

This interim report tries to find the cracks in the fundamentals and suggests repair procedures

This leads to unconventional solutions and a new model of Physics.

The Hilbert book model.

# On the origin of physical dynamics

Report of a research project

Hans van Leunen

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# ON THE ORIGIN OF PHYSICAL DYNAMICS

## Colophon

Collection of papers written by Ir J.A.J. van Leunen

The main subject of this collection is a new model of physics

The collection also contains papers about related subjects

This collection is written as an e-book. It contains hyperlinks that become active with Adobe Reader Pro in the electronic version, which can be accessed at <http://www.crypts-of-physics.eu>. Last update of this (printed) version: **Monday, July 18, 2011**

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# **ON THE ORIGIN OF PHYSICAL DYNAMICS**

## *ACKNOWLEDGEMENTS*

I thank my wife Albertine, who tolerated me to work days and nights on a subject that can only be fully comprehended by experts in this field. For her I included the tale that makes the stuff a bit more comprehensible to those that do not eat formulas for breakfast, lunch and dinner. For several years she had to share me with my text processor. She stimulated me to bring this project to a feasible temporary end, because this project is in fact a never ending story.

I also have to thank my friends and discussion partners that listened to my lengthy deliberations on this non society chat suitable subject and patiently tolerated that my insights changed regularly.

## *DETAILS*

This is a report of a running research project.  
The project started in 2009.

The continuing status of the project can be followed at

<http://www.crypts-of-physics.eu>

Software related subjects are treated at

<http://www.scitech.nl/MyENHomepage.htm#software>

The author's e-print site is:

[http://vixra.org/author/Ir\\_J\\_A\\_J\\_van\\_Leunen](http://vixra.org/author/Ir_J_A_J_van_Leunen) .

## Contents

Preface .....	23
Logic model.....	24
PART ONE .....	26
Cracks of fundamental quantum physics .....	26
Abstract.....	27
History .....	27
Cracks in the fundamentals .....	28
Fist scratches .....	28
Severe defects.....	28
Back to the future .....	28
Dynamic way out .....	29
Granularity .....	29
Coping with granularity.....	29
Underpinning for strands .....	30
Background coordinates.....	30
Strand operator .....	30
Strand space .....	31
Fields .....	32
Function of the field .....	32
Basic field constituent .....	32
Fields influence the strand .....	33
Particles.....	33
Curvature, torsion and chirality.....	33

Extended Hilbert space.....	34
Extended quantum logic .....	34
Covering field .....	34
Curvature field.....	34
About the field concept .....	35
Canonical coordinates.....	35
Ideal coordinates .....	35
Fourier transform .....	36
Use of the Fourier transform .....	37
Fourier transform habits.....	37
Actual coordinates.....	38
Coherent state .....	39
Distributions and fields .....	40
Hilbert distributions.....	40
Hilbert field .....	40
Optics and quantum physics .....	40
Dynamics .....	41
Spacetime.....	41
Relativity.....	42
Strand model.....	43
Information detection .....	44
Discussion.....	44
Rigged Hilbert space.....	45
Quantum clouds .....	46
Testing theories.....	46
Summary of scratches.....	47

Summary of repairs.....	47
Open issues.....	48
REFERENCES .....	49
PART TWO.....	51
On the origin of physical dynamics.....	51
Abstract.....	52
Introduction.....	54
The fundament.....	54
Equations of motion .....	55
The logic of the model .....	55
Recapitulation and extrapolation.....	63
GPS coordinates.....	66
Test proposition.....	66
Numbers .....	67
Prospect.....	69
Comments.....	69
Version 3 .....	69
Project.....	69
References.....	70
Equation editor .....	70
EM fields.....	71
Strands .....	71
Notation note .....	71
Acquired indications.....	73
Studying physics .....	73
Classical versus quantum physics .....	73

The rediscovery of quaternions.....	73
Representation restriction .....	74
Extended quantum logic .....	75
Dynamic quantum logic .....	75
Curved space.....	75
Intensified imaging .....	75
Granular GPS .....	79
Progression step details .....	79
Release and removal of quanta .....	80
Fields and probability amplitude distributions.....	80
A detailed list of indications and considerations .....	80
First conclusion.....	86
Logic.....	87
Logics .....	87
Example proposition.....	87
Atomic predicates.....	89
Type definitions.....	90
Strands as type definitions.....	91
Items .....	92
Representation of items .....	94
Vacuum.....	95
Hilbert spaces.....	96
Dual views of a Hilbert space .....	96
Position .....	96
Physical coordinates.....	97
Generating a Hilbert space GPS .....	98

Canonical conjugate .....	100
The Hilbert space GMS.....	101
The fourth dimension .....	101
Time and dynamics.....	103
Hilbert functions.....	104
Limitedness .....	105
Countability.....	105
Granularity .....	105
Investigating an alternative operator .....	106
Strand operator.....	107
Strand interpretation.....	109
Fundamental measures and units.....	110
Numbers .....	111
Sign selections.....	111
Product rule.....	113
Product sub-terms .....	115
Operators.....	115
Matrices.....	116
Construction.....	116
Colors .....	117
Path characteristics.....	117
Path equations.....	118
Curve length.....	118
Reparameterization.....	119
Functions and fields.....	120
Distributions in quaternionic Hilbert space .....	120

Hilbert field .....	121
Sampled Hilbert field.....	122
Blur function .....	123
Bypassing granularity.....	125
The basic constituent and private field .....	127
Covering field .....	130
Decomposition .....	131
Decomposition and quaternionic Fourier transform .....	133
Curvature field.....	134
About the field concept .....	134
Functions in quaternionic Hilbert space .....	135
Elementary Hilbert distribution.....	136
Characteristic functions .....	136
Differentiation.....	137
Canonical conjugate .....	139
Complex Fourier transform .....	140
Heisenberg's uncertainty .....	141
The quaternionic displacement generator .....	142
Idealized field conditions.....	142
Affine space.....	144
Quaternionic Fourier transform split .....	144
Alternative transverse plane.....	147
Alternative approach to Fourier transform .....	147
Weil-Brezin transform .....	149
Fourier transform .....	149
Functions invariant under Fourier transform .....	150

Harmonic oscillating Hilbert field .....	162
Rotational symmetry.....	165
Spherical harmonics.....	165
Spherical harmonics eigenvalues.....	169
Special Fourier transform pairs .....	179
Complex Fourier transform invariance properties .....	180
Fourier transform properties .....	180
Parseval's theorem .....	180
Convolution.....	181
Differentiation.....	181
Vacuum expectation value.....	181
Hilbert field equations .....	181
Statics and dynamics.....	182
The quaternionic nabla .....	183
Blurring the charges.....	184
Continuity equation for charges .....	184
Another interpretation of $\rho$ .....	186
Example potential.....	187
Differential potential equations.....	190
Maxwell equations .....	193
Conservation laws.....	195
States.....	202
State definition.....	204
Pure state .....	205
Item state.....	205
Probability density .....	208

States and blurs.....	209
Observables and field values.....	211
Numbers .....	211
$2^n$ -on construction.....	213
Waltz details.....	213
Sign selections.....	216
Need for spinors .....	217
Influence .....	218
The universe of items.....	218
Inertia .....	218
Nearby items .....	224
Rotational inertia .....	224
Storage, sign selections and virtual items.....	225
The proposition.....	225
The origin of dynamics.....	227
Extended quantum logic .....	228
Interpretation in logical terms .....	228
Minkowski signature .....	230
Dynamics .....	231
Schrödinger or Heisenberg picture.....	231
Unitary transform.....	231
Trail of infinitesimal transforms.....	232
Unitary transform with full set of eigenvectors.....	233
Fourier transform as unitary transform .....	233
Single infinitesimal step .....	235
Relativity.....	238

Inertia and progression step .....	241
Redefinition .....	241
Trails .....	243
Cycles .....	244
Redefiner .....	244
Equations of motion .....	246
Private continuity equation .....	246
Particles .....	246
Interactions .....	247
Schrödinger equation .....	247
Pauli equation .....	248
Dirac equation .....	249
Fields .....	249
More fields .....	250
Lagrangian .....	250
Path through field .....	251
Calculus of variations .....	253
Affine geometry .....	253
Christoffel symbols .....	255
The action along the live path .....	256
Optics .....	259
Veiling glare and halo .....	263
Strands .....	264
Schiller's strands .....	264
Planck values .....	266
Strand basics .....	266

Strand table.....	267
Unique aspects of the model.....	269
Fundament .....	269
Insights.....	271
Discussion.....	273
Macro and micro.....	273
Dynamic logic .....	273
Conclusion.....	274
Appendix .....	277
History of quantum logic .....	277
Quantum logic .....	277
Lattices .....	278
Proposition .....	281
Observation .....	282
Quaternion coordinates .....	282
Polar coordinates .....	282
3 sphere .....	283
Hopf coordinates .....	285
Group structure .....	286
Versor .....	287
Symplectic decomposition .....	287
$2^n$ -on construction.....	287
$2^n$ -ons.....	288
Regular quaternionic functions .....	291
The separable Hilbert space $\mathbb{H}$ .....	297
Notations and naming conventions.....	297

Quaternionic Hilbert space .....	297
Derivation of the one dimensional Euler Lagrange equation.....	309
Equation.....	309
Derivation.....	310
Zak transform .....	313
Definition.....	313
Elementary properties. ....	313
Analytic properties.....	315
Inversion formulas. ....	315
Applications. ....	315
Thoughts.....	317
Spin and dyadic product.....	317
Wave package .....	318
Systems .....	320
Measurement .....	326
Hamilton-Jacobi.....	327
The Lagrangian.....	328
The world's action.....	330
Representing multiple fields.....	330
Planck limits for all physical observables .....	332
Basic measures .....	332
Fundamentals.....	332
Elementary particles .....	333
Virtual particles .....	334
EM limits.....	334
Derived limits .....	335

Cosmological limits.....	337
Limit quality.....	339
References:.....	339
PART THREE.....	342
How the brain works .....	342
Abstract.....	343
Pre-processing.....	343
Processing.....	343
Image intensification.....	344
Imaging quality characteristics .....	344
Vision of noisy images.....	345
Information association .....	345
Noise filter .....	346
Brain waves .....	346
Reasoning .....	346
Other species .....	347
Humans.....	347
Science .....	347
Physical reality.....	347
Theories.....	348
Inventions of the human mind.....	348
History .....	349
Dreams .....	349
PART FOUR.....	351
A law of nature .....	351
Abstract.....	352

Complexity .....	352
Potential complexity .....	353
Physical relations.....	353
Modularization .....	354
Modular system design .....	354
Intelligent modularization .....	356
Coupling .....	356
Interfaces.....	357
Principle of minimal action.....	358
The power of modularization.....	358
Abusing modularization .....	359
Software .....	359
PART FIVE .....	362
Story of a war against software complexity .....	362
Abstract.....	363
Prelude .....	363
Analysis .....	363
Setting.....	364
History .....	364
Strategy .....	365
Approach.....	365
What happened.....	366
Attack .....	366
Set-back .....	367
Remnants .....	367
Goal .....	368

Lessons.....	368
Conclusions .....	369
Way out.....	369
Discussion.....	370
PART SIX.....	372
Managing the software generation process.....	372
Abstract.....	373
Software complexity .....	373
Introduction.....	373
Managing complexity .....	374
Breaking level.....	374
Measure of complexity .....	374
Extreme complexity .....	375
The modular approach .....	375
Modularization .....	375
Modular system design .....	376
Interfaces.....	376
Proper modules .....	378
Properties and actions .....	379
Costs of modularization .....	379
Abuse.....	380
Modularization success cases .....	380
Requirements for success .....	381
Difficulties posed by modularization.....	382
Diversity .....	382
Compatibility .....	382

Platforms.....	383
Hiding intellectual property .....	383
Availability .....	384
Specification .....	384
Hardware versus software.....	384
History .....	384
Basic architecture trends .....	385
Functional analysis.....	385
Abstract data types.....	385
Object orientation .....	386
Current software components.....	386
State of affairs.....	387
Coupling the market and the design and creation of software modules and interfaces .....	388
Standardization and marketing .....	388
Designing and generating components .....	388
Versions and diversity .....	389
Hiding intelligent property.....	389
Automating system integration .....	389
Publishing.....	390
A fully fledged software components industry .....	390
Sketch .....	390
The demo .....	391
Software availability .....	396
PART SEVEN .....	398
Physics and religion .....	398

Universe as a creator .....	399
The upper creator .....	400
PART EIGHT.....	405
The Hilbert book model .....	405
Introduction.....	407
What's playing? .....	407
Quantum Theory .....	407
History .....	408
Clarification .....	408
Hilbert model.....	409
Numbers .....	409
Hilbert operators .....	410
Graininess .....	410
GPS .....	411
Grain chains .....	411
Anchor Points .....	412
Fields .....	412
Private Fields.....	413
Field Theory .....	413
The field model.....	414
Hilbert sandwich .....	414
Hilbert book .....	414
Discussion.....	415
References.....	415
PART NINE.....	417
A tall quantum tale.....	417

Story .....	418
Prelude .....	418
The encounter .....	420
Interpretation .....	434
INDEX .....	438

## Preface

This book contains a collection of interrelated papers. The title of the book equals the title of the main paper. The main paper (part 2) describes a research project in which the origin of dynamics must be uncovered. As a starting point, cracks in the fundamentals of physics are detected and suggestions are given for the repair of these deficiencies (part 1).

The paper that describes how the brain works (part 3) also gives information on how the visual trajectory of vertebrates optimizes the perception of low light level signals. It indicates that this system during a billion years has helped vertebrates to survive low light level conditions. The observed radiation is generated by Poisson processes. The conclusion is that information comes to us in the form of clouds of quanta, rather than in waves.

The paper that introduces a new law of nature (part 4) explains that nature has a built-in tendency to reduce complexity via modularization. As a result it achieves to construct very complicated items that include intelligent species.

Two other papers (part 5 and 6) describe what happens when the merits of this new law are not exploited and how that situation can be cured.

The next paper (part 7) treats the relation between physics and religion.

