Special Smarandache Curves According To Darboux Frame In E^3

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Abstract

In this study, we determine some special Smarandache curves according to Darboux frame in E^3 . We give some characterizations and consequences of Smarandache curves.

Keywords: Smarandache curves, Darboux Frame, Normal curvature, Geodesic Curvature, Geodesic torsion.

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1 Introduction

In differential geometry, there are many important consequences and properties of curves. Researchers follow labours about the curves. In the light of the existing studies, authors always introduce new curves. Special Smarandache curves are one of them. This curve is defined as, a regular curve in Minkowski space-time, whose position vector is composed by Frenet frame vectors on another regular curve, is called a Smarandache Curve [6]. Special Smarandache curves have been studied by some authors [1, 3, 6]. M. Turgut and S. Yilmaz have defined a special case of such curves and call it Smarandache $\mathbf{TB_2}$ Curves in the space E_1^4 [6]. They have dealed with a special Smarandache curves which is defined by the tangent and second binormal vector fields. Besides, they have computed formulas of this kind curves by the method expressed in [6]. A. T. Ali has introduced some special Smarandache curves in the Euclidean space [1]. Special Smarandache curves such as Smarandache curves $\mathbf{TN}_1$, $\mathbf{TN}_2$, $\mathbf{N}_1\mathbf{N}_2$ and $\mathbf{TN}_1\mathbf{N}_2$ according to Bishop frame in Euclidean 3-space have been investigated by Çetin et al [3]. Furthermore, they studied differential geometric properties of these special curves and they calculated first and second curvature (natural curvatures) of these curves. Also they found the centers of the curvature spheres and osculating spheres of Smarandache curves.

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In this study, we investigate special Smarandache curves such as Smarandache **Tg**, **Tn**, **gn** and **Tgn** according to Darboux frame in Euclidean 3-space. Furthermore, we find some properties of these special curves and we calculate normal curvature, geodesic curvature and geodesic torsion of these curves.

2 Preliminaries

In this section, we give an information about special Smarandache curves and Darboux frame. Let M be an oriented surface and let consider a curve $\alpha(s)$ on the surface M . Since the curve $\alpha(s)$ is also in space, there exists Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ at each points of the curve where $\mathbf{T}$ is unit tangent vector, $\mathbf{N}$ is principal normal vector and $\mathbf{B}$ is binormal vector, respectively. The Frenet equations of the curve $\alpha(s)$ is given by

$$\begin{cases} \mathbf{T}^{'}=\kappa\mathbf{N} \\ \mathbf{N}^{'}=-\kappa\mathbf{T}+\tau\mathbf{B} \\ \mathbf{B}^{'}=-\tau\mathbf{N} \end{cases}$$

where κ and τ are curvature and torsion of the curve $\alpha(s)$, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve. Since the curve $\alpha(s)$ lies on the surface M there exists another frame of the curve $\alpha(s)$ which is called Darboux frame and denoted by $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$. In this frame $\mathbf{T}$ is the unit tangent of the curve, $\mathbf{n}$ is the unit normal of the surface M and $\mathbf{g}$ is a unit vector given by $\mathbf{g}=\mathbf{n}\times\mathbf{T}$. Since the unit tangent $\mathbf{T}$ is common in both Frenet frame and Darboux frame, the vectors $\mathbf{N},\mathbf{B}$,$\mathbf{g}$, $\mathbf{n}$ lie on the same plane. So that the relations between these frames can be given as follows

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} \tag{2.1}$$

where φ is the angle between the vectors $\mathbf{g}$ and $\mathbf{n}$. The derivative formulae of the Darboux frame is

$$\begin{bmatrix} \dot{\mathbf{T}} \\ \dot{\mathbf{g}} \\ \dot{\mathbf{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{T} \\ \mathbf{g} \\ \mathbf{n} \end{bmatrix} \tag{2.2}$$

where , k_g is the geodesic curvature, k_n is the normal curvature and τ_g is the geodesic torsion of $\alpha(s)$.

The relations between geodesic curvature, normal curvature, geodesic torsion and κ , τ are given as follows

$$k_g=\kappa\cos\varphi,\ k_n=\kappa\sin\varphi\ ,\ \tau_g=\tau+\frac{d\varphi}{ds}. \tag{2.3}$$

In the differential geometry of surfaces, for a curve $\alpha(s)$ lying on a surface M the followings are well-known

i) $\alpha(s)$ is a geodesic curve $\Leftrightarrow k_g = 0$,
ii) $\alpha(s)$ is an asymptotic line $\Leftrightarrow k_n = 0$,
iii) $\alpha(s)$ is a principal line $\Leftrightarrow \tau_g = 0$, [7].

Let $\alpha = \alpha(s)$ and $\beta = \beta(s^*)$ be a unit speed regular curves in E^3 and $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be moving Serret-Frenet frame of $\alpha(s)$. Smarandache curves **TN** are defined by $\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{N})$, Smarandache curves **NB** are defined by $\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{N}+\mathbf{B})$ and Smarandache curves **TNB** are defined by $\beta(s^*) = \frac{1}{\sqrt{3}}(\mathbf{T}+\mathbf{N}+\mathbf{B})$.

3 Special Smarandache Curves According To Darboux Frame In E^3

In this section, we investigate special Smarandache curves according to Darboux frame in E^3. Let $\alpha = \alpha(s)$ and $\beta = \beta(s^*)$ be a unit speed regular curves in E^3 and defined by $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$ and $\{\mathbf{T}^*, \mathbf{g}^*, \mathbf{n}^*\}$ be Darboux frame of these curves, respectively.

3.1 Tg− Smarandache Curves

Definition: Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha = \alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$ $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$.

Tg- Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{g}). \tag{3.1}$$

Now, we can investigate Darboux invariants of **Tg** − Smarandache curve according to $\alpha = \alpha(s)$. Differentiating (3.1) with respect to s, we get

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left(k_g\mathbf{T} - k_g\mathbf{g} - (k_n+\tau_g)\,\mathbf{n}\right), \tag{3.2}$$

and

$$\mathbf{T}^*\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left(k_g\mathbf{T} - k_g\mathbf{g} - (k_n+\tau_g)\,\mathbf{n}\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2k_g^2 + (k_n+\tau_g)^2}{2}}. \tag{3.3}$$

The tangent vector of curve β can be written as follow,

$$\mathbf{T}^* = \frac{-1}{\sqrt{2k_g^2 + (k_n+\tau_g)^2}}\left(k_g\mathbf{T} - k_g\mathbf{g} - (k_n+\tau_g)\,\mathbf{n}\right). \tag{3.4}$$

