.e., $\boldsymbol{u}\not\equiv\boldsymbol{w}$.

Imposing the boundary condition at infinity $\boldsymbol{u}|_{r\to\infty}=0, r=\sqrt{x^2+y^2+z^2}$, the velocity $\boldsymbol{w}=A(t)\boldsymbol{u}$ obey the same boundary condition, for $A(0)=1,\ A(t)\not\equiv 1$ finite for all $t\ge 0$, i.e. $\boldsymbol{w}(x,y,z,t)=A(t)\boldsymbol{u}(x,y,z,t)$ and $\boldsymbol{u}(x,y,z,t)$ obey the same initial and boundary conditions, so there is no uniqueness solutions for Navier-

Stokes (and Euler) equations in this case of incompressible and potential flows with velocity zero at infinity, if $\boldsymbol{u} \neq 0$.

§ 3

Sobolev[4] (pp. 12, 13, 18, 19) is very assured to affirm that *the problem of the motion of an incompressible fluid is equivalent to that of finding an unknown function* V *(the velocity-potential) such that*

$$\boldsymbol{v} = grad\, V, \qquad v_x = \frac{\partial V}{\partial x}, \quad v_y = \frac{\partial V}{\partial y}, \quad v_z = \frac{\partial V}{\partial z}.$$

Continuing his citation, *substituting these expressions for the velocity components in the continuity equation, we get*

$$\varrho\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) = 0$$

or

$$\nabla^2 V = 0. \tag{1.17}$$

(...)

Later we shall write down the complete set of equations of motion for a fluid and we shall show that any function V *which satisfies (1.17) does indeed describe a possible motion of the fluid. Thus to solve a problem of fluid motion it suffices to know to find the requisite solutions of equation (1.17).*

In some circumstances, the velocity $\boldsymbol{v}$ *and so also the function* V *do not depend on the time* t*; the motion is then one of steady flow.*

(...)

We can now verify what was said earlier about the potential flow of an incompressible fluid: namely, that

$$\boldsymbol{v} = grad\, V,$$

$$\nabla^2 V = 0,$$

do actually satisfy the complete set of equations (Euler equations with mass density coefficient ϱ and external force (X, Y, Z), note mine)*, if the function* ϱ *is defined correspondingly, and if further*

$$X = \frac{\partial U}{\partial x}, \; Y = \frac{\partial U}{\partial y}, \; Z = \frac{\partial U}{\partial z},$$

i.e., if the external force have a potential.

It suffices to show that if we take

$$v_x = \frac{\partial V}{\partial x}, \quad v_y = \frac{\partial V}{\partial y}, \quad v_z = \frac{\partial V}{\partial z},$$

then the equations (1.22) (the Euler equations) *allow the function* p *to be constructed. When the expressions for* v_x, v_y, v_z *are substituted, these equations yield explicit expressions for*

$$\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z}.$$

And it is known from the theory of partial differential equations of the first order that the equations will be compatible provided that the mixed second-order derivatives

$$\frac{\partial^2 p}{\partial x \partial y}, \frac{\partial^2 p}{\partial y \partial z}, \frac{\partial^2 p}{\partial z \partial x}$$

determined from the different equations have the same values. (...)

Then, following Sobolev, if the external force is gradient, if it have a potential, the solutions for velocity in the Euler's equations in case of incompressible flows are given by Laplace's equation, the velocity is a harmonic function in the three orthogonal directions, not only one possibility among others, but in fact they are the unique possible cases of solution, only harmonic functions, when the external force is gradient (for example also without external force, $X = Y = Z = 0$) and the fluid is incompressible.

The same argument used by Sobolev for solve Euler's equations can be used for solve the Navier-Stokes equations: *Thus to solve a problem of fluid motion it suffices to know to find the requisite solutions of equation (1.17),* $\nabla^2 V = 0$. This is like the conclusion of Lagrange, viewed in section § 1, for Euler's equations in potential flow case.