The last two papers are more light weight descriptions of the main subject. The Hilbert book model is the name of a comprehensible paper (part 8) that describes the new fundament.

The last paper (part 9) describes in the form of a fairytale how the universe works.

## Logic model

It is always possible to build a potentially usable theory and a corresponding model by starting from a consistent set of axioms. Classical logic represents such a set. It is a theory on itself. However, in the realm of quantum physics nature cheats with one of the axioms of classical logic. So it is sensible to adapt classical logic and change the corresponding axiom such that this adapted logic fits better. The adapted version of the theory and the corresponding model build on this new logic. The adapted logic is known as traditional quantum logic.

Traditional quantum logic and as a consequence the new model still do not fit most of the features that we know from physical phenomena. The reason is that the new model cannot handle fields and it cannot treat dynamics. So, as a next step, the traditional quantum logic must be extended to a new version of quantum logic that can cope with fields. The trick required for this extension blurs a subset of the propositions. The blur represents the sticky resistance of these propositions against change. In practice this extension is done via the isomorphic companion of traditional quantum logic, which is a separable Hilbert space. The new logic does not yet have a generally accepted name. So, we leave the name at extended quantum logic.

Still this extended quantum logic can only handle static situations. Thus the obtained model is not a dynamic model. This situation can be cured by taking a sequence of these extended quantum logics such that each subsequent element represents a static status quo of the dynamic universe that the final model is aimed to describe.

The result is called the Hilbert book model. In this model the progression is made in universe wide steps.

Due to its foundation on adapted and extended classical logic and by including guidance from physical concepts the resulting model can be considered as an abstraction of physical reality.



# Part one

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**Cracks of fundamental quantum physics**

# Cracks of fundamental quantum physics

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## Abstract

The fundamentals of quantum physics are still not well established. This paper tries to find the cracks in these fundamentals and suggests repair procedures. This leads to unconventional solutions and a new model of physics. As part of this enterprise an underpinning of the existence of strands is provided. Another innovation is the derivation of a curvature field from the superposition of all other fields. The most revolutionary introduction is the representation of dynamics by a sequence of separable Hilbert spaces. Together, this embodies a repair of fundamentals that does not affect the building.

## History

In its first years the development of quantum physics occurred violently <sup>[1]</sup>. As a consequence some cracks sneaked into the fundamentals of this branch of physics. A careful investigation brings these cracks to the foreground. The endeavor to repair these cracks delivers remarkable results.

In the early days of quantum physics much attention was given to equations of motion that were corrections of classical equations of motion. The Schrödinger approach was one and the Heisenberg approach was another. Schrödinger used a picture in which the state of a particle changes with time. Heisenberg uses a picture in which the operators change with time. For the observables this difference makes no difference. This fact is important. It shows that time is just a parameter instead of a property of physical items<sup>1</sup>. Later Garret Birkhoff, John von Neumann

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<sup>1</sup> Later this fact is used in order to apply the progression step counter as a parameter that characterizes the members of a sequence of Hilbert spaces.

and Constantin Piron found a more solid foundation that was based on quantum logic. They showed that the set of propositions of this logic is isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space, whose inner product is defined with the numbers taken from a division ring. The ring can be the real numbers, the complex numbers or the quaternions. Since then many physicists do their quantum physics in the realm of a Hilbert space.

## Cracks in the fundamentals

### Fist scratches

These physicists quickly encountered the obstinate character of the separable Hilbert space. Its normal operators have countable eigenspaces. This can still correspond to a dense coverage of the corresponding hyper complex number space. However, this space is no continuum. Thus, functions defined using these eigenspaces as parameter domains cannot be differentiated. In order to cope with this defect, most physicists resorted to the corresponding rigged Hilbert space, but in doing so they neglect that in this way the relation with quantum logic gets lost.

### Severe defects

Further, it appears that the separable Hilbert space cannot represent physical fields and cannot represent dynamics. This is a severe drawback and it looks as if the switch to the rigged Hilbert space becomes mandatory. For example quantum field theory represents fields as operators that reside in this rigged Hilbert space.

### Back to the future

On the other hand there are more and more signals that nature is fundamentally granular and rigged Hilbert spaces do not provide that feature. This guides backwards to the separable Hilbert space. But in that case we must learn to live with its granularity. In addition we must find other ways to represent fields.

## Dynamic way out

The rigged Hilbert space gave similar problems with representing dynamics as the separable Hilbert space does. There is no place for time as an eigenvalue of an operator neither in separable Hilbert space nor in rigged Hilbert space. For that reason, it is better to accept that the separable Hilbert space can only represent a static status quo.

## Granularity

Nature is fundamentally granular. The so called Planck units<sup>2</sup> are designed using dimension analyses, but it is generally accepted that below these limits (Planck length, Planck time, Planck constant = unit of action and Boltzmann's constant = unit of entropy) no discerning observation is possible. One could say that below these limits nature does not exist or that nature just steps over these regions. The Planck length and Planck time are related to the Planck constant, the speed of light  $c$  and the gravitational constant. It is not said that nature's granularities have exactly these sizes. It is rather an order of magnitude indication, but these measures are useful and we do not have a better estimate. The mutual relation between these units is important. For example, the ratio between the Planck length and the Planck time equals the speed of light. If you reckon that at every time step a physical item can at the utmost take one space step, then the maximum speed of all physical items is automatically set at the speed  $c$ .

## Coping with granularity

A solution must be found for the fact that GPS-like normal operators in separable Hilbert space possess granular eigenspaces. A GPS operator would have a lattice-like eigenspace of densely packed granules. The lattice would possess preferred directions. It means that it is impossible to define an operator that acts like a proper global positioning system (GPS),

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<sup>2</sup> [http://en.wikipedia.org/wiki/Planck\\_units](http://en.wikipedia.org/wiki/Planck_units)

which is required in the positioning of field values or when we want to relate Hilbert vectors with position.

## Underpinning for strands

The separable Hilbert space can provide a GPS-like operator that offers a dense coordinate system as its eigenspace. An eigenspace consisting from all rational quaternionic numbers would be countable and thus it can be an eigenspace of a normal operator.

However this eigenspace does not support differentiation and is not granular. In nature the space of the positions is granular and the size of the granules is of the order of the Planck length, which is  $1.6 \cdot 10^{-35}$  m. The lattice formed by the densely packed granules in the eigenspace of the granular GPS-like operator would immediately show unnatural preferred directions. Such situations can only occur in condensed matter. This fact holds for any multidimensional subset of eigenvalues, even if it contains values on the real axis.

## Background coordinates

A background coordinate system exists in rigged Hilbert space, but when we insist on granularity of the GPS-like coordinates, it cannot be directly used in separable Hilbert space in order to locate Hilbert vectors in a regular way. As signaled before, this will introduce anomalies. So, we must find an indirect way. This is delivered by the strand operator, which resides in separable Hilbert space and has an equivalent in the rigged Hilbert space. There it can be coupled to the background coordinate system. The strand operator does not possess multidimensional sets of eigenvalues. Thus, it avoids the mentioned problems in separable Hilbert space.

## Strand operator

It is possible to define a normal operator in separable Hilbert space whose eigenspace consists out of a set of chains that are put together from granules. In the chain the granules are ordered. In each chain one granule is exceptional. We call it the **current granule**. The part of the chain that ends just before the current granule is called the past sub-chain. The part

that starts just after the current granule is the future sub-chain. Via the background coordinate system that is delivered by the rigged Hilbert space each granule gets its own position. That value becomes the eigenvalue of the Hilbert vector that corresponds with the granule. In this way the chain defines a trail of Hilbert vectors.

One could ask whether having only the current granule could be sufficient. For the model, the direct neighborhood of the current granule is the most relevant part of the chain. The rest of the chain is hardly used. It only gives a reflection of a possible past and a possible future which is derived from the current field configuration. However, the step to the next version of the “current granule” is taken inside the chain. At a given time step maximally one space step is allowed. That step has the size of one granule.

The embedding of the chain in the field gives a stronger control by the field on the position of the current granule. If this assumption is true, then the chain must quite accurately follow a field line. The fluctuations of the chain must follow the fluctuations of the corresponding field line.

With other words the chain couples the path of the current granule to a field line (geodesic?).

The eigenspace only contains imaginary eigenvalues. Any mix with real or complex values would render the “chain” either discontinuous or multidimensional. (?)

## **Strand space**

The operator has an outer horizon. Outside this horizon its eigenspace does not contain chains. It might also have inner horizons such that inside these inner horizons no chains exist.

The inner horizons are borders of black holes. Since the unit sphere of the Hilbert space is an affine space and all eigenvectors of the GPS operator

are represented in that space, the GPS can be considered to have no origin. In that case the outer horizon is just the border of another black hole.

The chains are closed or they start and end at a horizon. They may also reflect tangentially against a horizon. These chains have much in common with the strands in Schiller's strand model. However, they are not exactly the same.

At field excitations the chains may be generated or annihilated. This occurs for example in field configurations that are locally invariant under Fourier transformation, such as linear and spherical harmonics.

## Fields

Fields do not fit inside a separable Hilbert space. Any field would cover the whole Hilbert space. Every Hilbert vector would touch a value of the field. Which value is touched, depends on the functionality of this vector. When the vector is one of the eigenvectors of a normal operator and when the field can be expressed as a function of the eigenvalues of this operator, then the field value would correspond with the eigenvalue that corresponds to eigenvector. In that case, the considered field value will be connected to the considered vector.

## Function of the field

The function of the physical fields is to take care of minimizing changes during dynamical steps. This function becomes evident when [dynamics](#) is implemented. Fields keep the shape of the chains of the strand operator smooth.

## Basic field constituent

A probability amplitude distribution that is attached to the current granule takes care of the fact that the chain in the neighborhood of the current granule stays sufficiently smooth. This becomes important when dynamics is implemented because with each dynamic step the current

granule either stays at its current position or it moves one place ahead in the chain.

It must be noticed that exactly this restriction is the reason why speed has a maximum! The ratio of the space step and the time step equals the speed of light.

The squared modulus of the probability amplitude distribution is a probability density distribution. It determines the probability of the position of the current granule. The probability is large when the position is close to the position of the previous current granule.

### **Fields influence the strand**

The rest of the chain may be influenced by the probability distributions of the current granules of other chains. Taken over a sequence of dynamic steps, the chain appears to fluctuate. The fluctuation determines the probability distribution and vice versa the dynamic changes of the probability density distribution determine the fluctuations of the chains. This relation is instantaneous. There is no causal relation.

If the chains would be observable, then the probability distribution could be determined by averaging the fluctuations over some period. However, neither the chains, nor the probability amplitude function are directly observable items. Only their effects become observable.

### **Particles**

Depending on its type, an elementary particle relates to one, two or three of these chains. In this way the current granules of these chains are related to the current section of the path of the particle.

### **Curvature, torsion and chirality**

In contrast to torsion, curvature relates to mass. For example, according to strand model, the chain that represents a massless photon has a helix shape. The chains that represent the massive W bosons have the shape of an overhand knot. Since this knot shows chirality, it possesses electric charge. The chains that represent the massive Z bosons have the shape of a figure eight knot. Because the figure eight knot features no handedness,

it does not possess electric charge. In a similar way Schiller <sup>[2]</sup> attributes properties to all elementary particles.

## Extended Hilbert space

The addition of probability amplitude distributions to the current granules extends the separable Hilbert space to a new construct. For that reason we call this construct an extended separable Hilbert space.

## Extended quantum logic

Via the relation between the separable Hilbert space and traditional quantum logic we can extend quantum logic to an extended quantum logic that includes physical fields in a similar way as the extended Hilbert space model does.

## Covering field

The probability amplitude distribution that is connected to the current granule is a basic field constituent. The superposition of all these basic constituents forms a covering field. The configuration of the covering field depends on the configuration of the elementary particles. When the configuration of chains changes, then the configuration of particles changes and the covering field changes accordingly.

## Curvature field

According to Helmholtz theorem, the static covering field decomposes into a rotation free part and one or two divergence free parts. The corresponding local decomposition into a one dimensional longitudinal part and a transverse part defines a local curvature. This curvature can be used to define a derived field. We will call this the curvature field. It has all aspects of the gravitation field. When split back into curvature fields that are private to the particles the private curvature field can be used to attach the property "mass" to the corresponding particle.

## About the field concept

It is common practice to treat the EM fields and the gravitation field as different and independent subjects. In this interpretation, the gravitation field generates the curvature of the coordinate system in which the other fields must operate.

This paper takes a different approach. It puts the reason for the curvature of the coordinates in the properties and configuration of the covering field. The curvature that exists in this way is used to derive the curvature field. The wave function is also interpreted as a constituent of the covering field. In this way it also contributes to the curvature field. This picture unifies all fields.

## Canonical coordinates

We start with the situation in which we can select ideal coordinates. What that means will become clear soon.

## Ideal coordinates

The inner product of the Hilbert space can be used to relate two orthogonal bases that are each-other's canonical conjugate. In a quaternionic Hilbert space this is not a straightforward procedure. Luckier wise, the quaternionic number space can be divided into a series of complex number spaces. We just chose one imaginary direction and do as if we are in complex Hilbert space. However, this singles out that particular direction. We may choose the direction in which the local longitudinal direction of the covering field runs. The definition of longitudinal is in fact taken in the canonical coordinate space. It can be any radial direction taken from the origin of that space. This may give problems when this space is curved, thus when this direction changes with location. For the moment we assume that we have selected a coordinate system for which the selected longitudinal field direction runs along a straight line. We do not bother about granularity, because we will base our investigations on the background GPS and GMS coordinate

systems. So we pick the eigenspace of a normal GPS-like operator  $\check{Q}$  that resides in rigged Hilbert space as our coordinate system. It has an equivalent GPS-like coordinate operator  $Q$  in separable Hilbert space. The operator  $\check{Q}$  is selected such that the selected longitudinal direction of the field runs along one of the imaginary base vectors of the eigenspace. The set of eigenvectors  $\{|q\rangle\}$  of operator  $Q$  forms an inner product with another normal operator  $P$  which is the canonical conjugate of  $Q$ . The eigenvector  $|q\rangle$  corresponds to an eigenvalue  $q$  and similarly the eigenvector  $|p\rangle$  of  $P$  corresponds to an eigenvalue  $p$ . The inner products are now given by:

(1)

$$\langle p|q\rangle = \exp(i\hbar qp)$$

The constant  $\hbar$  in  $\hbar = 2\pi\hbar$  is Planck's constant. The imaginary 3D base vector  $\mathbf{i}$  of the quaternionic number space is the imaginary base number of the selected complex number space.