Differentiating (3.4) with respect to s, we obtain

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds}=\frac{-1}{\left(2k_g^2+(k_n+\tau_g)^2\right)^{\frac{3}{2}}}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{g}+\Gamma_3\mathbf{n}\right) \tag{3.5}$$

where

$$\begin{cases} \Gamma_1=(k_n+\tau_g)\left\{k_g\left(k_n^{'}+\tau_g^{'}-k_nk_g-\tau_gk_g\right)-k_g^{'}(k_n+\tau_g)-k_n\left[2k_g^2+(k_n+\tau_g)^2\right]\right\}-2k_g^4 \\ \Gamma_2=(k_n+\tau_g)\left\{-k_g\left(k_n^{'}+\tau_g^{'}+k_nk_g+\tau_gk_g\right)+k_g^{'}(k_n+\tau_g)-\tau_g\left[2k_g^2+(k_n+\tau_g)^2\right]\right\}-2k_g^4 \\ \Gamma_3=k_g(k_n+\tau_g)\left[-2k_g^{'}-k_n+\tau_g(k_n+\tau_g)\right]+2k_g^2\left(k_g\tau_g+k_n^{'}+\tau_g^{'}\right). \end{cases}$$

Substituting (3.3) in (3.5), we get

$$\dot{\mathbf{T}}^*=\frac{\sqrt{2}}{\left(2k_g^2+(k_n+\tau_g)^2\right)^2}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{g}+\Gamma_3\mathbf{n}\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa^*=\left\|\dot{\mathbf{T}}^*\right\|=\frac{\sqrt{2\left(\Gamma_1^2+\Gamma_2^2+\Gamma_3^2\right)}}{\left(2k_g^2+(k_n+\tau_g)^2\right)^2}$$

and

$$\mathbf{N}^*=\frac{1}{\sqrt{\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\left(\Gamma_1\mathbf{T}+\Gamma_2\mathbf{g}+\Gamma_3\mathbf{n}\right).$$

On the other hand, we express

$$\mathbf{B}^*=\mathbf{T}^*\times\mathbf{N}^*=\frac{1}{\sqrt{2k_g^2+(k_n+\tau_g)^2}\sqrt{\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\begin{vmatrix} \mathbf{T} & \mathbf{g} & \mathbf{n} \\ -k_g & k_g & k_n+\tau_g \\ \Gamma_1 & \Gamma_2 & \Gamma_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$\mathbf{B}^*=\frac{1}{\sqrt{2k_g^2+(k_n+\tau_g)^2}\sqrt{\Gamma_1^2+\Gamma_2^2+\Gamma_3^2}}\left(\mu_1\mathbf{T}+\mu_2\mathbf{g}+\mu_3\mathbf{n}\right)$$

where

$$\begin{cases} \mu_1=k_g\Gamma_3-(k_n+\tau_g)\Gamma_2 \\ \mu_2=(k_n+\tau_g)\Gamma_1+k_g\Gamma_3 \\ \mu_3=-k_g\Gamma_2-k_g\Gamma_{1.} \end{cases}$$

We differentiate (3.2) with respect to s in order to calculate the torsion

$$\beta^{''}=\frac{-1}{\sqrt{2}}\left\{\left(k_g^{'}+k_g^2+k_n(k_n+\tau_g)\right)\mathbf{T}+\left(-k_g^{'}+k_g^2+\tau_g(k_n+\tau_g)\right)\mathbf{g}+\left(k_nk_g-k_g\tau_g-k_g^{'}+\tau_g^{'}\right)\mathbf{n}\right\},$$

and similarly

$$\beta^{'''} = \frac{-1}{\sqrt{2}} \left(\eta_1 \mathbf{T} + \eta_2 \mathbf{g} + \eta_3 \mathbf{n} \right),$$

where

$$\begin{cases} \eta_1 = k_g^{''} + 2k_n \left(k_n^{'} + \tau_g^{'} \right) + k_g \left(3k_g^{'} - \tau_g^2 - k_n^2 - k_g^2 \right) + k_n^{'} \left(k_n + \tau_g \right) \\ \eta_2 = -k_g^{''} + 2\tau_g \left(k_n^{'} + \tau_g^{'} \right) + k_g \left(3k_g^{'} + \tau_g^2 + k_n^2 + k_g^2 \right) + \tau_g^{'} \left(k_n + \tau_g \right) \\ \eta_3 = -k_g^{''} - \tau_g^{''} + k_g \left(k_n^{'} - \tau_g^{'} \right) + \left(k_n + \tau_g \right) \left(\tau_g^2 + k_n^2 + k_g^2 \right) + 2k_g^{'} \left(k_n - \tau_g \right). \end{cases}$$

The torsion of curve β is

$$\tau^* = \frac{-1}{\sqrt{2}} \frac{\left(\eta_1 + \eta_2 \right) \left[k_g \left(k_n^{'} + \tau_g^{'} \right) - k_g^{'} \left(k_n + \tau_g \right) \right] + \left(2k_g^2 + \left(k_n + \tau_g \right)^2 \right) \left[\tau_g \eta_1 - k_n \eta_2 + k_g \eta_3 \right]}{2 \left(k_g^{'2} + k_g^4 \right) + \left(k_n + \tau_g \right) \left[\left(k_n + \tau_g \right) \left(k_n^2 + \tau_g^2 \right) + 2k_n \left(k_g^{'} + k_g^2 \right) + 2\tau_g \left(k_g^{'} - k_g^2 \right) \right] + \left(k_n^{'} + \tau_g^{'} \right) \left[k_n^{'} + \tau_g^{'} - 2k_g \left(k_n - \tau_g \right) \right] + k_g^2 \left(k_n - \tau_g \right)^2}.$$

The unit normal vector of surface M and unit vector $\mathbf{g}$ of β are as follow. Then, from (2.1) we obtain

$$\mathbf{g}^* = \frac{1}{\sqrt{\varsigma}\varepsilon} \left\{ \left(\sqrt{\varsigma} \cos\varphi^* \Gamma_1 + \sin\varphi^* \mu_1 \right) \mathbf{T} + \left(\sqrt{\varsigma} \cos\varphi^* \Gamma_2 + \sin\varphi^* \mu_2 \right) \mathbf{g} + \left(\sqrt{\varsigma} \cos\varphi^* \Gamma_3 + \sin\varphi^* \mu_3 \right) \mathbf{n} \right\},$$

and

$$\mathbf{n}^* = \frac{1}{\sqrt{\varsigma}\varepsilon} \left\{ \left(\mu_1 \cos\varphi^* - \sqrt{\varsigma} \sin\varphi^* \Gamma_1 \right) \mathbf{T} + \left(\mu_2 \cos\varphi^* - \sqrt{\varsigma} \sin\varphi^* \Gamma_2 \right) \mathbf{g} + \left(\mu_3 \cos\varphi^* - \sqrt{\varsigma} \sin\varphi^* \Gamma_3 \right) \mathbf{n} \right\}.$$

where $\varepsilon = \sqrt{\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2}$, $\varsigma = 2k_g^2 + \left(k_n + \tau_g \right)^2$ and φ^* is the angle between the vectors $\mathbf{g}^*$ and $\mathbf{n}^*$. Now, we can calculate geodesic curvature, normal curvature, geodesic torsion of curve , so from (2.3) we get