All solution of Euler equations is solution of Navier-Stokes equations for potential and incompressible flows, when $\nabla^2 \boldsymbol{u} = 0$. If $\boldsymbol{u} = \nabla \phi$ then $\nabla \times \boldsymbol{u} = 0$, because

$$\text{(3.1)} \qquad \nabla \times \boldsymbol{u} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial k \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix} = 0$$

being $\frac{\partial^2 \phi}{\partial x_i \partial x_j} = \frac{\partial^2 \phi}{\partial x_j \partial x_i}$, $1 \le i, j \le 3$.

If $\nabla \times \boldsymbol{u} = 0$ (potential flow) and $\nabla \cdot \boldsymbol{u} = 0$ (incompressible flow) then

(3.2) $$\nabla^2 \boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times (\nabla \times \boldsymbol{u}) = 0,$$

i.e. the derivatives of second order in Navier-Stokes equations vanishes in case of potential and incompressible flows and the Navier-Stokes equations reduced to the Euler equations, whose respective solutions are harmonic functions. In this case, solve Euler equations implies solve Navier-Stokes equations and if the Navier-Stokes equations has unique solution at least in a small and not null time interval $[0, T]$, with the boundary condition $\lim_{r\to\infty} |\boldsymbol{u}| = 0\,, r = \sqrt{x^2 + y^2 + z^2}$, then this first solution in time is also the solution of Euler equations and the velocity satisfies the Laplace's equation.

§ 4

How the condition

(4.1) $$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}, 1 \le i, j \le 3,$$

solve the Euler equations, equivalent to $\nabla \times \boldsymbol{u} = 0$, so the condition of irrotational or potential flow is a necessary condition for solution of these equations for non-steady flows. This has been rigorously proven by Lagrange[3], for a gradient external force. Including the incompressibility condition, we have the Laplace's equation in vector form, $\nabla^2 \boldsymbol{u} = 0$ and $\nabla^2 \boldsymbol{u^0} = 0$, where $\boldsymbol{u^0}$ is the initial velocity, even without uniqueness solution, as viewed in section § 2.

For steady flows, where $\frac{\partial \boldsymbol{u}}{\partial t} \equiv 0$ and $\boldsymbol{u} = \boldsymbol{u^0}$ for all $t \ge 0$, the condition for existence of solution is that $\frac{\partial S_i}{\partial x_j} = \frac{\partial S_j}{\partial x_i}$ for all pair (i, j), $1 \le i, j \le 3$, defining $S_i = f_i - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j}$, where $\boldsymbol{f} \equiv \boldsymbol{f^0}$ is the stationary external force. This is a common condition for existence of solution for a system $\nabla p = \boldsymbol{S}$, representing the stationary Euler's equations, that is $\nabla \times \boldsymbol{S} = 0$.

Then now is possible go to the solution related to the breakdown of the Euler equations, corresponding to the cases (C) and (D) of [1]: without external force or with an external force which have a potential, $\boldsymbol{f} = \nabla U$, $\boldsymbol{f} \in S(\mathbb{R}^3 \times [0, \infty))$, S representing the Schwartz space, if the initial velocity $\boldsymbol{u^0} \in \mathbb{R}^3$ is not a potential flow and (considering also the steady flows) $\frac{\partial S_i}{\partial x_j} \neq \frac{\partial S_j}{\partial x_i}$ for any pair (i, j) such that $1 \le i, j \le 3$, with $S_i = f_i^0 - \sum_{j=1}^{3} u_j^0 \frac{\partial u_i^0}{\partial x_j}$, $x_1 \equiv x$, $x_2 \equiv y$, $x_3 \equiv z$, $\boldsymbol{f^0} = \boldsymbol{f}(x, y, z, 0)$, there is no solution $(\boldsymbol{u}, p)$ for the Euler equations, belonging to C^∞ or no, periodic solution or no. In special, when $\boldsymbol{u^0} \in S(\mathbb{R}^3)$ and $\boldsymbol{u^0}$ is not a gradient function, with $\frac{\partial S_i}{\partial x_j} \neq \frac{\partial S_j}{\partial x_i}$ for any (i, j), there is no solution for Euler equations, in the mentioned conditions for $\boldsymbol{f}$. Besides that the unique initial velocity $\boldsymbol{u^0} \in S(\mathbb{R}^3)$, harmonic and gradient function is $\boldsymbol{u^0} = 0$, which provide only the trivial solution $\boldsymbol{u} = 0$ for velocity in Schwartz space and infinite solutions for pressure, $p = U + \theta(t)$, $p \in C^\infty(\mathbb{R}^3 \times [0, \infty))$.