This procedure can be performed for the two operators and three mutually perpendicular imaginary base vectors of the eigenspace. We have defined the procedure for the operators  $P$  and  $Q$  that reside in separable Hilbert space, but with respect to its application to Fourier transforms, it makes more sense for the equivalent operators  $\check{P}$  and  $\check{Q}$  in rigged Hilbert space. Their eigenspaces form a continuum.

### Fourier transform

It can easily be seen that the specified inner product also defines a complex Fourier transform. We start with the separable Hilbert space. By taking all three dimensions the specified inner product defines the imaginary part of a quaternionic Fourier transform.

(2)

$$\langle q|f\rangle = \langle f|q\rangle^* = f^*(q) = \sum_p (\langle q|p\rangle \cdot \langle p|f\rangle)$$

And reversely:

(3)

$$\langle p|f \rangle = \sum_q (\langle p|q \rangle \cdot \langle q|f \rangle)$$

It must be reckoned that these are discrete transforms. Here the **Hilbert function**

(4)

$$f(q) = \langle f|q \rangle$$

is transformed into its Fourier partner.

In rigged Hilbert space the sum becomes an integral.

## Use of the Fourier transform

In separable Hilbert space, Hilbert functions are sampled functions and are constructed from the eigenvectors and eigenvalues of a normal operator and a selected Hilbert vector. See formula (4).

The discrete transform and the Hilbert functions do not have many usages. In practice the Fourier transform is applied to [Hilbert fields](#)<sup>3</sup> rather than to Hilbert functions.

The Fourier transform of a quaternionic field must be performed with a quaternionic Fourier transform that acts in a continuous number space <sup>[3]</sup>. The Fourier transformation of [a private field](#)<sup>4</sup> of a particle does two things. It shifts from a GPS-like coordinate system to its canonical conjugate GMS-like coordinate system. Apart from that it transforms the private field from a quantum cloud into a wave package. This new probability distribution tells about momentum rather than about position.

## Fourier transform habits

A Fourier transform keeps inner products invariant. Thus it is a unitary transformation. It has no eigenvectors and as a consequence it has no eigenvalues. However, in rigged Hilbert space functions exist that apart from a multiplication factor are invariant under Fourier transformation. Examples of these are the functions that describe linear and spherical quantum harmonic oscillators. The multiplication factor can be 1, i, -1, or -i.

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<sup>3</sup> Distributions and fields: Hilbert fields

<sup>4</sup> Distributions and fields: Hilbert fields: Private field

In separable Hilbert space, the Fourier transform converts an orthogonal base into another orthogonal base, which is completely distinct from the original base. Any member of the second base is a linear combination of all members of the first base. The modulus of all coefficients in this linear combination is equal to unity. In rigged Hilbert space the function  $\exp(i p q \hbar)$  and the Dirac delta function  $\delta(q)$  form Fourier transform pairs. In separable Hilbert space the Kronecker delta replaces Dirac's delta function.

This reveals that the existence of canonical conjugation is the reason of the weakening of the modular law that makes the difference between classical logic and quantum logic.

A very important property of Fourier transforms is, that it transforms a differentiation into a multiplication with the canonical conjugated coordinate. This only works in rigged Hilbert space. In practice it is applied to Hilbert fields<sup>5</sup>.

### Actual coordinates

In practice the ideal conditions are seldom valid and if they are valid, they are only valid locally. It means that the inner product (1) has only local validity and the same holds for the Fourier transforms that are specified with the aid of that inner product. In actual situations the coordinate system in the environment gets curved. Only an appropriate coordinate transformation can bring us back to the ideal situation. This is a purely mathematical activity and the required transform changes with the field configuration that resides in the current static status quo. It does not affect physical reality. So if we know how to perform this coordinate transformation then physics in this static status quo becomes trivial. This is the reason why particles move along geodesics.

The presented picture supposes that nowhere the field excitations are so violently that it becomes impossible to define a local curvature. With this restriction, the required coordinate transformation stays possible.

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<sup>5</sup> Distributions and fields: Hilbert fields

## Coherent state

A **coherent state** is a specific kind of state of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system.

The coherent state  $|\alpha\rangle$  is defined to be the 'right' eigenstate of the annihilation operator  $\mathcal{A}$ . Formally, this reads:

(5)

$$|\mathcal{A} \alpha \rangle = \alpha |\alpha \rangle$$

Since  $\mathcal{A}$  is not Hermitian,  $\alpha$  is a hyper complex number that is not necessarily real, and can be represented as

(6)

$$\alpha = |\alpha| \exp(i\theta)$$

where  $\theta$  is a real number.  $|\alpha|$  is the amplitude and  $\theta$  is the phase of state  $|\alpha\rangle$ .

This formula means that a coherent state is left unchanged by the annihilation or the creation of a particle. The eigenstate of the annihilation operator has a [Poissonian](#)<sup>6</sup> number distribution. A Poisson distribution is a necessary and sufficient condition that all annihilations are statistically independent. (Shot noise is characterized by a Poisson distribution. See [information detection](#).)

The coherent state's location in the complex plane ([phase space](#)<sup>7</sup>) is centered at the position and momentum of a classical oscillator of the same phase  $\theta$  and amplitude. As the phase increases the coherent state circles the origin and the corresponding disk neither distorts nor spreads. The disc represents Heisenberg's uncertainty. This is the most similar a quantum state can be to a single point in phase space.

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<sup>6</sup> <http://en.wikipedia.org/wiki/Poissonian>

<sup>7</sup> [http://en.wikipedia.org/wiki/Phase\\_space](http://en.wikipedia.org/wiki/Phase_space)

## Distributions and fields

The concepts that have been introduced so far invite the introduction of Hilbert distributions and Hilbert fields.

### Hilbert distributions

Hilbert distributions are sets of Hilbert vectors, in which each vector corresponds to the current granule of a member of a set of chains. Thus, these vectors are eigenvectors of the strand operator. All granules correspond with a Hilbert vector, but the vectors of a Hilbert distribution correspond with the current granule. An **elementary Hilbert distribution** is a set of Hilbert vectors that belong to an elementary particle.

### Hilbert field

A Hilbert field is a blurred Hilbert distribution. The blurs are the probability amplitude distributions that are attached to the corresponding current granules.

A **private Hilbert field** is a Hilbert field that belongs to an elementary Hilbert distribution. However, if a complicated particle consists of a set of elementary particles, then we consider the superposition of the private fields of the elementary particles as the private field of the complicated particle. The Hilbert field is a skew field. Its values are taken from a division ring.

The covering field is the superposition of all private fields.

### Optics and quantum physics

If all probability amplitude distributions would be similar, then the Hilbert field can be considered as the convolution of this probability amplitude distribution and a distribution of Dirac delta functions that correspond to the Hilbert distribution. This picture resembles ray optics and if we take the Fourier transform then it resembles Fourier optics. This is the reason that wave mechanics has so much similarity with optics. The Optical Transfer Function characterizes the information transfer capability of an imaging system. In similar way the Fourier transform of

the probability amplitude distribution characterizes the information transfer capability of a physical system. Nothing is said yet about detecting the information that is carried by the particles. That will be treated [later](#)<sup>8</sup>.

## Dynamics

The extended separable Hilbert space model can only represent a **static status quo**. By using this ingredient, dynamics can be implemented by a model that consists of an ordered sequence of such extended Hilbert spaces. It corresponds to an equivalent sequence of extended quantum logics.

In order to give this model a name, we can call it the Hilbert sequence model or more illustrative, the **Hilbert book model**. Passing through the sequence is like glancing through a book, where each page describes a static status quo.

## Spacetime

This procedure introduces a new parameter that acts as a global progression step counter. This parameter must not be confused with our common notion of time, which only has local validity.

The dynamic model implies that space is not the only granular quantity. It also means that progression occurs in discrete steps. Further, it indicates that against general acceptance, fundamentally, space and progression have little to do with each other. With other words, no support exists for a fundamental physical spacetime quantity.

That does not say that no relation between the fundamental space step and the fundamental time step can exist. The Minkowski signature is a clear prove of such relation. It can already be understood from the ratio between the Planck length and the Planck time. The explanation of the Minkowski signature must be sought in what happens during an infinitesimal progression step.

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<sup>8</sup> Information detection

When the smallest possible space step  $l_{pl} = \sqrt{\hbar G/c^3}$  and the smallest possible coordinate time step  $t_{pl} = \sqrt{\hbar G/c^5}$  are put into the Minkowski signature,

(7)

$$\Delta t^2 = \Delta \tau^2 - \Delta q^2/c^2$$

then the corresponding spacetime step  $\Delta \tau$  is zero.

The number of Planck time steps equals the number of global progression steps. The number of Planck length steps must always be equal to or lower than the number of Planck time steps. A free photon never takes a non-zero spacetime step. The number of its space steps always equals the number of its time steps.

Any particle that does not travel with light speed skips some of its space steps. Any particle can take a space step in a direction that differs from the direction of a previous step.

## Relativity

Wiki states: One Planck time is the time it would take a photon travelling at the speed of light to cross a distance equal to one Planck length.

Theoretically, this is the smallest time measurement that will ever be possible, roughly  $10^{-43}$  seconds. Within the framework of the laws of physics as we understand them today, for times less than one Planck time apart, we can neither measure nor detect any change.

Nothing occurs in that period. It is as if universe does not exist in that period. Nature just steps over this period. These steps are taken in synchrony. In the model this follows from the fact that a separable Hilbert space can only represent a static status quo. This also holds for a Hilbert space that is extended with static fields. Dynamics occurs when each extended Hilbert space is followed by a subsequent extended Hilbert space.

Relativity must fit in that picture. General relativity easily follows from the curvature of the longitudinal part of the covering field.

Wikipedia states: Special relativity incorporates the principle that the speed of light is the same for all inertial observers regardless of the state of motion of the source.

This is the case when all free photons take one Planck-length step when they suffer a Planck-time step. This may require observed length contraction and observed time dilatation. These effects would not occur when we measure our observations in integer amounts of Planck units. The trick is that the Planck units suffer the same contraction/dilatation as the observed items. Thus the number of Planck unit steps is independent of the observer.

With other words, if measurements are done using Planck units, then the integer measuring results will be observer independent. The measuring units travel with the observed item.

## Strand model

The main difference between the Hilbert space approach that is taken here and Schiller's approach lays in the interpretation of the source of the observable data. The principle fundamental postulate of Schiller's strand model is that the crossing switches of strands deliver the observable data. In the Hilbert book model the cloud of quanta that corresponds with the moving and rotating probability amplitude distributions that are connected to the current granules carry the observable data. There must be no difference between the results of the two models.

Further, Schiller's strand model derives fields from strand tangles. In the Hilbert approach the shape and the fluctuation of the chains are controlled by fields. As indicated before there is no difference between the two pictures because the relation is instantaneous and involves no causality.

In both pictures the described concepts form the basis of a consistent model. This model delivers the proper equations of movement. <sup>[2]</sup> <sup>[3]</sup>. The

reason of this conformance lays in the similarity of the basic field constituents.

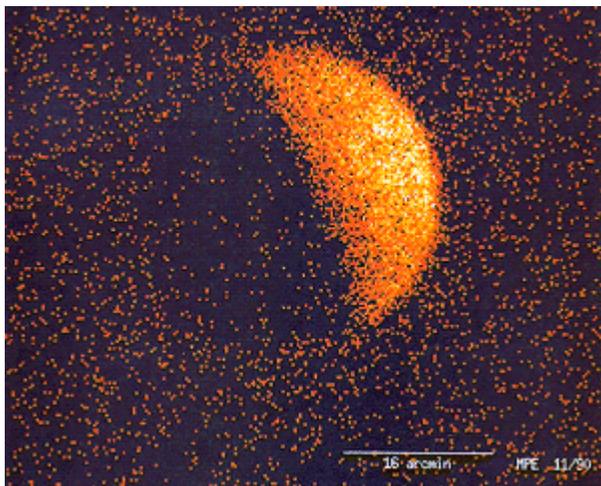


Figure 1: Intensified low dose image of the moon

## Information detection

All information that is transmitted by nature is carried by clouds of information carrying quanta (see figure 1). The clouds themselves carry secondary information in their shape and their movement characteristics. It looks as if all quanta are generated by a series of Poisson processes. These facts become apparent when observations or measurements are done at very low dose rates <sup>[3]</sup>. The shape of the cloud is set by the corresponding probability amplitude distribution.

As indicated before, coherent states act as Poisson processes. This is only an example. It does not say that coherent states are the only forms of states that act as Poisson processes.

## Discussion

This model of physical reality does not contain continuous waves. It does not contain singularities. Nor does it contain infinities. The only infinity it uses is the infinity of the dimension of the separable Hilbert space.

The model is fundamentally granular. The only continuities that the extended Hilbert space uses are the continuity of the background coordinate system that it borrows from its rigged partner and the continuity of the shapes of the probability amplitude distributions. Schiller's strand model takes the conclusions still further. He claims that this theory fully explains the standard model and that no further particles than those specified by the standard model exist.

Apart from the difference with respect to the main postulate of strand model, an important difference exists between the approach presented here and Schiller's strand model. Schiller presents the gravitation field as a separate field that is mainly determined by distant fluctuations of tangle tails. In this paper, the gravitation field is treated as a derived field. It has long range effects due to the fact that its charges (the curvature values) do not get compensated by opposite charges as happens with electric charges. Proof is given by the existence of inertia, which can only be explained by analyzing the influence of the universe of particles on a local particle <sup>[3], [4]</sup>. Locally this influence causes an enormous potential, which according to Sciama can be characterized by the gravitational constant  $G$ . Uniform movement of a particle does not raise other field activity than a field reconfiguration, but any acceleration of the particle goes together with an extra vector field <sup>[4]</sup>.