$$k_g^* = \frac{\sqrt{2 \left(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 \right)}}{\left(2k_g^2 + \left(k_n + \tau_g \right)^2 \right)^2} \cos\varphi^*,$$

and

$$k_n^* = \frac{\sqrt{2 \left(\Gamma_1^2 + \Gamma_2^2 + \Gamma_3^2 \right)}}{\left(2k_g^2 + \left(k_n + \tau_g \right)^2 \right)^2} \sin\varphi^*,$$

and

$$\tau_g^* = \frac{-1}{\sqrt{2}} \frac{-1}{\sqrt{2}} \frac{\left(\eta_1 + \eta_2 \right) \left[k_g \left(k_n^{'} + \tau_g^{'} \right) - k_g^{'} \left(k_n + \tau_g \right) \right] + \left(2k_g^2 + \left(k_n + \tau_g \right)^2 \right) \left[\tau_g \eta_1 - k_n \eta_2 + k_g \eta_3 \right]}{2 \left(k_g^{'2} + k_g^4 \right) + \left(k_n + \tau_g \right) \left[\left(k_n + \tau_g \right) \left(k_n^2 + \tau_g^2 \right) + 2k_n \left(k_g^{'} + k_g^2 \right) + 2\tau_g \left(k_g^{'} - k_g^2 \right) \right] + \left(k_n^{'} + \tau_g^{'} \right) \left[k_n^{'} + \tau_g^{'} - 2k_g \left(k_n - \tau_g \right) \right] + k_g^2 \left(k_n - \tau_g \right)^2} + \frac{d\varphi^*}{ds^*}.$$

Corollary 1 *Consider that $\alpha(s)$ is a geodesic curve, then the following equation holds,*

Consider that $\alpha(s)$ is a geodesic curve, then the following equation holds,

i) $\kappa^* = \frac{\sqrt{2\left(k_n^2+\tau_g^2\right)}}{(k_n+\tau_g)}$,

ii) $\tau^* = \frac{-1}{\sqrt{2}} \frac{(k_n+\tau_g)^2+(k_n+\tau_g)\left(k_n' \tau_g-k_n\tau_g'\right)}{(k_n+\tau_g)^3\left(k_n^2+\tau_g^2\right)+\left(k_n'+\tau_g'\right)^2}$,

iii) $k_g^* = \frac{\sqrt{2\left(k_n^2+\tau_g^2\right)}}{(k_n+\tau_g)}\cos\varphi^*$,

iv) $k_n^* = \frac{\sqrt{2\left(k_n^2+\tau_g^2\right)}}{(k_n+\tau_g)}\sin\varphi^*$,

v) $\tau_g^* = \frac{-1}{\sqrt{2}} \frac{(k_n+\tau_g)^2+(k_n+\tau_g)\left(k_n' \tau_g-k_n\tau_g'\right)}{(k_n+\tau_g)^3\left(k_n^2+\tau_g^2\right)+\left(k_n'+\tau_g'\right)^2} + \frac{d\varphi^*}{ds^*}$.

3.2 Tn− Smarandache Curves

Definition: Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha = \alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$ $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$.

Tn- Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{T}+\mathbf{n}). \tag{3.6}$$

Now, we can investigate Darboux invariants of **Tn** − Smarandache curve according to $\alpha = \alpha(s)$. Differentiating (3.6) with respect to s, we get

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left(k_n\mathbf{T}+(\tau_g - k_g)\,\mathbf{g} - k_n\mathbf{n}\right), \tag{3.7}$$

and

$$\mathbf{T}^*\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left(k_n\mathbf{T}+(\tau_g - k_g)\,\mathbf{g} - k_n\mathbf{n}\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2k_n^2 + (\tau_g - k_g)^2}{2}}. \tag{3.8}$$

The tangent vector of curve β can be written as follow,

$$\mathbf{T}^* = \frac{-1}{\sqrt{2k_n^2 + (\tau_g - k_g)^2}}\left(k_n\mathbf{T}+(\tau_g - k_g)\,\mathbf{g} - k_n\mathbf{n}\right). \tag{3.9}$$

Differentiating (3.9) with respect to s, we obtain

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\left(2k_n^2 + (\tau_g - k_g)^2\right)^{\frac{3}{2}}}\left(\gamma_1\mathbf{T}+\gamma_2\mathbf{g} + \gamma_3\mathbf{n}\right) \tag{3.10}$$

where

$$\begin{cases} \gamma_1 = (\tau_g - k_g)\left\{k_n\left(-k_g^{'} + \tau_g^{'} + k_n k_g - \tau_g k_n\right) - k_n^{'}(\tau_g - k_g) + k_g\left[2k_n^2 + \tau_g - k_g\right]\right\} - 2k_n^4 \\ \gamma_2 = k_n(\tau_g - k_g)\left[2k_n^{'} + \tau_g\left(k_g^2 - \tau_g^2\right)\right] - 2k_n^2\left(k_n k_g + \tau_g^{'} - k_g^{'} - k_n\tau_g\right) \\ \gamma_3 = (\tau_g - k_g)\left\{-k_n\left(-k_g^{'} + \tau_g^{'} - k_n k_g + \tau_g k_n\right) + k_n^{'}\left(\tau_g - k_g\right) - \tau_g\left[2k_n^2 + \tau_g - k_g\right]\right\} - 2k_n^4. \end{cases}$$

Substituting (3.8) in (3.10), we get

$$\dot{\mathbf{T}}^* = \frac{\sqrt{2}}{\left(2k_n^2 + (\tau_g - k_g)^2\right)^2}\left(\gamma_1\mathbf{T} + \gamma_2\mathbf{g} + \gamma_3\mathbf{n}\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa^* = \left\|\dot{\mathbf{T}}^*\right\| = \frac{\sqrt{2\ \left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right)}}{\left(2k_n^2 + (\tau_g - k_g)^2\right)^2}$$

and

$$\mathbf{N}^* = \frac{1}{\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}\left(\gamma_1\mathbf{T} + \gamma_2\mathbf{g} + \gamma_3\mathbf{n}\right).$$

On the other hand, we express

$$\mathbf{B}^* = \mathbf{T}^* \times \mathbf{N}^* = \frac{1}{\sqrt{2k_n^2 + (\tau_g - k_g)^2}\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}\begin{vmatrix} \mathbf{T} & \mathbf{g} & \mathbf{n} \\ -k_n & k_g - \tau_g & k_n \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$\mathbf{B}^* = \frac{1}{\sqrt{2k_n^2 + (\tau_g - k_g)^2}\sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}}\left(\nu_1\mathbf{T} + \nu_2\mathbf{g} + \nu_3\mathbf{n}\right)$$

where

$$\begin{cases} \nu_1 = (k_g - \tau_g)\,\gamma_3 - k_n\gamma_2 \\ \nu_2 = k_n\gamma_1 + k_n\gamma_3 \\ \nu_3 = -k_n\gamma_2 + (\tau_g - k_g)\gamma_1. \end{cases}$$