I finish this work with a qualitative discussion of this result. Any student of physics, Gravitation or Electromagnetism, knows that the most well-known non trivial solution of the Laplace's equation is of the form $1/r$, which diverges in origin and goes to zero at infinity. According Liouville's Theorem[4], a harmonic function which is limited is constant, and equal to zero if it tends to zero at infinity. How the Millennium Problem requires a limited solution in all space for velocity and an limited initial velocity which goes to zero at infinity (in cases (A) and (C)), then we are forced to choose $\boldsymbol{u^0} = 0$.

Without these requisites we can obtain other solutions for velocity, for example, $\boldsymbol{u} = \boldsymbol{A}(t)$, as well as potential flows in general, including spatially periodic functions of unitary period without singularities in the cube $[0,1]^3$, which refers to case (B). Initial velocities spatially periodic but non potential flows lead to case (D) if the external force is gradient, no time dependence and $\frac{\partial s_i}{\partial x_j} \neq \frac{\partial s_j}{\partial x_i}$ for any (i, j).

As pointed by Lagrange and Sobolev, the solution of Laplace's equation is essential in the solution of Euler's equations. According Courant[5] (p.241), for $n = 2$ the "general solution" of the potential equation (or Laplace's equation) is the real part of any analytic function of the complex variable $x + iy$. For $n = 3$ one can also easily obtain solutions which depend on arbitrary functions. For example, let $f(w, t)$ be analytic in the complex variable w for fixed real t. Then, for arbitrary values of t, both the real and imaginary parts of the function

$$u = f(z + ix\cos t + iy\sin t\,, t)$$

of the real variables x, y, z are solutions of the equation $\nabla^2 u = 0$. Further solutions may be obtained by superposition:

$$u = \int_a^b f(z + ix\cos t + iy\sin t\,, t)dt.$$

For example, if we set

$$f(w, t) = w^n e^{iht},$$

where n and h are integers, and integrate from $-\pi$ to $+\pi$, we get homogeneous polynomials

$$u = \int_{-\pi}^{\pi} (z + ix\cos t + iy\sin t)^n\, e^{iht} dt$$

in x, y, z, following example given by Courant. Introducing polar coordinates $z = r\cos\theta, x = r\sin\theta\cos\phi, y = r\sin\theta\sin\phi$, we obtain

$$u = 2r^n e^{ih\phi}\int_0^{\pi}(\cos\theta + i\sin\theta\cos t)^n \cos ht\ \ dt$$
$$= r^n e^{ih\phi}P_{n,h}(\cos\theta),$$

where $P_{n,h}(\cos\theta)$ are the associated Legendre functions.

A musician must make music,
an artist must paint,
a poet must write,
if he is to be ultimately at peace with himself.
What a man can be, he must be.

Abraham H. Maslow

Um músico deve compor,
um artista deve pintar,
um poeta deve escrever,
caso pretendam deixar seu coração em paz.
O que um homem pode ser, ele deve ser.

A essa necessidade podemos
dar o nome de autorrealização.
Refere-se ao desejo do homem
de autopreenchimento, isto é,
à tendência que ele apresenta
de se tornar, em realidade,
o que já é em potencial.

Abraham H. Maslow

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