## Rigged Hilbert space

The rigged partner  $\mathbf{H}$  of a separable Hilbert space  $\mathbf{H}$  is not a separable Hilbert space, but a Gelfand triplet. It is an ordered set  $(\Phi, \mathbf{H}, \Phi^x)$ , where  $\mathbf{H}$  is the Hilbert space used to generate  $\Phi$  and  $\Phi^x$ . The eigenspaces of normal operators in a Gelfand triplet need not be countable. They can be continuous spaces such as the full set of quaternions. The name of Hilbert is misused to identify the Gelfand triplet as a rigged Hilbert space. This paper uses the Gelfand triplet  $\mathbf{H}$  in order to provide a background GPS system and to couple the equivalent of the separate Hilbert space strand operator to the corresponding GPS operator. Both the equivalent strand operator and the GPS operator reside in the rigged Hilbert space  $\mathbf{H}$ . In this way the granules of the chains that reside in separable Hilbert space

get their position. Another use of the background GPS operator is the coupling of field values to a position value. For that purpose the field values must be attached to the corresponding eigenvectors in rigged Hilbert space  $\mathbf{H}$ .

## Quantum clouds

The notion of quantum cloud needs clarification. The quantum cloud that corresponds to the private field of an elementary particle only contains the current granules of that particle as information carrying quanta. A field that consists of a superposition of the private fields of a set of elementary particles corresponds to a quantum cloud that contains quanta that correspond to the current granules of these particles. Not only the quanta carry information. Also the shape of the cloud that contains the quanta contains interpretable information.

When the cloud consists of emitted particles, then the process that controls the emission can be considered as a Poisson process. Upon detection an elementary particle is fully absorbed or it is converted into other particles from which at least one is absorbed. A detected particle was emitted by some body. During its travel it may have been reflected against or deflected by other bodies. The corresponding quantum clouds are affected correspondingly.

A quantum cloud can gain and lose quanta. An emission generates quanta and the corresponding private fields, which then form the shape of the cloud. The quantum cloud that corresponds to a private field disappears with its last quantum.

## Testing theories

You can falsify a theory when its conclusions according to a selected logic are not consistent with its presumptions. If you take classical logic as a criterion then QM is a wrong theory. If you take quantum logic as a criterion, then more of quantum physics will pass, but you will have difficulty in checking quantum field theory. Only after extending

quantum logic, such that it includes fields, you can handle quantum field theory as well. Still this equipment does only reach to test static situations.

## Summary of scratches

The following scratches have been treated here.

1. Due to its link with traditional quantum logic quantum physics must be based on separable Hilbert spaces, but quite often it is based on a rigged Hilbert space.
2. Neither the separable Hilbert space nor the rigged Hilbert space can represent dynamics. They can only represent a static status quo.
3. The separable Hilbert space cannot represent physical fields. It must be extended in order to cope with fields. In rigged Hilbert space fields are often represented by operators.
4. Nature is fundamentally granular. The rigged Hilbert space cannot represent this properly.
5. It is impossible to represent a GPS-like operator in separable Hilbert space.
6. Gravitation field is seen as an independent field. In fact it can be derived from other fields.

## Summary of repairs

The following repairs have been suggested.

1. Base quantum physics on a book of Hilbert spaces, where each page is an extended infinite dimensional separable Hilbert space that represents the current static status quo. The extension is done by blurring a subset of the Hilbert vectors.
2. Introduce a strand operator whose eigenspace consists of one dimensional chains of granules, where each granule gets its position from a background GPS coordinate system that is generated by a GPS operator that houses in rigged Hilbert space.

3. One of the granules of each chain is special. It is blurred. A corresponding eigenvector gives it its position. The blur is a probability amplitude distribution. It anchors on the granule and on the corresponding Hilbert vector. During dynamic steps the blur keeps the chain smooth.
4. Elementary particles are anchored on the special granules of one or more chains. The corresponding blurs together form the particle's private field, which is also its wave function.
5. Together the private fields form an overall covering field.
6. The static covering field can be decomposed into a rotation free longitudinal part and a divergence free transverse part. This decomposition runs along curved lines.
7. The curvature can be used to define a derived curvature field.
8. The private curvature field of a particle enables the determination of the mass of the particle.
9. Glancing through the pages of the book of Hilbert spaces reveals the dynamics of the system. Dynamics couples the static parts of the fields.

## Open issues

Not treated here are the equations of motion. They are intimately related to what happens during the infinitesimal progression steps that link the members of the sequence of extended Hilbert spaces.

They are also closely related to the dynamic equations that treat Hilbert fields and as can be expected show great resemblance with the Maxwell equations.

One thing can already be stated:

All equations of motion are in fact continuity equations that treat the local information generation, annihilation and transfer.

(8)

Total change within  $V$  = flow into  $V$  + production inside  $V$

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# Part two

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**On the origin of physical dynamics**

## Abstract

When physics must be based on an axiomatic foundation then the law set of traditional quantum logic is a valid candidate. However, at first sight, these axioms do not treat physical fields and they do not treat dynamics. It only prescribes the static relations that exist between quantum logical propositions that treat static subjects. These subjects are considered to be physical subjects or their properties. Amongst these propositions statements exist that describe everything that can be said about the static condition of a given physical item. Such propositions represent that item.

Traditional quantum logic is lattice isomorphic to the set of closed subspaces of an infinite dimensional separable Hilbert space  $\mathbf{H}$ . That is why quantum mechanics is usually done with the aid of Hilbert space features.

The representation of a physical field does not fit in a Hilbert subspace. Physical fields have a universe wide range and their presentation would cover all of a whole Hilbert space.

Piron has shown that a candidate Hilbert space can be defined by using one of three division rings for the specification of the inner products. The choice comprises the real numbers, the complex numbers and the quaternions. The choice for the quaternions means that manipulations of the Hilbert space, such as Fourier transforms, in general use such multi-dimensional numbers.

According to Helmholtz decomposition theorem, the quaternionic Fourier transform can be divided in a complex longitudinal Fourier transform and a transverse Fourier transform. For quaternionic functions this means that they can be *locally* split into a one-dimensional rotation free part and a two-dimensional divergence free part.

This e-paper indicates that traditional quantum logic can be expanded to extended quantum logic, which includes the influences of physical fields in the form of potential propositions that concern virtual items. Dynamic extended quantum logic is lattice isomorphic with the set of subspaces of *a set of* Hilbert spaces. The fields take care of the coherence between these Hilbert spaces.

In this complicated way the axioms of traditional quantum logic form the constraints of the dynamics of quantum physics. When the dynamics of the universe would be put to a hold, then the axioms of extended quantum logic would describe all static constraints that are put to that universe. In the developed model, dynamics means that universe steps from one static status quo to the next. After the step the conditions are changed and the static constraints are reestablished. If we find the laws that control the steps, then we have found a complete axiomatic foundation of physics. Classical physics forms another constraint of dynamical quantum physics. This e-paper studies what happens during the step.

Solutions are given for coping with the inherent countability of the eigenspaces of operators in the separable Hilbert space and for coping with the apparent graininess of some physical quantities. A classification of skew Hilbert fields will be generated that corresponds closely to the Maxwell fields. Further, this e-paper investigates what happens in the infinitesimal steps that nature takes in order to arrive at the next static status quo. In this way the origin of dynamics and the origin of special relativity may be revealed.

The e-paper unifies particle states with physical fields and treats the equivalent of the gravitation field as a derived curvature field.

## Introduction

The aim of this paper is to build upon a fundament consisting of a minimal set of axioms and then derive as much as is possible from fundamental physics by using only purely mathematical methods.

Observations and measurements at low dose rates learn that ALL information that we receive comes to us in small packages that we call quanta. The distribution of the probability of these information carrying quanta determines the picture that we get from reality.

The development of quantum physics in its early days went violently. The consequence is that many of the fundamentals of this theory are not constructed carefully. Fundamental repair is required.

This e-paper repairs the fundamentals without disturbing the building. All equations of motion keep their validity.

## The fundament

This e-paper builds on the following postulates:

1. The rock fundament of physics is an ordered sequence of instances of traditional quantum logic.
2. All physical information is transferred in the form of clouds of information carrying quanta.
3. The shape of this cloud is determined by a probability amplitude distribution that generates a tendency to keep these quanta together.

The first point suggests the name **Hilbert book model** for the model that is described in this paper. Each next page of the book describes a subsequent static status quo.

## Equations of motion

All equations of motion are in fact continuity equations that treat the local information generation, annihilation and transfer.

Total change within  $V$  = flow into  $V$  + production inside  $V$

## The logic of the model

This e-paper is not about reality. It is about a model that could be a reflection of part of reality.

When reasoning about physical reality, it is safe to follow the rules of classical logic. If one starts with a true statement and these rules are followed, then the path of reasoning stays with truth. Classical logic is based on about [25 axioms](#)<sup>9</sup>. A significant part of these axioms defines the structure of the logic as a half-ordered set and some other axioms expand this to define the set as a lattice. The other axioms have more to do with the rules that must be followed in order to reason logically. May be it is a good starting point to use logic itself as a fundament of physics.

In the first decades of the last century it was discovered that nature itself cheats with classical logic. Numerous observations of the behavior of small particles revealed that some of the interrelations between these observations are in conflict with classical logic. Birkhoff and von Neumann interpreted this conflict and came to the conclusion that nature obeys its own kind of logic. They named this logic **quantum logic**.

The model that is discussed here builds its foundation on traditional quantum logic. This e-paper is not about quantum logic. It uses quantum logic because [traditional quantum logic](#)<sup>10</sup> defines the **static** framework in which quantum dynamics takes place. Traditional quantum logic prescribes the potential relations that may exist between quantum

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<sup>9</sup> Appendix: Quantum logic

<sup>10</sup> Appendix: History of quantum logic

logical propositions. Amongst these propositions statements exist that describe everything that can be said about the static condition of a given physical item. Such propositions represent that item. These propositions form the top of a hierarchy of propositions that treat the current values of the properties of the considered item. It means that traditional quantum logic can represent physical items.

Traditional quantum logic is lattice isomorphic to the set of closed subspaces of an infinite dimensional separable Hilbert space  $\mathbf{H}$ . That is why quantum mechanics is usually done with the aid of Hilbert space features. The representation of a physical field does not fit in a Hilbert subspace. Physical fields have a universe wide territory and their presentation would cover all of a complete Hilbert space.

Piron has shown that a candidate Hilbert space can be defined over one of three division rings. The choice comprises the real numbers, the complex numbers and the quaternions. The choice for the quaternions means that manipulations of the Hilbert space, such as the Fourier transforms, in general operate on these multi-dimensional numbers. In the model the representations of physical fields are Hilbert fields. Hilbert fields are blurred Hilbert distributions. Hilbert distributions are sets of Hilbert vectors. The blur is a local field excitation that is attached to a Hilbert vector. The blur is characterized by a continuous spread function. This spread function represents a probability amplitude distribution. The values of this function may reach all Hilbert vectors. In this way these fields not only cover the whole separable Hilbert space, but they also become differentiable.

Apart from Hilbert fields the much simpler Hilbert functions exist. Hilbert functions can be defined with the help of a normal operator. Using the eigenvalues and the inner products of the eigenvectors with a selected Hilbert vector that vector can be converted in a hyper complex function. Hilbert functions can be interpreted as sampled versions of continuous functions. They are not differentiable. The eigenspace of a

quaternionic normal operator may consist of a number set that is everywhere dense in the quaternionic number space. For example it may consist out of all rational quaternions. When the eigenspace is granular it can still cover the whole quaternionic number space, but it will have a lattice structure and this lattice will show preferred directions. Another possibility is the eigenspace of the strand operator. It does not cover the whole quaternionic number space. It possesses horizons. Within the outer horizon and outside the inner horizons its chains can move freely through the imaginary part of the quaternionic number space. All these operators can be used to construct Hilbert functions. In order to represent the anchors of physical particles Hilbert fields seem to be better suited than Hilbert functions. By using suitable blurs the Hilbert fields are differentiable. The Hilbert functions are not differentiable. At the utmost they are quasi differentiable. Still Hilbert functions can act as Hilbert distributions that are used to form Hilbert fields.

As a consequence, the theory that is derived here is largely based on the properties of these multidimensional transforms and on the properties of Hilbert fields. Any Hilbert field can be split in a rotation free longitudinal part and a divergence free transverse part. The direction in which a field is rotation free may change with the values of the local coordinates. As long as the direction stays stationary, the corresponding coordinates can be considered as belonging to a complex plane that is embedded in a quaternionic space. Selection of another coordinate system gives a different topology of the field decomposition.

The blurs that constitute the Hilbert fields do not fit inside the realm of an infinite dimensional separable Hilbert space, but their values can be temporary attached to Hilbert vectors. On the other hand the separable Hilbert space  $\mathbf{H}$  can be embedded in a rigged Hilbert space  $\mathbf{H}$  in which the blurs of a Hilbert field get a better supported place. This picture extends the Gelfand triple to a **Hilbert sandwich** that apart from the Gelfand triple consists of a GPS coordinate system and a covering field. The combination of GPS coordinate system and the covering field

decomposes the **static** covering field into a patch cover. The decomposition defines a curvature. That curvature defines a derived field which is also part of the sandwich<sup>11</sup>. The sandwich only represents a static status quo.

Fourier transforms can be defined in a separable Hilbert space, but there they expose sampling characteristics that do not occur in a corresponding rigged Hilbert space  $\mathbf{H}$ . Similarly the notions of differentiation and integration are easily implemented in a rigged Hilbert space  $\mathbf{H}$ , but do not fit in the corresponding separable Hilbert space  $\mathbf{H}$ . Without the blurring trick, differentiation is impossible in the realm of a separable Hilbert space  $\mathbf{H}$ .

However, the set of closed subspaces of a rigged Hilbert space  $\mathbf{H}$  is no longer lattice isomorphic with the set of propositions in a traditional quantum logic system. We do not want to offer the isomorphism with quantum logic in order to achieve differentiability of functions. This differentiability is already introduced by the blurs that are attached to the Hilbert vectors. This approach delivers a cleaner model that becomes even better comprehensible when later on we interpret the blur as a probability amplitude distribution. Further, nature has a fundamental granular character, which fits naturally to a separable Hilbert space, but does not naturally match with a rigged Hilbert space.