We differentiate (3.7) with respect to s in order to calculate the torsion

$$\beta^{''} = \frac{-1}{\sqrt{2}}\left\{\left(k_g^{'} + k_n^2 - k_g\left(\tau_g - k_g\right)\right)\mathbf{T} + \left(k_n k_g + \left(\tau_g^{'} - k_g^{'}\right) + k_n\tau_g\right)\mathbf{g} + \left(k_n^2 + \left(\tau_g - k_g\right)\tau_g - k_n^{'}\right)\mathbf{n}\right\},$$

and similarly

$$\beta^{'''} = \frac{-1}{\sqrt{2}}\left(\omega_1\mathbf{T} + \omega_2\mathbf{g} + \omega_3\mathbf{n}\right),$$

where

$$\begin{cases} \omega_1 = k_n^{''} - 2k_g\left(\tau_g^{'} - k_g^{'}\right) + k_n\left(3k_n^{'} - \tau_g^2 - k_n^2 - k_g^2\right) + k_g^{'}\left(k_g - \tau_g\right) \\ \omega_2 = k_g^{''} - \tau_g^{''} + k_n\left(k_g^{'} + \tau_g^{'}\right) + \left(k_g - \tau_g\right)\left(\tau_g^2 + k_n^2 + k_g^2\right) + \tau_g\left(2k_n^{'} - k_n^2\right) + 2k_g k_n^{'} \\ \omega_3 = -k_n^{''} + 2\tau_g\left(\tau_g^{'} - k_g^{'}\right) + k_n\left(3k_n^{'} + \tau_g^2 + \tau_g\left(k_g + \tau_g\right)\right) + \left(k_g - \tau_g\right)\left(k_n k_g - \tau_g^{'}\right). \end{cases}$$

The torsion of curve β is

$$\tau^* = \frac{-1}{\sqrt{2}} \frac{\begin{array}{c}\left(\omega_1 + \omega_3\right)\left[k_n\left(\tau_g^{'} - k_g^{'}\right) - k_n^{'}\left(\tau_g - k_g\right) + k_n^2\left(\tau_g + k_g\right)\right] + \left(\tau_g - k_g\right)^2\left(\tau_g\omega_1 + k_g\omega_3\right) + \\ k_n\left(\tau_g - k_g\right)\left[k_n\left(\omega_1 - \omega_3\right) + \omega_2\left(k_g - \tau_g\right)\right] - 2k_n\left(k_n^{'} + k_n^2\right)\omega_2\end{array}}{\begin{array}{c}2\left(k_n^{'2} + k_n^4\right) + \left(\tau_g - k_g\right)\left[2\left(k_n^2 - k_n^{'}\right) - 2\left(k_n^2 - k_n^{'}\right)k_g\right] + \left(\tau_g - k_g\right)^2\left(1 + k_g^2\right) + \\ \left(\tau_g - k_g^{'}\right)\left[\left(\tau_g^{'} - k_g^{'}\right) + 2\left(k_n k_g + k_n\tau_g\right)\right] + k_n^2\left(k_g + \tau_g\right)^2\end{array}}.$$

The unit normal vector of surface M and unit vector $\mathbf{g}$ of β are as follow. Then, from (2.1) we obtain

$$\overrightarrow{g}^* = \frac{1}{\sqrt{\psi}\pi}\left\{\left(\sqrt{\psi}\cos\varphi^*\gamma_1 + \sin\varphi^*\nu_1\right)\mathbf{T} + \left(\sqrt{\psi}\cos\varphi^*\gamma_2 + \sin\varphi^*\nu_2\right)\mathbf{g} + \left(\sqrt{\psi}\cos\varphi^*\gamma_3 + \sin\varphi^*\nu_3\right)\mathbf{n}\right\},$$

and

$$\overrightarrow{N}^* = \frac{1}{\sqrt{\psi}\pi}\left\{\left(\nu_1\cos\varphi^* - \sqrt{\psi}\sin\varphi^*\gamma_1\right)\overrightarrow{t} + \left(\nu_2\cos\varphi^* - \sqrt{\psi}\sin\varphi^*\gamma_2\right)\overrightarrow{g} + \left(\nu_3\cos\varphi^* - \sqrt{\psi}\sin\varphi^*\gamma_3\right)\overrightarrow{N}\right\}.$$

where $\pi = \sqrt{\gamma_1^2 + \gamma_2^2 + \gamma_3^2}$, $\psi = 2k_n^2 + \left(\tau_g - k_g\right)^2$ and φ^* is the angle between the vectors $\mathbf{g}^*$ and $\mathbf{n}^*$.Now, we can calculate geodesic curvature, normal curvature, geodesic torsion of curve , so from (2.3) we get

$$k_g^* = \frac{\sqrt{2\left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right)}}{\left(2k_n^2 + \left(\tau_g - k_g\right)^2\right)^2}\cos\varphi^*,$$

and

$$k_n^* = \frac{\sqrt{2\left(\gamma_1^2 + \gamma_2^2 + \gamma_3^2\right)}}{\left(2k_n^2 + \left(\tau_g - k_g\right)^2\right)^2}\sin\varphi^*,$$

and

$$\tau_g^* = \frac{-1}{\sqrt{2}} \frac{\begin{array}{c}\left(\omega_1 + \omega_3\right)\left[k_n\left(\tau_g^{'} - k_g^{'}\right) - k_n^{'}\left(\tau_g - k_g\right) + k_n^2\left(\tau_g + k_g\right)\right] + \left(\tau_g - k_g\right)^2\left(\tau_g\omega_1 + k_g\omega_3\right) + \\ k_n\left(\tau_g - k_g\right)\left[k_n\left(\omega_1 - \omega_3\right) + \omega_2\left(k_g - \tau_g\right)\right] - 2k_n\left(k_n^{'} + k_n^2\right)\omega_2\end{array}}{\begin{array}{c}2\left(k_n^{'2} + k_n^4\right) + \left(\tau_g - k_g\right)\left[2\left(k_n^2 - k_n^{'}\right) - 2\left(k_n^2 - k_n^{'}\right)k_g\right] + \left(\tau_g - k_g\right)^2\left(1 + k_g^2\right) + \\ \left(\tau_g - k_g^{'}\right)\left[\left(\tau_g^{'} - k_g^{'}\right) + 2\left(k_n k_g + k_n\tau_g\right)\right] + k_n^2\left(k_g + \tau_g\right)^2\end{array}} + \frac{d\varphi^*}{ds^*}.$$

Corollary 2 *Consider that $\alpha(s)$ is a geodesic curve, then the following equation holds,*

Consider that $\alpha(s)$ is an asymptotic line, then the following equation holds,

i) $\kappa^* = \frac{\sqrt{2\left(k_g^2+\tau_g^2\right)}}{(\tau_g-k_g)^2}$,

ii) $\tau^* = \frac{-1}{\sqrt{2}}\frac{(k_g-\tau_g)\left(k_g'\tau_g-k_g\tau_g'\right)}{\left(1+k_g^2\right)\left(\tau_g-k_g'\right)\left(\tau_g'-k_g'\right)(\tau_g-k_g)^{-2}}$,

iii) $k_g^* = \frac{\sqrt{2\left(k_g^2+\tau_g^2\right)}}{(\tau_g-k_g)^2}\cos\varphi^*$,

iv) $k_n^* = \frac{\sqrt{2\left(k_g^2+\tau_g^2\right)}}{(\tau_g-k_g)^2}\sin\varphi^*$,

v) $\tau_g^* = \frac{-1}{\sqrt{2}}\frac{(k_g-\tau_g)\left(k_g'\tau_g-k_g\tau_g'\right)}{\left(1+k_g^2\right)\left(\tau_g-k_g'\right)\left(\tau_g'-k_g'\right)(\tau_g-k_g)^{-2}}+\frac{d\varphi^*}{ds^*}$.