In a three dimensional vector space a Fourier transform of a vector field can locally be divided in a one-dimensional longitudinal, (locally) complex Fourier transform and a two-dimensional transverse Fourier transform. The longitudinal transform works only on the longitudinal part of the field that is being transformed. The transverse transform works only on the transverse part of the field that is being transformed. This also applies to the case where this vector space is formed by the imaginary quaternions and the fields have quaternionic values.

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<sup>11</sup> In this way the sandwich starts to resemble a club sandwich.

The division in a longitudinal part and a transverse part of a function or a field *has only a local validity*. It holds as long as the longitudinal direction is sufficiently (= within accepted inaccuracy) stationary. The split is the subject of the Helmholtz decomposition theorem. Multi-dimensional Dirac delta functions show the same decomposition as the multi-dimensional Fourier transform. The splits lose their significance when the field gets too wild.

The fact that this field categorization has only local validity and that it is related to an imaginary direction causes that the quaternionic Fourier transform must be considered to operate in a curved coordinate space. The differentiability of quaternionic functions and Hilbert fields also offers this categorization. Interestingly, Fourier transformation converts differentiation into multiplication with the canonical coordinate.

For a given field this situation can be solved by using two coordinate systems. One in which the coordinates show a curvature of the field that is set by the longitudinal direction and one in which the field has no curvature. The field values stay the same, but the coordinates that act as parameters change. This concept can be extended to a covering field, which is the superposition of all Hilbert fields that exist in the Hilbert space. Using the coordinate system for which the covering field has no curvature the **universe wide Fourier transform** can be taken.

For a given field and a given coordinate system it is possible to define a decomposition related local curvature. That curvature can be used to define a derived field. We will call this partner field the **curvature field** of that combination.

In a given coordinate system the fields can be categorized according to their symmetry properties. These categorizations must also cope with the curvature that is related to that coordinate system.

In the described way, traditional quantum logic and the Helmholtz decomposition theorem together form a set of laws that define the relational static status quo that would exist in nature when dynamics could be put to a hold. This paper points out that traditional quantum logic can be expanded such that it includes the representation of *static* physical fields.

The indifference of the properties of physical items for which picture is used, the Schrödinger picture or the Heisenberg picture<sup>12</sup> indicates that time is not a property of physical items but instead it is a parameter that characterizes the progress of dynamics. This is the reason why dynamics can be included into the model by representing nature by a sequence of such extended quantum logics. The fields regulate the coherence between subsequent quantum logics. This also means that the model can include dynamics by representing nature by a sequence of Hilbert spaces. The blurs in the Hilbert fields regulate the coherence between subsequent Hilbert spaces. It means that the blurs are smooth functions of the progression step counter. The progression step counter is a global parameter! It differs from our common notion of time. The blur acts as a probability density distribution. When the parameter is a position coordinate, then the probability density specifies the chance that during the next change the current position changes to this new coordinate. The form of the probability density distribution is such, that this change is has a tendency to be minimal. The probability density is the squared modulus of the hyper complex probability amplitude. This last value contributes to the local field value.

As a consequence of these blur properties, the subsequent Hilbert spaces do not differ much. The same holds for the configuration of the fields in sub-sequent stages of the static status quo. In fact the fields can be seen as a storage place for the conditions that determine the relation between the past, the current and the future static status quo.

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<sup>12</sup> Dynamics: Schrödinger or Heisenberg picture

In this view the fields represent relations between sets of potential propositions. These propositions are statements that say everything that can be said about the static condition of virtual items. The set of propositions in dynamic extended quantum logic is no longer isomorphic with the set of closed subspaces of a single Hilbert space. It is isomorphic with the closed subspaces of a series of Hilbert spaces. One member of this set of Hilbert spaces is the currently actual Hilbert space. It contains the representatives of actual physical items. The other members are *virtual* Hilbert spaces. They contain the representations of virtual physical items.

Each virtual Hilbert space corresponds to an instance of a virtual (traditional) quantum logic. These virtual quantum logics represent potential replacements of the actual traditional quantum logic. A virtual quantum logic differs from other virtual quantum logics in the fact that their propositions have a different configuration in terms of their atomic predicates or in terms of their sub-ordered propositions. In a similar sense they will differ from the actual quantum logic.

The replacement will be made between logics that have a great resemblance with each other and the values of the predicates within equivalent propositions will be the same or close to each other. A replacement can be seen as a combined annihilation and creation. Annihilation must not be followed by creation and creation must not be preceded by annihilation. With other words annihilation and creation is done during progression steps.

A redefiner, which steps from the actual Hilbert space to a virtual one, implements dynamics. The redefinition step exchanges the actual Hilbert space against a virtual Hilbert space whose selection is determined by the current fields.

In order to be able to control the coherence between subsequent Hilbert spaces, the blurs that constitute the Hilbert fields act as probability

density distributions. In fact, they are probability amplitude distributions whose squared modulus is a probability density distribution. These distributions have a form that minimizes change during the step from the current Hilbert space to its successor. As a consequence physical quantities do not become observable as continuous objects. Observables become available in the form of information carrying quanta that form the outcome of stochastic processes. The form of the clouds of information quanta is described by the probability amplitude distributions that together form the Hilbert fields.

Dynamics can be interpreted as a sequence of steps in which each step leads nature from the conditions of one static status quo to the conditions of the next static status quo. The laws that define the static status quo are fairly clear. During the steps several things happen. The laws that govern the dynamics are rather obscure. The steps couple the static ingredients into a dynamic mixture. For example, the step couples the longitudinal part of the field with its transverse part. The steps are taken universe wide. A redefiner with a universe wide domain controls these steps. The step counter presents a universe-wide progression parameter. This parameter must not be confused with our common notion of time, but it cannot be denied that it has some relation with it.

During the step particles move forced by their own momentum and by the surrounding covering field to their new position. As a consequence their private fields get redistributed in space. Thus the covering field and its derived partner the curvature field will change. This delivers the preconditions for the next step. These activities are all infinitesimal.

Inertia shows that fields do not interfere with uniform movement. However, acceleration goes together with an extra field contribution. The words "goes together with" mean that no causal relation exists. Inertia represents the influence of the whole universe on the condition of a local physical item. In fact it is a bilateral relation. Only the curvature field

takes part in inertia. The primary fields have charges that compensate each other's universe wide contributions.

Thus, the changes of the curvature field go together with acceleration of the particles. It is already indicated that uniform movement causes a reconfiguration of the covering field. This is all that happens during the infinitesimal progression step.

We must now analyze what acceleration and reconfiguration does during an infinitesimal progression step.

[Inertia](#)<sup>13</sup> can guide the way. Roughly, the driving force comes from the difference  $\Delta\mathbf{E}(\mathbf{r}, t)$  between the current curvature field and the previous curvature field. It is contained in an equivalent of the stripped version of one of [Maxwell's equations](#)<sup>14</sup>.

(1)

$$\Delta\mathbf{E}(\mathbf{r}, t) = G \cdot \frac{\partial\mathbf{v}}{\partial t}$$

$G$  is the gravitational constant.

During the progression step, the described infinitesimal adventure happens to all particles.

Please notice the switch from covering field to curvature field. It is essential!

### Recapitulation and extrapolation

In summary: Traditional quantum logic is usually defined via its structure as an orthomodular lattice. This logic only defines part of the *static* skeleton of the frame in which quantum physics operates. It does not state anything about physical fields. The Helmholtz/Hodge decomposition theorem defines the structure of static physical fields. In that way this theorem plays a similar role as traditional quantum logic. However, the

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<sup>13</sup> Influence: Inertia

<sup>14</sup> Dynamics: Unitary transform: Inertia and the progression step

decomposition has only local validity, where quantum logic has global validity. Extended quantum logic encompasses both law sets. These law sets do not specify or even touch the source of *dynamics*. Dynamics couples the static fields. The coupling not only applies to parts of the same field. It also concerns different fields. For example dynamics couples electrostatic fields with magnetostatic fields into dynamical electromagnetic fields. The gravitation field describes the curvature of observable space that is caused by the decomposition properties of the other fields. Thus, instead of a separate field the gravitation field can be considered as the result of the rotation properties of the other fields. [Inertia<sup>15</sup>](#) reveals the importance of the gravitation field.

Both the propositions about a quantum physical system and physical fields are closely related. However, this relation only gets relevant when dynamics comes into play. Dynamics causes a continuing redefinition of the propositions. This converts the current static status quo into the next one. When one proposition is changed it interchanges its constituting atomic predicates with other predicates. The change can even involve the exchange of atomic predicates against atomic predicates that are of another type. It is also possible that the configuration of a complex system that consists of simpler components is altered.

The static physical fields can be interpreted as storage of the preconditions for the next step. The physical fields are the representatives of the influences that go together with the sticky resistance of the set of propositions against the changes that occur due to the redefinitions of the propositions that describe physical items. This sticky resistance also occurs in propositions that are sub-ordered to other propositions. Inertia is a feature that shows this resistance explicitly.

The propositions about quantum physical items can be represented by closed subspaces of a Hilbert space. The presence of dynamics means that

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<sup>15</sup> Influence: Inertia

the relations between these subspaces are not stationary. They change between subsequent Hilbert spaces. It is also possible to give the physical fields a representation by attaching them to the Hilbert spaces. Their values can be attached to Hilbert vectors. However, it must be clear that quantum physical items and physical fields are not the same stuff. Physical fields cannot be represented by closed Hilbert subspaces. They cover the whole universe and as a consequence they cover the whole Hilbert space. However, the strength of individual fields may be concentrated around separate excited places that are represented by single Hilbert vectors or a small set of Hilbert vectors. Such private fields diminish with distance. Together the private fields form a covering field. For a given coordinate system that covering field has a partner curvature field that can be interpreted as gravitational field.

The actual physical items are distributed in space and are surrounded by potentials that act as a kind of blur. This is why quantum physics has much in common with [optics](#)<sup>16</sup>. The superposition of the separate blurs characterizes the information transfer quality of the corresponding field. For each particle a separate blur characterizes the quantum generation properties of that particle. At not too short distances the electromagnetic fields have the same shape as gravitational field. Locally, the EM fields and the gravitation field are based on the same Hilbert distributions. As is indicated above, the gravitation field is a derived field. The main difference lays in the fact that the charges of electromagnetic fields have the same size but may have different sign such that they may partly compensate each other's influence. The charge (mass) of the gravitational field is always positive, but it may differ in size. Another difference is that the gravitation field is the consequence of the decomposition properties of the other fields. Mass appears to be an expression of space curvature and on its turn this curvature is an expression of the rotation properties of the non-gravitational fields. The curvature fields that correspond with private fields do not compensate each other's influence. The masses of all

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<sup>16</sup> Optics

physical items work together in order to create the immense potential that causes [inertia](#).

## GPS coordinates

One of the most intriguing facts is that a [GPS operator](#)<sup>17</sup> that resides in the separable Hilbert space  $\mathbf{H}$  cannot be used to define the position of particles. Due to the granularity of its eigenspace it would immediately introduce unnatural preferred directions. In contrast its equivalent, the GPS operator that resides in rigged Hilbert space  $\mathbf{H}$  is very useful as a coordinate system for determining the location of field values. In this way it can be used to locate the field excitations that go together with particles. This continuous GPS operator cannot be used directly in order to locate the Hilbert vectors that represent particles. Thus there exists no GPS like operator that can be used to locate particles in Hilbert space. An alternative is formed by the [strand operator](#)<sup>18</sup>. The strand operator uses the continuous GPS operator as a background coordinate system. Its eigenspace depends on the configuration of the covering field.

## Test proposition

It is an elucidating experience to try to implement a complicated quantum logical proposition in the representation of quantum logic in Hilbert space. In that way we may discover how dynamics emerges in this static skeleton. For that reason, we choose as an example a predicate with quantifiers rather than a clean proposition.

The selected example proposition ( $\spadesuit$ ) is

“All items in universe influence each other’s position”.

We will already give the final conclusion of this experiment here: A well-ordered replacement of atomic predicates in an enveloping proposition

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<sup>17</sup> The Hilbert GPS

<sup>18</sup> Hilbert spaces: Strand operator

appears to occur without extra field activity, but any deviation of a well ordered replacement causes an extra field activity in the form of influence of the complete set of all propositions.

This explains the interaction between fields and physical items. A local deviation of the uniformity of the distribution of physical items can still cause a slight influence of neighboring items. At small distances the influences can be large. The influence of fields can be implemented in the separable Hilbert space. Via an action = reaction game the interaction between fields and Hilbert subspaces form the source of dynamics.

What further happens during the implementation of our example proposition (♠) is completely governed by mathematics. Thus, *for our example* no further extension of quantum logic is needed to transform it into a useful version of dynamic quantum logic. However, nothing is said yet of what occurs during the infinitesimal progression steps. During this step one static status quo is converted to the next static status quo. This will be the main subject of this e-paper.

## Numbers

As number spaces we use the  $2^n$ -ons of Warren Smith rather than the hyper complex numbers based on the Cayley-Dickson construction. Up to the octonions the corresponding number spaces are similar. (See <http://www.math.temple.edu/~wds/homepage/nce2.pdf><sup>19</sup>). For higher  $n$  the  $2^n$ -ons behave in a nicer way. They keep more of their number characteristics. We use the quaternions ( $n=2$ ) as the number space that is used to define the inner product of the Hilbert space. However, we tolerate operators to have eigenvalues that are higher dimensional  $2^n$ -ons. We also use  $2^n$ -ons in order to set the values of physical fields.

When we use these numbers as eigenvalues or as field values, then we apply their number characteristics as well as their storage capacity. A  $2^n$ -

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<sup>19</sup> Appendix:  $2^n$ -on construction

on contains  $2^n$  real numbers. We also tolerate that eigenvalues of operators and values of fields support multiple sign selections, such as the inversion of the real axis and the handedness (chirality) of external vector products for their eigenvalues.  $2^n$ -ons offer  $n$  sign selections and contain  $n$  independent imaginary base numbers. Each new independent base number introduces a new sign selection. The sign selections translate into an  $n$ -fold hyper complex conjugation and  $2^n$  different skew fields.

With  $n > m$ , the  $2^n$ -ons act like  $2^m$ -ons in their lower  $m$  dimensions. Further, the  $2^n$ -ons contain several subspaces of  $2^m$ -ons. We may use smoothly curved manifolds that are crossed by curves which form trails of  $2^n$ -on numbers and that are locally touched by tangent spaces that can be interpreted as  $2^n$ -on number spaces.

When the members of a set  $2^n$ -ons approach zero, then in their mutual arithmetic actions they are getting more and more the characters of lower dimensional  $2^m$ -ons. In the same sense, when two  $2^n$ -ons approach each other, their mutual arithmetic actions are getting more and more the characteristics the arithmetic of lower dimensional  $2^m$ -ons.

The implementation of the proposition ( $\spadesuit$ ) leads to a story of manipulators and manipulated observables. The number waltz feature ( $c=ab/a$ ) of the  $2^n$ -ons that becomes a noticeable effect for  $n>1$  seems to play a significant role in our model. If this model applies to quantum physics, then it may reveal why special relativity exists and brings clearness in the different notions of time that exist in quantum physics. The curvature introduced by the spatial variance of what the longitudinal direction is reveals how the mentioned influences can be implemented as component fields which are defined on a curved coordinate system. This holds for gravitational fields as well as for the other fields such as electromagnetic fields.

Implementing quantum physics in a complex Hilbert space hides these interesting features and diminishes the insight that higher dimensional  $2^n$ -ons can reveal.

## Prospect

The article shows that there is a need to extend traditional quantum logic such that it not only includes the representations of fields but also includes axioms, which specify the dynamic underpinning of quantum physics.

In the course of this project several fundamental aspects of physics get uncovered.

## Comments

### Version 3

Version 3 builds on the content of previous versions. This new version stands on itself, but it reorganizes and extends the contents of versions 1 and 2. One reason is that the text in some paragraphs relies on the text of a series of other paragraphs, so that it is impossible to configure the paragraphs in a streaming order without repeating much of the content. Here we may take the solution to refer to future paragraphs and add a “back to XXX” reference after the target text.

## Project

This project is far from finished. Most parts I have rewritten several times. Some ingredients are already included before they are finalized and before they are put at the proper position in the context. I try to make the whole paper consistent with its parts and I try to keep my goal to include nothing that did not follow directly or indirectly from the axioms of traditional quantum logic. I only tolerate mathematics as a valid tool and ingredient. I will not use or accept intuition as a reason to include a subject in the text. However, I will use indications retrieved from previous experiences. I will also not tolerate the usefulness of a concept or its acceptance in the physical community as a valid argument to include that concept. If you encounter places where I did not succeed in that goal, then you may conclude that I still have to work on that section. When the paper gets its final version, then no deviations of my goal should result. However, partly due to my progressed age (~70), I might never reach that

condition. Then, *you* reader might take over and finish the job. But first think of the possibility that we will succeed. *What does it mean that all of fundamental physics is based on mathematics and on the ~25 axioms of traditional quantum logic?! Well, another input is the observation that ALL information about nature arrives in the form of clouds of information carrying quanta. The cloud as a whole carries secondary information.*

## References

This e-paper contains no lengthy reference list. References to other documents are usually presented inline and are mostly put in the form of hyperlinks. The appendix and a sometimes referenced [toolkit<sup>20</sup>](#) contains a collection of stuff that otherwise must be grasped from internet. As is done in this article, much of the contents of the toolkit are directly or indirectly obtained from Wikipedia or from publicly accessible publications. In that case the text is adapted to the requirements of the papers that use this information. Most texts on internet are based on complex Hilbert spaces, so where necessary I have converted these texts into quaternionic versions.

## Equation editor

This paper is prepared with MS Word 2010. This word processor version contains a rather capable equation editor and a large series of fonts including Cambria Math. However, the equation editor does not cooperate with the paragraph indexing in order to automatically enumerate the out of line equations. For that reason equations are enumerated manually and relative to the current paragraph header. References inside that paragraph just use the equation number. References from outside of the paragraph are hyperlinks that refer to the paragraph header. The hyperlink text may then include the equation number. In that case, you must move manually to the referred equation inside the target paragraph.

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<sup>20</sup> <http://www.crypts-of-physics.eu/Toolkit.pdf>

## EM fields

This paper draws significantly from the book on electromagnetic field theory of Bo Thidé. That book has a different goal and uses different premises. The book does not use the quaternionic field approach as is done here, but its contents easily translate to quaternions. Further its formulation is very precise, it links formulas to physical concepts and most of all it is online:

[http://www.plasma.uu.se/CED/Book/EMFT\\_Book.pdf](http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf)<sup>21</sup>.

## Strands

I took many ideas from the research of Christoph Schiller as it is presented in his online book

<http://www.motionmountain.net/research.html> .

If strand theory is a valid approach to a model of physical reality and if the theory presented here is also a valid approach, then strands must be representable in this model. For that reason examples of possible embedded definitions are presented. First strands are specified as [item type definitions](#)<sup>22</sup>. There a single strand closely equals a photon type.

Next a normal operator is defined that has [eigenvalue sets in the form of strands](#)<sup>23</sup>.

## Notation note

This paper uses  $\{ \}$  in order to indicate a set or a function.

Depending on the context  $\{ |f_s\rangle \}_s$  means an ordered set of vectors  $|f_s\rangle$  where  $s$  is the ordering index. In other contexts  $\{ |f_s\rangle \}_s$  means a vector function  $|f(s)\rangle$  where  $s$  is the (discrete or quaternionic) parameter.

Continuous functions are presented in the normal way.

$f(\{q_j\}_j)$  is a function  $f(q_1, q_2, q_3, q_4, \dots, q_n)$  of the set of parameters  $(\{q_j\}_j)$ , where  $j = 1, 2, \dots, n$ . The index constraint  $n$  might be infinity. Nature itself is finite, however it lives in a model that has an infinite scope.

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<sup>21</sup> Hilbert field equations

<sup>22</sup> Logic: Item types: Strands as type definitions

<sup>23</sup> Hilbert space: Limitedness: Strand operator

The appendix and the toolkit contain information about other notation and naming conventions that are used in this paper.

## Acquired indications

Several indications stimulated the development of the theory that is presented in this paper. They are listed in this section.

### Studying physics

When I was studying physics I was triggered by two facts that have significant influence on my current insights.

### Classical versus quantum physics

After finishing the semesters that treated classical physics I started taking lessons in quantum mechanics and I was immediately amazed by the large difference in the way that classical mechanics was handled and the way that quantum mechanics was done. Questioning the teachers did not bring much relief. Their explanation was that the difference is due to the superposition principle. Investigating this reply reveals quickly that this is an alternative description of the different way of working, but no explanation. So, I dived into the library and into scientific bookshops until I finally found a booklet from P. Mittelstaedt: (*Philosophische Probleme der modernen Physik*, BI Hochschultaschenbücher, Band 50, 1963) that contained a chapter on quantum logic. I concluded that this produced the answer that I was looking for. Small particles obey a kind of logic that differs from classical logic. As a result their dynamic behavior differs from the behavior of larger objects. I searched further and encountered papers from Garret Birkhoff and John von Neumann that explained the correspondence between quantum logic and Hilbert spaces. In those years C. Piron wrote his papers that finalized my insight in this subject, but first I must explain the other fact that triggered me.

### The rediscovery of quaternions

Quantum physics appeared to be done in the realm of Hilbert spaces. Operators in those spaces delivered the eigenvalues that played the role of values of observable quantities.

I had problems with the fact that according to the in those days commonly accepted theory the operators, which deliver observable values

as their eigenvalues or as their expectation values, had to be self-adjoint and as a consequence these operators could only deliver real valued eigenvalues. Nature has a clear 3+1 dimensional structure and there was no logical indication in the quantum theory that was lectured in those days that explained why four observable values must cling together. I started searching for a number system that could deliver this extra connectivity and I quickly discovered a number system with 3+1 dimensions that supported addition, multiplication and division. It took me more time to discover that this number system was already discovered more than a century before by [William Rowan Hamilton](#)<sup>24</sup> when he was walking with his wife over a bridge in Dublin. He was so glad about his discovery that he carved the corresponding formula into the sidewall of the bridge. The inscription has faded away, but it is now molded in bronze and fixed to the same wall. When an assistant professor told me the story I started to read papers on quaternions and discovered the work of Constantin Piron.

Birkhoff and von Neumann already discovered that the set of propositions in a traditional quantum logic system is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space  $\mathbf{H}$ . Piron proved that the inner product of this Hilbert space must be specified with members of a division ring. There are only three division rings: the real numbers, the complex numbers and the quaternions. I went for the widest choice and started studying quaternionic Hilbert spaces.

### Representation restriction

After discovering that traditional quantum logic can be represented inside an infinite dimensional separable Hilbert space, it is a disappointment to discover that this presentation does not cover physical fields and does not cover dynamics.

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<sup>24</sup> [http://nl.wikipedia.org/wiki/William\\_Rowan\\_Hamilton](http://nl.wikipedia.org/wiki/William_Rowan_Hamilton)

### Extended quantum logic

However, it appears that it is possible to expand traditional quantum logic in a way that corresponds to adding blurs to a subset of the Hilbert vectors. The blurred subsets may represent particles. In this way the superposition of the blurs may represent the physical fields. This leads directly to the existence of Maxwell-like fields in the realm of such an extended Hilbert space. This also leads to an extended quantum logic that covers physical fields.

### Dynamic quantum logic

Dynamics can be implemented by representing dynamic quantum logic as a sequence of extended traditional quantum logics that each represents a static status quo. This dynamic step can be detailed further. See [Progression step details<sup>25</sup>](#). With respect to the Hilbert space the dynamic model uses a sequence of extended Hilbert spaces. It can be compared to a book, where each page represents a static status quo.

### Curved space

When quaternions are taken as the division ring, then Fourier transforms become quaternionic Fourier transforms. The ideal Euclidean formulation of the multi-dimensional Fourier transform cannot cope with a variable direction in which the Fourier analyzed function or field is rotation free. This can be circumvented by converting the field by a coordinate transformation such that the resulting field has a stationary direction in which it is rotation free. This corresponds with accepting the existence of a curved coordinate space. This curved space is subject of general relativity. With other words, extended quantum logic supports general relativity.

### Intensified imaging

After finishing my study I started my career in the development laboratory for high-tech electronic appliances of a big electronics company. My task consisted of the analysis and measurement of the

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<sup>25</sup> Acquired indications: Progression step details

visual trajectory, starting from the radiation source and ending after interpretation of the image in the brain of the observer. At those times (~1975) this was fundamental research, because both the measuring methods and the modeling methodology in this area were still in their childhood. The target products for the laboratory were night vision devices and X-ray image intensifiers.

Intensified imaging is required at low dose rates and in situations where the radiation detection capability of the human eye is unsuitable. This occurs with starlight scene imaging and with X-ray shadow imaging. The low dose rate is necessary due to the fact that no active scene lighting can be supplied or due to the fact that hazardous gamma ray effects must be avoided.

When the snowing image produced by image intensifying equipment is observed, then it becomes immediately clear that this image is built up from a large number of separate spots that together form a rather noisy picture of the object.

The research not only concerned perception experiments and measurement. We also had to devise the standards for the measurements as well. So we took part in the establishment of develop worldwide standards for the specification and measurement of the Optical Transfer Function (OTF) and its modulus the MTF. We also took part in the committees that created the standards for the Detective Quantum Efficiency (DQE).

The fact that these standards were not only required but were also successful is in itself very astonishing. We needed these standards because we could model the visual trajectory as a chain of which the first elements consisted of a set of Poisson generators. The generators are characterized with their efficiency and a spatial, angular and chromatic distribution.

Next in the chain are attenuating binomial processes. Statistically a blur can also be considered as a binomial process. The information is spread over a larger area. A Poisson process can be combined with a subsequent binomial process into a generalized Poisson process that has a lower efficiency.

The chain also contains light lenses and particle lenses. Further, the equipment aided chain contains detection surfaces that convert radiation quanta into electrons or electrons into radiation. This chain might also contain scintillation layers that convert high energy X-ray quanta in large series of low energy light quanta. It might contain fiber optic plates that just transport images, usually from a curved to a flat surface. It might contain channel plates that convert single electrons into clouds that contain about hundred thousand electrons. It might contain image receivers that convert the image into an electric signal or into a photographic plate.

The equipment aided chain may and the unaided chain will also contain the eye of a human observer. Intensified images are detected by the cones in the fovea. At very low light levels the adapted eye detects the images via the rods in the fovea. Rods have a much lower acuity than cones. Therefore they have a much longer integration time. In general, measures that reduce noise have both positive and negative effects on the information content of the signal. There is an optimum condition. My task was to find that condition.

After the detection in the fovea the received signal is handled by a large series of preprocessors that act in parallel as well as in sequence. The preprocessors associate the signals that are received by receptors that lay in each other's neighborhood. The association tests a detail pattern that is typical for the considered preprocessor. The associated signal is only passed further when its signal to noise ratio surpasses a given boundary level. In this way the higher regions of the information processing are not disturbed by unnecessary noise.

All preprocessors work in this way as noise filtering decision centers. The association results in a categorization of the encoded image. The signal that reaches the folded fourth layer of the visual cortex represents the completely coded version of the received image. In the human brain, a folded surface of about four square millimeters is devoted to each image receptor in the fovea. This code is interpreted further in the brain. As early as possible the filtering process stops noise and details of the image that do not carry useful information from proceeding further in the chain.

Due to this design, already the unaided brain-eye combination is well suited to perceive and interpret images in a very large dynamic range of circumstances. Apart from the fact that the visual channel can adapt from somewhat above starlight conditions until bright daylight conditions, the visual trajectory appears to be optimized for handling signals that enter the eye in the form of clouds of quanta that are generated by Poisson processes.