3.3 gn− Smarandache Curves

Definition: Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha=\alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$ $\{\mathbf{T},\mathbf{g},\mathbf{n}\}$.

gn - Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{2}}(\mathbf{g}+\mathbf{n}). \tag{3.11}$$

Now, we can investigate Darboux invariants of **gn** − Smarandache curve according to $\alpha=\alpha(s)$. Differentiating (3.11) with respect to s, we get

$$\beta' = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left((k_n+k_g)\mathbf{T}+\tau_g\mathbf{g}-\tau_g\mathbf{n}\right), \tag{3.12}$$

and

$$\mathbf{T}^*\frac{ds^*}{ds} = \frac{-1}{\sqrt{2}}\left((k_n+k_g)\mathbf{T}+\tau_g\mathbf{g}-\tau_g\mathbf{n}\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{2\tau_g^2+(k_n+k_g)^2}{2}}. \tag{3.13}$$

The tangent vector of curve β can be written as follow,

$$\mathbf{T}^* = \frac{-1}{\sqrt{2\tau_g^2+(k_n+k_g)^2}}\left((k_n+k_g)\mathbf{T}+\tau_g\mathbf{g}-\tau_g\mathbf{n}\right). \tag{3.14}$$

Differentiating (3.14) with respect to s, we obtain

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\left(2\tau_g^2+(k_n+k_g)^2\right)^{\frac{3}{2}}}\left(\lambda_1\mathbf{T}+\lambda_2\mathbf{g}+\lambda_3\mathbf{n}\right) \tag{3.15}$$

where

$$\begin{cases} \lambda_1 = 2\tau_g\tau_g^{'}\left(k_n+k_g\right)-2\tau_g^2\left(k_n^{'}+k_g^{'}\right)+\tau_g\left(k_g-k_n\right)\left(2\tau_g^2+\left(k_n+k_g\right)^2\right) \\ \lambda_2 = -2\tau_g^4+\tau_g\left(k_n+k_g\right)\left[\left(k_n^{'}+k_g^{'}\right)-2\tau_g k_g\right]-\left(k_n+k_g\right)^2\left(\left(k_n+k_g\right)k_g+\tau_g^{'}+\tau_g^2\right) \\ \lambda_3 = -2\tau_g^4+\tau_g\left(k_n+k_g\right)\left[\left(k_n^{'}+k_g^{'}\right)-2\tau_g k_n\right]+\left(k_n+k_g\right)^2\left(-\left(k_n+k_g\right)k_n+\tau_g^{'}-\tau_g^2\right) \end{cases}$$

Substituting (3.13) in (3.15), we get

$$\dot{\mathbf{T}}^* = \frac{\sqrt{2}}{\left(2\tau_g^2+\left(k_n+k_g\right)^2\right)^2}\left(\lambda_1\mathbf{T}+\lambda_2\mathbf{g}+\lambda_3\mathbf{n}\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa^* = \left\|\dot{\mathbf{T}}^*\right\| = \frac{\sqrt{2\ \left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)}}{\left(2\tau_g^2+\left(k_n+k_g\right)^2\right)^2}$$

and

$$\mathbf{N}^* = \frac{1}{\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\lambda_1\mathbf{T}+\lambda_2\mathbf{g}+\lambda_3\mathbf{n}\right).$$

On the other hand, we express

$$\mathbf{B}^* = \mathbf{T}^*\times\mathbf{N}^* = \frac{1}{\sqrt{2k_n^2+\left(\tau_g-k_g\right)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\begin{vmatrix} \mathbf{T} & \mathbf{g} & \mathbf{n} \\ -(k_n+k_g) & -\tau_g & \tau_g \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$\mathbf{B}^* = \frac{1}{\sqrt{2\tau_g^2+\left(k_n+k_g\right)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\rho_1\mathbf{T}+\rho_2\mathbf{g}+\rho_3\mathbf{n}\right)$$

where

$$\begin{cases} \rho_1 = -\tau_g\lambda_3-\tau_g\lambda_2 \\ \rho_2 = \tau_g\lambda_1+\left(k_n+k_g\right)\lambda_3 \\ \rho_3 = -\left(k_n+k_g\right)\lambda_2+\tau_g\lambda_1. \end{cases}$$

We differentiate (3.7) with respect to s in order to calculate the torsion

$$\beta^{''} = \frac{-1}{\sqrt{2}}\left\{\left(k_g^{'}+k_n^{'}+\tau_g\left(k_n-k_g\right)\right)\mathbf{T}+\left(k_g(k_n+k_g)+\tau_g^{'}+\tau_g^2\right)\mathbf{g}+\left(k_n(k_n+k_g)-\tau_g^{'}+\tau_g^2\right)\mathbf{n}\right\},$$

and similarly

$$\beta^{'''} = \frac{-1}{\sqrt{2}}\left(\chi_1\mathbf{T}+\chi_2\mathbf{g}+\chi_3\mathbf{n}\right),$$

where

$$\begin{cases} \chi_1 = k_n^{''} + k_g^{''} - 2\dot{\tau_g}\left(k_n - k_g\right) + \tau_g\left(k_n^{'} - k_g^{'}\right) - \left(k_n + k_g\right)\left(\tau_g^2 + k_n^2 + k_g^2\right) \\ \chi_2 = \tau_g^{''} + 2k_g\left(k_n^{'} + k_g^{'}\right) + 3\tau_g\tau_g^{'} + \left(k_n + k_g\right)\left(k_g^{'} - \tau_g k_n\right) + k_g\tau_g\left(k_n - k_g\right) - \tau_g^3 \\ \chi_3 = \tau_g^{''} + 2k_n\left(k_n^{'} + k_g^{'}\right) + 3\tau_g\tau_g^{'} + \left(k_n + k_g\right)\left(k_g^{'} + \tau_g k_g\right) + k_n\tau_g\left(k_n - k_g\right) + \tau_g^3. \end{cases}$$

The torsion of curve β is

$$\tau^* = \frac{-1}{\sqrt{2}} \frac{\begin{array}{c}\left(k_n + k_g\right)^2 [\left(\left(\tau_g^{'} + \tau_g\right)^2 + k_g\left(k_n + k_g\right)\right)\chi_3 - \left(\left(\tau_g^{'} - \tau_g\right)^2 - k_n\left(k_n + k_g\right)\right)\chi_2 + \\ \tau_g\left(k_n + k_g\right)\chi_1] - \tau_g\left(\chi_2 + \chi_3\right)\left[\left(k_n^{'} + k_g^{'}\right) + \tau_g\left(k_n - k_g\right)\right]^2\end{array}}{\begin{array}{c}\left(k_n + k_g\right)^2\left(k_n^2 + k_g^2\right) + 2\left(k_n + k_g\right)\left[k_g\left(\tau_g^{'} + \tau_g^2\right) + 2k_n\left(\tau_g^2 - \tau_g^{'}\right)\right] + \\ 2{\tau_g^{'}}^2 + 2\tau_g^4 + \left[\left(k_n^{'} + k_g^{'}\right) + \tau_g\left(k_n - k_g\right)\right]^2\end{array}}.$$

The unit normal vector of surface M and unit vector $\mathbf{g}$ of β are as follow. Then, from (2.1) we obtain

$$\mathbf{g}^* = \frac{1}{\sqrt{\Delta}\Omega}\left\{\left(\sqrt{\Delta}\cos\varphi^*\lambda_1 + \sin\varphi^*\rho_1\right)\mathbf{T} + \left(\sqrt{\Delta}\cos\varphi^*\lambda_2 + \sin\varphi^*\rho_2\right)\mathbf{g} + \left(\sqrt{\Delta}\cos\varphi^*\lambda_3 + \sin\varphi^*\rho_3\right)\mathbf{n}\right\},$$

and

$$\mathbf{n}^* = \frac{1}{\sqrt{\Delta}\Omega}\left\{\left(\rho_1\cos\varphi^* - \sqrt{\Delta}\sin\varphi^*\lambda_1\right)\mathbf{T} + \left(\rho_2\cos\varphi^* - \sqrt{\Delta}\sin\varphi^*\lambda_2\right)\mathbf{g} + \left(\rho_3\cos\varphi^* - \sqrt{\Delta}\sin\varphi^*\lambda_3\right)\mathbf{n}\right\}.$$

where $\Omega = \sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}$, $\Delta = 2\tau_g^2 + \left(k_n + k_g\right)^2$ and φ^* is the angle between the vectors $\mathbf{g}^*$ and $\mathbf{n}^*$. Now, we can calculate geodesic curvature, normal curvature, geodesic torsion of curve , so from (2.3) we get

$$k_g^* = \frac{\sqrt{2\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)}}{\left(2\tau_g^2 + \left(k_n + k_g\right)^2\right)^2}\cos\varphi^*,$$

and

$$k_n^* = \frac{\sqrt{2\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)}}{\left(2\tau_g^2 + \left(k_n + k_g\right)^2\right)^2}\sin\varphi^*,$$

and

$$\tau^* = \frac{-1}{\sqrt{2}} \frac{\begin{array}{c}\left(k_n + k_g\right)^2 [\left(\left(\tau_g^{'} + \tau_g\right)^2 + k_g\left(k_n + k_g\right)\right)\chi_3 - \left(\left(\tau_g^{'} - \tau_g\right)^2 - k_n\left(k_n + k_g\right)\right)\chi_2 + \\ \tau_g\left(k_n + k_g\right)\chi_1] - \tau_g\left(\chi_2 + \chi_3\right)\left[\left(k_n^{'} + k_g^{'}\right) + \tau_g\left(k_n - k_g\right)\right]^2\end{array}}{\begin{array}{c}\left(k_n + k_g\right)^2\left(k_n^2 + k_g^2\right) + 2\left(k_n + k_g\right)\left[k_g\left(\tau_g^{'} + \tau_g^2\right) + 2k_n\left(\tau_g^2 - \tau_g^{'}\right)\right] + \\ 2{\tau_g^{'}}^2 + 2\tau_g^4 + \left[\left(k_n^{'} + k_g^{'}\right) + \tau_g\left(k_n - k_g\right)\right]^2\end{array}} + \frac{d\varphi^*}{ds^*}$$

Corollary 3 *Consider that $\alpha(s)$ is a geodesic curve, then the following equation holds,*

Consider that $\alpha(s)$ is a principal line, then the following equation holds,

i) $\kappa^* = \frac{\sqrt{2\left(k_n^2+\tau_g^2\right)}}{(k_n+\tau_g)}$,

ii) $\tau^* = \frac{-1}{\sqrt{2}} \frac{(k_n+\tau_g)^2+\left(k_g k_n^{'} - k_n k_g^{'}\right)}{\left(k_n^2+k_g^2\right)+\left(k_n^{'}+\tau_g^{'}\right)^2 (k_n+k_g)^{-2}}$,

iii) $k_g^* = \frac{\sqrt{2\left(k_n^2+\tau_g^2\right)}}{(k_n+\tau_g)} \cos\varphi^*$,

iv) $k_n^* = \frac{\sqrt{2\left(k_n^2+\tau_g^2\right)}}{(k_n+\tau_g)} \sin\varphi^*$,

v) $\tau_g^* = \frac{-1}{\sqrt{2}} \frac{(k_n+\tau_g)^2+\left(k_g k_n^{'} - k_n k_g^{'}\right)}{\left(k_n^2+k_g^2\right)+\left(k_n^{'}+\tau_g^{'}\right)^2 (k_n+k_g)^{-2}} + \frac{d\varphi^*}{ds^*}$.

3.4 Tgn− Smarandache Curves

Definition 1 *Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha = \alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$ $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$.*

Let M be an oriented surface in E^3 and let consider the arc - length parameter curve $\alpha = \alpha(s)$ lying fully on M. Denote the Darboux frame of $\alpha(s)$ $\{\mathbf{T}, \mathbf{g}, \mathbf{n}\}$.

Tgn- Smarandache curve can be defined by

$$\beta(s^*) = \frac{1}{\sqrt{3}} (\mathbf{T} + \mathbf{g} + \mathbf{n}). \tag{3.16}$$

Now, we can investigate Darboux invariants of **Tgn** − Smarandache curve according to $\alpha = \alpha(s)$. Differentiating (3.16) with respect to s, we get

$$\beta^{'} = \frac{d\beta}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\sqrt{3}} \left((k_n + k_g)\mathbf{T} + (\tau_g - k_g)\mathbf{g} - (\tau_g + k_n)\mathbf{n}\right), \tag{3.17}$$

and

$$\mathbf{T}^* \frac{ds^*}{ds} = \frac{-1}{\sqrt{3}} \left((k_n + k_g)\mathbf{T} + (\tau_g - k_g)\mathbf{g} - (\tau_g + k_n)\mathbf{n}\right)$$

where

$$\frac{ds^*}{ds} = \sqrt{\frac{(k_n + k_g)^2 + (\tau_g - k_g)^2 + (\tau_g + k_n)^2}{3}}. \tag{3.18}$$