All vertebrate visual trajectories work according to the sketched principles. Over billions of years evolution has exploited the fact that information that comes to living species is generated by Poisson processes. The visual trajectory of vertebrates is optimized for handling this information for the benefit of the survival of the owner of this channel. See: <http://www.crypts-of-physics.eu/Howthebrainworks.pdf><sup>26</sup>.

This fact is a strong indication that all visual information comes to us in the form of clouds of quanta. When looking at low dose rates through an intensified viewer, it becomes clear that this assumption is valid. The perceived noisy image is built from separate dots that represent the detected quanta. No radiation wave is visible. What you see is just a streaming cloud of quanta.

The fact that visual information is generated by Poisson processes indicates a more general feature of physics. ALL information that is

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<sup>26</sup> Part three: How the brain works

transferred by electromagnetic radiation can be considered to be generated by Poisson processes. This becomes apparent when measurements are done at very low dose rates. In the static model the Poisson processes represent a lateral distribution. In addition, taken over a sequence of Hilbert spaces the Poisson processes represent a temporal distribution.

## Granular GPS

In separable Hilbert space a normal operator has a countable eigenspace. This allows to the usage of the rational quaternions as the eigenspace of a normal operator. However, nature appears to support a minimum for the distance between two positions. This renders a position related operator granular. In separable Hilbert space  $\mathbf{H}$  the granularity of the eigenspace of a GPS-like coordinate operator presents problems with the fact that a dense packaging of the granules generates unrealistic preferred directions. Its equivalent in the corresponding rigged Hilbert space  $\mathbf{H}$  does not suffer this restriction. It can be used as coordinate system for fields, but it cannot be used to locate particles inside the separable Hilbert space  $\mathbf{H}$ . The fields are attached to a subset of the Hilbert vectors and all Hilbert vectors touch their values.

## Progression step details

After the former indications the theory reaches the stage that it becomes sensible that the model of nature, which takes its foundation on quantum logic, steps from one static status quo to the next. It dawns that this is the way that dynamics is implemented. What happens during these steps is still mysterious. The Hilbert space itself only suggests a Euclidean signature of observable space time. However, Einstein proved that observable spacetime has a Minkowski signature. This discrepancy must have its origin in what occurs inside the dynamic step. An early conclusion is that coordinate time does not play the role of the fourth dimension in the quaternionic eigenspace of a spacetime-like operator. It also differs from the role of the counter of the progression steps.

Both inertia and Feynman's approach of the path integral may guide what happens during an [infinitesimal dynamical step](#)<sup>27</sup>.

### **Release and removal of quanta**

During the step interactions take place and particles are emitted or absorbed. The information is carried by clouds of quanta. The quanta carry the information that they collect GPS and GMS related data in the dynamical step.

An indication for this fact houses in the structure of the creation and annihilation operators. These operators consist of a part that relates directly to the GPS operator and a part that directly relates to the GMS operator.

### **Fields and probability amplitude distributions**

Some subsets of Hilbert vectors represent elementary particles. It means that they are blurred. The blur is a probability amplitude distribution whose form is typical for the elementary particle type. Elementary particles combine to form more complex particles.

The superposition of all probability amplitude distributions that correspond to the separate particles forms the covering field.

A repositioning of a particle means a reconfiguration of the covering field and vice versa.

### **A detailed list of indications and considerations**

1. All information comes to us in the form of clouds of quanta.
2. These clouds get their shape via a combination of probability amplitude distributions.
3. Each type of elementary particle is characterized by a set of Hilbert vectors and a particular kind of probability amplitude distribution.
4. The information contained in the quanta and in the cloud is the only information that becomes observable.

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<sup>27</sup> Dynamics: Unitary transform: Infinitesimal dynamical step

5. This information consists of the information that is carried by the separate quanta and by the probability distribution that describes the cloud.
6. Each quantum in the cloud carries a set of information data.
7. This set contains a 3D position, a 3D momentum and a binary chirality.
8. The information that is carried by the quanta becomes available via an interaction process.
9. The information is measured in Planck units, eventually related via physical constants, such as the speed of light.
10. The probability amplitude distribution that characterizes a particle becomes part of the field that exists in the surroundings of the particle.
11. Physical fields consist from the superposition of the probability amplitude distributions of the separate particles.
12. Curvature and torsion of the path of the particle are secondary characteristics, which are introduced via the probability distributions that make up the field that exists in the direct environment of the particle.
13. In contrast to torsion, curvature appears to be linked with gravity. The photon path has a helix structure. It has no mass.
14. Curvature in the path of a particle is caused by the local rotation that exists in the surrounding field(s).
15. The rotation properties of the field determine the local decomposition of the static field.
16. This local decomposition determines a curvature of observable space.
17. On its turn this curvature specifies a curvature field.

18. The curvature field has all the characteristics of the gravitation field.
19. The generation of a given kind of quantum has a typical probability.
20. There exist anti-quanta. The generation of an anti-quantum is equivalent to the annihilation of the corresponding quantum.
21. Creation and annihilation operators have probability amplitude distributions as their eigenfunctions.
22. In their simplest form these probability distributions are Poisson distributions.
23. The generation of shot noise is characterized by Poisson distributions.
24. At high dose rates the Poisson distributions become Gaussian (normal) distributions.
25. For more complicated configurations the probability amplitude distribution must be considered rather than its squared modulus: the probability density distribution.
26. Bosons are characterized by probability amplitude distributions that remain invariant under a rotation of  $2\pi$ .
27. The probability amplitude distribution of a two boson system is invariant under perturbation of the bosons.
28. The creation and annihilation operators of bosons are characterized by a non-zero commutator.
29. Photons form the simplest boson type. They have a helix form.
30. The probability distribution of the corresponding quanta resembles a Poisson distribution.
31. Fermions are characterized by probability amplitude distributions that change sign under a rotation of  $2\pi$ .

32. The probability amplitude distribution of a two fermion system changes sign under perturbation of the fermions.
33. With each fermion type an anti-type exists.
34. A quaternionic probability amplitude distribution can also contain chirality information.
35. When chirality is taken into account then a probability amplitude distribution must be used rather than a probability density distribution.
36. Electric charge is related to the chirality properties of the corresponding particle.
37. The creation and annihilation operators of fermions are characterized by a non-zero anti-commutator.
38. Creation and annihilation operators can be split in a part that resides in configuration space and a part that resides in Fourier space.
39. A quant can be emitted (created), absorbed (annihilated) and it can be virtual.
40. Virtual quanta belong to previous or future events.
41. Only actual quanta deliver observable information.
42. Emitted and absorbed actual quanta belong to the current version of events.
43. During each dynamical step information is collected both from configuration space related sources and from momentum space related sources.
44. The part of the collected information that resides in configuration space delivers the 3D position information to the emitted/absorbed/virtual quant.

45. The part of the collected information that resides in Fourier space delivers the 3D momentum information to the emitted/absorbed/virtual quant.
46. The sum of an even function and its Fourier transform is invariant under Fourier transformation.
47. The difference between an odd function and its Fourier transform is invariant under Fourier transformation.
48. Apart from a scale factor, the functions that characterize linear and spherical harmonics are invariant under Fourier transformation.
49. The scale factor is 1, i, -1 or -i.
50. The harmonic functions are also related by creation and annihilation operators.
51. The harmonic functions contain a factor that equals a Gaussian probability distribution.
52. Any knot can be represented topologically by equations in Cartesian coordinates  $x, y, z$  of the form:  $x = f(t), y = g(t), z = h(t)$ , where  $f(t), g(t)$  and  $h(t)$  are Fourier series with finitely many terms.
53. Only in 3D space knots cannot all be unknotted.
54. There exist three basic types of elementary particles that can be distinguished via the number of [strands](#)<sup>28</sup> / Hilbert vectors involved. These basic types are the bosons, the quarks and the leptons.
55. The bosons can be distinguished in four categories:
  - a. The photons have a helix form and no chirality. They have no mass and are involved in EM interaction.

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<sup>28</sup> Strands

- b. the  $W^{\pm}$  bosons have a knotted form (overhand knot) and possess chirality. They have mass and are involved in weak interaction.
  - c. the Z bosons have a knotted form (figure eight knot) and no chirality. They have mass and are involved in weak interaction.
  - d. The gluons have rectangle loop shape ( $\propto$ ) and no electric, but color charge. They have no mass and are involved in strong interaction.
56. The quarks and the leptons can be distinguished in three generations.
  57. The particles are distinguished via the probability amplitude distribution of the corresponding quanta and the kind of information that is carried by these quanta.
  58. There exist three basic forms of interaction that are distinguished via the number of strands/Hilbert vectors that are involved in the interaction event.
  59. These basic forms of interaction can be related to Reidemeister moves.
  60. Particles become observable via their interactions, thus via the quanta that are generated due to these interactions.
  61. All motion observed in nature minimizes action.
  62. Uniform motion preferably occurs via a geodesic and obeys the geodesic equation.
  63. The visual trajectory of vertebrates is devised in order to cope with a huge dynamical range of light conditions ranging from starlight conditions up to bright daylight conditions
  64. Over billions of years, evolution has exploited the fact that information that comes to living species is generated by Poisson processes. The visual trajectory of vertebrates is optimized for

handling this information for the survival of the owner of this channel.

65. The Banach–Tarski theorem states that a spherical surface can be split in five pieces that can form two spheres of the same volume. The statement does not hold in the eigenspace of a coordinate operator that resides in separable Hilbert space.
66. In separable Hilbert space, at least one coordinate operator has lattice sampling properties. Its eigenspace shows preferred directions.
67. At the lowest scale it is not clear how the granules of an eigenspace of a Hilbert position operator are geometrically arranged. On a larger scale they appear to be influenced by fields.
68. The geometric sampling of normal operators between subsequent Hilbert spaces may differ.
69. Particles can be considered as sources and drains of information carrying quanta.
70. These sources and drains play their role in a continuity equation that treats information carried by quanta.
71. The concept of measurement has no significance at Planck scales.

The indications and considerations that are treated in this chapter will steer the development of the theory that is subject of this e-paper.

## First conclusion

The standard model can be retrieved via categorization of the particle types and their interactions. This comes down to categorizing probability amplitude distributions and categorization of information packages that are carried by generated quanta.

# Logic

## Logics

Quantum logic differs from classical logic in one of its axioms. The set of propositions in classical logic is isomorphic with the set of [Venn diagrams](#)<sup>29</sup>. The set of propositions of traditional [quantum logic](#)<sup>30</sup> is far more complex. This significant difference is due to the weakening of just a single one of the set of more than 25 axioms. It is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable [Hilbert space](#)<sup>31</sup>  $\mathbb{H}$ . The isomorphism means that quantum logical propositions can be represented by closed subspaces of a Hilbert space. The inner products of that Hilbert space can be defined by using numbers of a  $2^n$ -on number space. Taking  $n > 2$  for that purpose raises numeric problems with the closure of the subspaces. Traditional quantum logic does not include any axioms that treat dynamics and it does not treat the influences of physical fields. It only specifies stationary relations that are possible between physical items and their properties.

## Example proposition

In order to discover the emergence of dynamics we will implement a quantum logical proposition in Hilbert space and test its truthfulness. We will introduce in this example proposition physical fields as well as dynamics.

The example proposition( $\spadesuit$ ) is:

*All items in universe influence each other's position. ( $\spadesuit$ )*

It can be answered with either yes or no. And, if we succeed, it can be implemented in Hilbert space. So, in that case it is a valid quantum logical proposition.

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<sup>29</sup> [http://en.wikipedia.org/wiki/Venn\\_Diagram](http://en.wikipedia.org/wiki/Venn_Diagram)

<sup>30</sup> Appendix: Quantum logic

<sup>31</sup> Appendix: The separable Hilbert space

Proving 'yes' is cumbersome, but the 'no' is hardly less difficult. It requires finding an item of which the position is not influenced by at least one of the other items. For this purpose it is necessary to implement notions of *items*, the *universe*, *influences* and *position* in Hilbert space.

The statement includes quantifiers (*position*) and dynamic operational elements (*influence*). The set of axioms of traditional quantum logic does not treat dynamic operational elements. At least it does not do that in a realistic way. As we will see, the *influence* of the *universe* of propositions (*items*) will put particular restrictions to the extension of quantum logic into the realm of an extended dynamic logic. This restriction is manifested in the occurrence of [physical fields](#)<sup>32</sup> and [inertia](#)<sup>33</sup>.

Translated in physical terms inertia means that in contrast to a *uniform movement*, the *acceleration* of an *item* will go together with the *action* of a *physical field*. Notice that we use the words "goes together with" instead of "generates" or "causes".

Translated in logical terms a conclusion of the analysis of inertia runs: "During a redefinition of a proposition the exchange of atomic predicates in that proposition must be done in well-ordered and controlled steps. Otherwise the *community of propositions* will *influence* the considered proposition."

Again it must be noticed that there is no causal relation between the event of being well-ordered and the event of influencing. With other words, the inertial interaction is instantaneous.

When nature's logic is put in axioms, then influences that correspond to physical fields must follow from the axioms. Together with the

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<sup>32</sup> Functions and fields

<sup>33</sup> Influence: Inertia

specification of the origin of dynamics this will then result in a dynamic version of quantum logic.

I assume that this category of logic does not yet exist in mathematics. There exists a version of [dynamic operational quantum logic](#)<sup>34</sup>, but it does not cover or mention the effects of the representation of physical fields in logic and it does not specify the *origin of dynamics*.

### Atomic predicates

Atomic propositions are statements that are either true or false and which cannot be broken down into other simpler propositions. When an atomic proposition concerns a property, then it may contain the value of that property. We will call that kind of atomic propositions atomic predicates. For example “The speed is 5.” The identity or the category of the property is “speed”. The value of the property is 5 with a certain inaccuracy. The inaccuracy is typical for the category of the atomic statement. Only discrete properties can be observed without inaccuracy. The dimension of the value is “meter per second”, but that is another atomic statement and it is a fixed statement. Both the dimension and the inaccuracy form extra information that is part of the type definition of the atomic predicate category “speed”.