The tangent vector of curve β can be written as follow,

$$\mathbf{T}^* = \frac{-1}{\sqrt{(k_n + k_g)^2 + (\tau_g - k_g)^2 + (\tau_g + k_n)^2}} \left((k_n + k_g)\mathbf{T} + (\tau_g - k_g)\mathbf{g} - (\tau_g + k_n)\mathbf{n}\right). \tag{3.19}$$

Differentiating (3.19) with respect to s, we obtain

$$\frac{d\mathbf{T}^*}{ds^*}\frac{ds^*}{ds} = \frac{-1}{\left((k_n+k_g)^2+(\tau_g-k_g)^2+(\tau_g+k_n)^2\right)^{\frac{3}{2}}}\left(\delta_1\mathbf{T}+\delta_2\mathbf{g}+\delta_3\mathbf{n}\right) \quad (3.20)$$

where

$$\begin{cases} \delta_1 = (k_n+k_g)^2\left[k_g(\tau_g-k_g)-k_n(\tau_g+k_n)\right]+(k_n+k_g)\left[(\tau_g-k_g)\left(\tau_g'-k_g'\right)+(\tau_g+k_g)\left(\tau_g'+k_n'\right)\right]+ \\ \left((\tau_g-k_g)^2+(\tau_g+k_n)^2\right)\left[k_g(\tau_g-k_g)-\left(k_g'+k_n'\right)-k_n(\tau_g+k_n)\right] \\ \\ \delta_2 = (\tau_g-k_g)^2\left[-k_g(k_n+k_g)-\tau_g(\tau_g+k_n)\right]+(\tau_g-k_g)\left[(k_g+k_n)\left(k_g'+k_n'\right)+(\tau_g+k_n)\left(\tau_g'+k_n'\right)\right]+ \\ \left((k_g+k_n)^2+(\tau_g+k_n)^2\right)\left[-k_g(k_g+k_n)+\left(k_g'-\tau_g'\right)-\tau_g(\tau_g+k_n)\right] \\ \\ \delta_3 = (\tau_g+k_n)^2\left[\tau_g(k_g-\tau_g)-k_n(k_g+k_n)\right]+(\tau_g+k_n)\left[-(k_g+k_n)\left(k_g'+k_n'\right)-(\tau_g-k_g)\left(\tau_g'-k_g'\right)\right]+ \\ \left((k_g+k_n)^2+(\tau_g-k_g)^2\right)\left[\tau_g(k_g-\tau_g)+\left(\tau_g'+k_n'\right)-k_n(k_g+k_n)\right] \end{cases}$$

Substituting (3.18) in (3.20), we get

$$\dot{\mathbf{T}}^* = \frac{\sqrt{3}}{\left((k_n+k_g)^2+(\tau_g-k_g)^2+(\tau_g+k_n)^2\right)^2}\left(\delta_1\mathbf{T}+\delta_2\mathbf{g}+\delta_3\mathbf{n}\right).$$

Then, the curvature and principal normal vector field of curve β are respectively,

$$\kappa^* = \left\|\dot{\mathbf{T}}^*\right\| = \frac{\sqrt{2\ \left(\delta_1^2+\delta_2^2+\delta_3^2\right)}}{\left((k_n+k_g)^2+(\tau_g-k_g)^2+(\tau_g+k_n)^2\right)^2}$$

and

$$\mathbf{N}^* = \frac{1}{\sqrt{\left(\delta_1^2+\delta_2^2+\delta_3^2\right)}}\left(\delta_1\mathbf{T}+\delta_2\mathbf{g}+\delta_3\mathbf{n}\right).$$

On the other hand, we express

$$\mathbf{B}^* = \mathbf{T}^*\times\mathbf{N}^* = \frac{1}{\sqrt{(k_n+k_g)^2+(\tau_g-k_g)^2+(\tau_g+k_n)^2}\sqrt{\left(\delta_1^2+\delta_2^2+\delta_3^2\right)}}\begin{vmatrix} \mathbf{T} & \mathbf{g} & \mathbf{n} \\ -(k_n+k_g) & k_g-\tau_g & \tau_g+k_n \\ \delta_1 & \delta_2 & \delta_3 \end{vmatrix}.$$

So, the binormal vector of curve β is

$$\mathbf{B}^* = \frac{1}{\sqrt{(k_n+k_g)^2+(\tau_g-k_g)^2+(\tau_g+k_n)^2}\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2}}\left(\sigma_1\mathbf{T}+\sigma_2\mathbf{g}+\sigma_3\mathbf{n}\right)$$

where

$$\begin{cases} \sigma_1 = (k_g - \tau_g)\,\delta_3 - (\tau_g + k_n)\,\delta_2 \\ \sigma_2 = (\tau_g + k_n)\,\delta_1 + (k_n + k_g)\delta_3 \\ \sigma_3 = -(k_n + k_g)\,\delta_2 - (k_g - \tau_g)\,\delta_1. \end{cases}$$

We differentiate (3.17) with respect to s in order to calculate the torsion

$$\beta'' = \frac{-1}{\sqrt{3}} \left\{ \begin{array}{l} \left(k_g' + k_n' + k_g\,(k_g - \tau_g) + k_n\,(\tau_g + k_n)\right)\mathbf{T} + \left(k_g(k_n + k_g) + \left(\tau_g' - k_g'\right) + \tau_g\,(\tau_g + k_n)\right)\mathbf{g} + \\ \left(k_n(k_n + k_g) + \tau_g\,(\tau_g - k_g) - \left(\tau_g' + k_n'\right)\right)\mathbf{n} \end{array} \right\},$$

and similarly

$$\beta''' = \frac{-1}{\sqrt{3}}\,(\xi_1\mathbf{T} + \xi_2\mathbf{g} + \xi_3\mathbf{n}),$$

where

$$\begin{cases} \xi_1 = k_n'' + k_g'' - 2k_g'\left(k_g' - \tau_g'\right) - (k_n + k_g)\left(k_n^2 + k_g^2\right) + (k_g - \tau_g)\left(k_g' + k_n\tau_g\right) + \\ 2k_n\left(k_n' + \tau_g'\right) + (k_n + \tau_g)\left(k_n' - k_g\tau_g\right) \\ \\ \xi_2 = \tau_g'' - k_g'' + 2\tau_g\left(k_n' + \tau_g'\right) + 2k_g\left(k_n' + k_g'\right) + (k_n + k_g)\left(k_g' - k_n\tau_g\right) + \\ (\tau_g + k_n)\left(k_nk_g + \tau_g'\right) + (k_g - \tau_g)\left(k_g^2 + \tau_g^2\right) \\ \\ \xi_3 = -\left(\tau_g'' + k_n''\right) + 2k_n\left(k_n' + k_g'\right) + (\tau_g - k_g)\left(\tau_g' - k_nk_g\right) + (k_n + k_g)\left(k_n' + k_g\tau_g\right) + \\ 2\tau_g\left(\tau_g' - k_g'\right) + (k_n + \tau_g)\left(k_n^2 + \tau_g^2\right). \end{cases}$$

The torsion of curve β is

$$\tau^* = \frac{-1}{\sqrt{3}} \frac{\begin{array}{l} (k_n + k_g)(k_g - \tau_g)\left(\tau_g\xi_2 - k_n\xi_1\right) + (k_g - \tau_g)^2\left(\tau_g\xi_1 + k_g\xi_3\right) + \\ \left(k_n' + \tau_g'\right)\left[(k_g - \tau_g)\,\xi_1 + (k_n + k_g)\,\xi_2\right] - \\ \left(k_n' + k_g'\right)\left[\left(k_n' + \tau_g\right)\xi_1 + (k_g - \tau_g)\,\xi_3 - (k_n + \tau_g)\,\xi_2\right] + \\ (k_n + \tau_g)^2\left(\tau_g\xi_1 - k_n\xi_2\right) + \\ (k_n + \tau_g)(k_n + k_g)\left(\tau_g\xi_3 + k_g\xi_1\right) + (k_n + \tau_g)^2\left(k_g\xi_3 - k_n\xi_2\right) + \\ (k_g - \tau_g)(k_n + \tau_g)(k_n\xi_3 - k_g\xi_2) + (k_n + k_g)\left(\tau_g' - k_g'\right) \end{array}}{\begin{array}{l} (k_g - \tau_g)^2\left(k_g^2 + \tau_g^2\right) + 2\left(k_n' + k_g'\right)\left[k_g\,(k_g - \tau_g) + k_n\,(k_n + \tau_g)\right] + \\ (k_n + \tau_g)^2\left(k_n^2 + \tau_g^2\right) + 2\,(k_n + \tau_g)\left[\tau_g\left(\tau_g' - k_g'\right) + k_g^2\,(k_n + \tau_g)\right] + \\ (k_n + k_g)^2\left(k_n^2 + \tau_g^2\right) + 2\,(k_n + k_g)\left[k_g\left(\tau_g' - k_g'\right) - k_n\left(\tau_g' + k_g'\right)\right] + \\ k_n\tau_g\,(\tau_g - k_g) + \left(k_n' + k_g'\right)^2 + \left(k_n^2 + \tau_g^2\right) + \left(\tau_g' - k_g'\right)^2 + \left(\tau_g' - k_n'\right)^2 + \\ -2\tau_g\,(\tau_g - k_g)\left(\tau_g' + k_n'\right). \end{array}}.$$

The unit normal vector of surface M and unit vector $\mathbf{g}$ of β are as follow. Then, from (2.1) we obtain

$$\mathbf{g}^* = \frac{1}{\sqrt{\Phi\Lambda}}\left\{\left(\sqrt{\Lambda}\cos\varphi^*\delta_1+\sin\varphi^*\sigma_1\right)\mathbf{T}+\left(\sqrt{\Lambda}\cos\varphi^*\delta_2+\sin\varphi^*\sigma_2\right)\mathbf{g}+\left(\sqrt{\Lambda}\cos\varphi^*\delta_3+\sin\varphi^*\sigma_3\right)\mathbf{n}\right\},$$

and

$$\mathbf{n}^* = \frac{1}{\sqrt{\Phi\Lambda}}\left\{\left(\sigma_1\cos\varphi^*-\sqrt{\Lambda}\sin\varphi^*\delta_1\right)\mathbf{T}+\left(\sigma_2\cos\varphi^*-\sqrt{\Lambda}\sin\varphi^*\delta_2\right)\mathbf{g}+\left(\sigma_3\cos\varphi^*-\sqrt{\Lambda}\sin\varphi^*\delta_3\right)\mathbf{n}\right\}.$$

where $\Phi=\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}$, $\Lambda=(k_n+k_g)^2+(\tau_g-k_g)^2+(\tau_g+k_n)^2$ and φ^* is the angle between the vectors $\mathbf{g}^*$ and $\mathbf{n}^*$. Now, we can calculate geodesic curvature, normal curvature, geodesic torsion of curve , so from (2.3) we get

$$k_g^* = \frac{\sqrt{2\left(\delta_1^2+\delta_2^2+\delta_3^2\right)}}{\left((k_n+k_g)^2+(\tau_g-k_g)^2+(\tau_g+k_n)^2\right)^2}\cos\varphi^*,$$

and

$$k_n^* = \frac{\sqrt{2\left(\delta_1^2+\delta_2^2+\delta_3^2\right)}}{\left((k_n+k_g)^2+(\tau_g-k_g)^2+(\tau_g+k_n)^2\right)^2}\sin\varphi^*,$$

and

$$\tau_g^* = \frac{-1}{\sqrt{3}}\frac{\begin{array}{l}(k_n+k_g)(k_g-\tau_g)\left(\tau_g\xi_2-k_n\xi_1\right)+(k_g-\tau_g)^2\left(\tau_g\xi_1+k_g\xi_3\right)+\\ \left(k_n^{'}+\tau_g^{'}\right)\left[(k_g-\tau_g)\xi_1+(k_n+k_g)\xi_2\right]-\\ \left(k_n^{'}+k_g^{'}\right)\left[\left(k_n^{'}+\tau_g\right)\xi_1+(k_g-\tau_g)\xi_3-(k_n+\tau_g)\xi_2\right]+\\ (k_n+\tau_g)^2\left(\tau_g\xi_1-k_n\xi_2\right)+\\ (k_n+\tau_g)(k_n+k_g)\left(\tau_g\xi_3+k_g\xi_1\right)+(k_n+\tau_g)^2\left(k_g\xi_3-k_n\xi_2\right)+\\ (k_g-\tau_g)(k_n+\tau_g)\left(k_n\xi_3-k_g\xi_2\right)+(k_n+k_g)\left(\tau_g^{'}-k_g^{'}\right)\end{array}}{\begin{array}{l}(k_g-\tau_g)^2\left(k_g^2+\tau_g^2\right)+2\left(k_n^{'}+k_g^{'}\right)\left[k_g(k_g-\tau_g)+k_n(k_n+\tau_g)\right]+\\ (k_n+\tau_g)^2\left(k_n^2+\tau_g^2\right)+2(k_n+\tau_g)\left[\tau_g\left(\tau_g^{'}-k_g^{'}\right)+k_g^2(k_n+\tau_g)\right]+\\ (k_n+k_g)^2\left(k_n^2+\tau_g^2\right)+2(k_n+k_g)\left[k_g\left(\tau_g^{'}-k_g^{'}\right)-k_n\left(\tau_g^{'}+k_g^{'}\right)\right]+\\ k_n\tau_g(\tau_g-k_g)+\left(k_n^{'}+k_g^{'}\right)^2+\left(k_n^2+\tau_g^2\right)+\left(\tau_g^{'}-k_g^{'}\right)^2+\left(\tau_g^{'}-k_n^{'}\right)^2+\\ -2\tau_g(\tau_g-k_g)\left(\tau_g^{'}+k_n^{'}\right).\end{array}}+\frac{d\varphi^*}{ds^*}.$$

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