In fact there exist no continuous properties. The smallest inaccuracy is set by Planck units. On the other hand the granularity of the properties must not cause a regular lattice structure of the property space. This need not lead to contradictions, but it leads to [special solutions](#)<sup>35</sup> for the operators that deliver the value of the observable properties.

The atomic predicates form a set with a particular lattice structure. In this set we only consider atomic predicates that are independent of all other atomic predicates. Several choices of such sets exist. A subset consisting of

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<sup>34</sup> Discussion: Dynamic logic

<sup>35</sup> Hilbert space: Limitedness: Investigating a special operator

members of a chosen set may be [canonical conjugates](#)<sup>36</sup> of members of another set. However, canonical conjugates are always dependent. So they cannot be member of the same selected set.

In Hilbert space the type definitions of atomic predicates that concern numeric variables are represented by **operators**. The values of the properties in the atomic predicates correspond to the eigenvalues of the operators or they are expectation values. Expectation values are statistically determined via a probability characteristic that characterizes both the operator and a physical item. See [Wave function](#)<sup>37</sup>. In separable Hilbert space  $\mathbf{H}$  the eigenspaces of all normal operators are granular.

## Type definitions

Type definitions are propositions that describe and categorize subjects without specifying their variable values. An **atomic predicate type** is the type definition of a category of atomic predicates and specifies the type of property that these propositions treat. The definition also contains the *physical dimension* (unit) of the property, the inaccuracy and the allowed *range* of the potential values of this property. For example, if that category is “speed”, then the definition contains the *physical dimension* meters per second. The minimum of the absolute value is zero and the maximum of the absolute value is  $c$ . Speed is an imaginary quaternion.

When the type definition concerns a more complex object that can act as an individual the definition will be called an **item type** definition. Item type definitions use **atomic predicate types**.

When that item cannot be broken into simpler objects that still can act as an individual, then the type definition is an **elementary type** definition. Elementary type definitions are constructed of type definitions of atomic predicates.

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<sup>36</sup> Functions and fields: Canonical conjugate

<sup>37</sup> Functions and fields: Characteristic function

The elementary types form (a rather small) subset of the whole set of type definitions. Elementary types appear to divide into two categories: **bosons** and **fermions**. The fermions can be divided in **leptons** and **quarks**.

If the item is not an elementary type, then its type specification is a **system or sub-system type** definition. A (sub)-system type definition is constructed of elementary item type definitions and atomic predicate types.

The type definitions form a set with a different lattice structure. Its structure is isomorph with the structure of classical logic.

In Hilbert space no representation for item type definitions exists. However, in Hilbert space atomic predicate types are represented by operators.

### Strands as type definitions

The notion of [strand](#)<sup>38</sup> is introduced by Christoph Schiller. Apart from its crossing switch events, a strand is not observable. In strand theory the boson types are all represented by a single strand. Reversely a strand is nearly equivalent to the type definition of the simplest boson, which is the photon. A photon has a helix shape. A strand does not need to have that shape.

This e-paper defines a [strand operator](#)<sup>39</sup> that has an eigenspace in which chains of granules reside. These chains come close to Schiller's strands. In each chain one granule is special and is called the current granule. Only the current granule and its immediate neighborhood can deliver observable values.

The shape of a chain has a direct relation with the configuration of the current covering field. Taken over an ordered sequence of Hilbert spaces

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<sup>38</sup> Strands

<sup>39</sup> Hilbert space: Limitedness: Strand operator

the strand fluctuates under the influence of the changing field configuration. The current granule separates the chain in a “virtual past sub-chain” and a “virtual future sub-chain”. The words “past” and “future” are misleading while these parts do not really correspond to the actual past or future of the chain. They depend on the current field configuration, rather than on the past or future configuration.

## Items

The first problem that is raised by constructing the representation of proposition ( $\spadesuit$ ) is to determine what in this representation stands for an item. The simplest solution is to attach a subspace of the Hilbert space to the item. The corresponding proposition can be phrased as: “*This is the item*”. Something either belongs to the subspace or it is outside that subspace. Everything that can be attributed to the item can also be attributed to this subspace. Each of these propositions belongs to a hierarchy for which the mentioned proposition forms the top. All sub-ordered propositions correspond to subspaces of the item’s subspace. In this way the universe of items can be represented by a set of mutual orthogonal subspaces of the Hilbert space. Rays that are spanned by a single Hilbert vector and that are connected with a numeric value can be considered as atomic predicates. Subspaces spanned by such rays that are related to the same type of value can be considered as statements with a wider scope. The rays can be subspace of an items subspace. The subspace that corresponds to a conglomerate of elementary items also represents that conglomerate as an item. The configuration of the subspace that represents an item will change as a function of the parameter that measures the progression of the dynamic behavior of the item. It is possible that not only the values of the atomic predicates change. The types of these atomic predicates may change as well. This happens for example with atomic types that are each other’s canonical conjugate. It is also possible that the configuration of the subspace changes more drastically.

In a set of subsequent Hilbert spaces the subspace that represents the item can be moved around with respect to a selected base consisting of eigenvectors of a normal operator. In this way it may be possible to implement the dynamics of items. This moving around does not mean that the vectors are moved around. It means that at each step of the move the set of vectors that span the considered subspace is **redefined**. The redefinition corresponds to a redefinition of the corresponding proposition. Alternatively, it is also possible to redefine the selected normal operator. Thus, redefinition and the laws that govern redefinition convert the static quantum logic into a dynamic version of quantum logic. It will be shown that physical fields play a significant role in this redefinition.

With his bra-ket notation Dirac has provided us with a marvelous symbolism for vectors and even for operators. He did not provide us with symbols for subspaces. However, it is easy to extend his symbolism and indicate a subspace with a set of vectors that spans that subspace. For example  $\{|f_s\rangle\}_s$  indicates a set of element vectors  $|f_s\rangle$  with enumerator  $s$  that span a closed subspace. This set identifies the subspace. Different sets may identify the same closed subspace.

It is sensible to have one vector inside the item's subspace that is considered as characteristic for the location of the representation of the item in Hilbert space. We reserve the name **locator** for this vector. When the item is redefined, that vector may be redefined as well. This characteristic vector can be used to obtain a precise location of the subspace in Hilbert space. The process via which the locator is determined depends on the requirements. The requirements may be set in relation to an operator. For example the vector that corresponds with the expectation value of the operator for that subspace can be chosen as the locator. In that case [the state vector](#)<sup>40</sup> that corresponds with that operator may play the role of the locator. Two or more bosons can share the same

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<sup>40</sup> States

locator. Fermions that possess the same property values cannot share the same vector as a locator.

Atomic predicates are not considered to be statements that fully describe a physical item. The statement “This is the item” forms the top of a hierarchy of statements that all say something about the item. The hierarchy contains statements that define the item’s type. Other members of the hierarchy specify the items constituents. Still other statements concern the item’s atomic variables that together with the type definition specify the item’s identity. For atoms the variables of the subsystems are hidden from the outside of the atom. This means that atoms can be considered as [modules](#)<sup>41</sup>.

## Representation of items

A subspace in a single separable Hilbert space  $\mathbf{H}$  cannot represent all properties of a physical item. The fact that the position of the item is known means that an eigenvector of the position operator resides within the subspace that represents the item. Say that this subspace covers position values in a certain region. Heisenberg’s uncertainty principle now states that the value of the momentum of the item is uncertain. Any values of this property must correspond with eigenvectors of the momentum operator that also reside in this subspace. For elementary particles the subspace will be too small in order to guarantee sufficiently sure property values. Sufficient information could be collected when the Hilbert space also contains past and future data, such that the momentum can be derived/estimated from those data. The physical fields contain such preconditions. For a free elementary particle the momentum can be derived from the Fourier transform of the probability amplitude distribution that controls the position of the particle. This probability amplitude distribution is the wave function of the particle. Together with the subspace that represents the particle, the wave function represents all

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<sup>41</sup> Part four or <http://www.cryps-of-physics.eu/ThereExistsATendencyInNatureToReduceComplexity.pdf>

information that can be retrieved from the particle. Since all particles have such probability amplitude distributions these private fields get intermixed. Thus in the neighborhood of other particles the superposition of the private fields must be reckoned rather than a single private field.

## Vacuum

Multidimensional subspaces exist that do not represent a dynamical item. They can be considered as **vacuum**. It is still possible that the subspace represents a [ground state](#)<sup>42</sup>. We will assume that on the average the ‘filled’ and the vacuum subspaces are evenly distributed over a connected part of the Hilbert space. The phrase “evenly distributed” means that the distance between the representations of items makes sense. Here we do not mean the distance related to the norm of Hilbert vectors, but the coordinate related distance.

“Vacuum” does not say that these subspaces are empty. It is rather an indication that the subspace does not represent a dynamical object. Instead the subspace may represent a ground state.

**Vacuum does not generate observable information quanta.** In vacuum the clouds of quanta are empty.

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<sup>42</sup> Functions and fields: Quaternionic Fourier transform split: Ladder operator: Ground state

## Hilbert spaces

### Dual views of a Hilbert space

In Hilbert space normal operators exist whose eigenvectors form an orthonormal base of the Hilbert space. The canonical conjugate of that normal operator has a set of eigenvectors that is completely disjoint of the former orthonormal base. This fact defines pairs of views of the same Hilbert space that are related via canonical conjugation.

The corresponding orthogonal bases do not touch. Every base vector is a linear combination with non-zero coefficients of all members of “the other base”. All coefficients have the same norm.

### Position

The original proposition (♠) speaks about the position of the item. The position must be related to something that is available in the separable Hilbert space. This Hilbert space is defined over a number space. Thus we might attach a number of this number space (or a higher  $2n$ -on) to the Hilbert subspace that represents the item. That number must represent position. The natural way of attaching numbers to subspaces of a Hilbert space is via the concept of eigenvalues of normal operators. Any symmetry transform of a selected normal GPS coordinate operator might meet the requirements.

However, there exists a significant drawback. The eigenspaces of all normal operators that reside in a separable Hilbert space  $\mathbf{H}_i$  are countable. In addition the eigenspace of the GPS-like operator in  $\mathbf{H}_i$  must be granular. The granularity means that the difference between two different positions must be larger than the Planck length. A GPS that is constructed from a dense package of fixed size granules generates preferred directions. If we want to avoid this, then we must use the equivalent operator that resides in the corresponding rigged Hilbert space  $\mathbf{H}$ . That operator has a continuum as its eigenspace. Apart from the real axis of the hyper complex number system it shows no preferred directions. So for

position observations we must take rescue in rigged Hilbert space  $\mathbf{H}$ . But the eigenspace of this operator does not show a natural granularity. We can only use the corresponding GPS operator for providing a background coordinate system. The GPS operator is not a part of the separable Hilbert space  $\mathbf{H}$ . It cannot be used to locate the vectors of the separable Hilbert space. However, we can use it to give field values a location. What we have obtained are two GPS-like operators. One resides in separable Hilbert space and has a countable eigenspace. The other resides in rigged Hilbert space and has an uncountable eigenspace. This continuum is at least usable as a background coordinate system. The eigenspace of the first GPS-like operator forms a dense coverage of the second GPS-like operator. Both GPS-like operators do not support granularity.

For a given field we may choose a position operator  $Q$ , which resides in separable Hilbert space and has an equivalent  $\check{Q}$  in rigged Hilbert space, such that for that field we can work with the ideal form of the quaternionic Fourier transform. That means that by using these coordinates as parameters, the field that will be analyzed has decompositions that run along straight lines in the eigenspaces of  $Q$  and  $\check{Q}$ .  $\check{Q}$  introduces a new coordinate system that is curved with respect to the original GPS-like coordinate system.

The new coordinates are characterized by the fact that the considered field when formulated using these coordinates shows a decomposition into static parts that runs along straight coordinate lines. A Fourier transform taken in these coordinates has universe wide validity. The canonical conjugate  $P$  of operator  $Q$  also shows a similar behavior for the Fourier transform of the analyzed field that was first stated in  $Q$  coordinates and after transformation is specified in  $P$  coordinates. The same relation holds for operator  $\check{Q}$  and the canonical conjugate  $\check{P}$ .

## Physical coordinates

Coordinates are not necessarily physical quantities. The physical coordinates of identifiable physical items are granular. The granularity

means that at a given progression step they can only change with a step that either is zero or is equal to a Planck length. Coordinates that are eigenvalues of normal operators in separable Hilbert space  $\mathbf{H}$  are countable. The set of rational quaternions is countable, but this set is not granular. In rigged Hilbert space  $\mathbf{H}$  the eigenspace of a normal operator may be uncountable. It means that this space forms a continuum. The set of all imaginary quaternions forms a continuum. In a given static status quo, only a countable and granular subset of these eigenvalues can be physical quantities.

We took the Planck length here as THE minimum distance between positions. The Planck length is derived via dimensional analysis. The important thing is that a minimum exists. Planck length is a proper name for it and the exact size is less important.

## Generating a Hilbert space GPS

The first step is the introduction of a suitable GPS system in Hilbert space. This can be done by taking an orthonormal base of Hilbert vectors and add quaternion values to them. Due to the separability of the Hilbert space this number set must be countable. Let us take the rational quaternions as an example. This construction defines a normal operator  $Q$  with countable infinite number of eigenvectors  $|q\rangle$  and corresponding eigenvalues  $q$ . We will use the name **coordinate space** for the eigenspace of the **coordinate operator**  $Q$ .

The quaternions clearly have an origin. In contrast, the unit sphere of the Hilbert space, which contains all eigenvectors of  $Q$  is an affine space. The eigenvectors of  $Q$  form an orthonormal base. This singles out the eigenvector that belongs to the origin of the eigenspace. It indicates that  $Q$  must only be used for relative locations.

When we speak about the ( $Q$ ) **coordinate distance** between two vectors  $|f\rangle$  and  $|g\rangle$  in Hilbert space, then we mean the numerical distance between the values of  $\langle f|Q|f\rangle/\langle f|f\rangle$  and  $\langle g|Q|g\rangle/\langle g|g\rangle$ .

$Q$  has an infinite but countable number of eigenvalues. A location in coordinate space represents a location on the unit sphere of Hilbert space.

The fact that  $Q$  must be bounded means that  $Q$  has a **boundary**  $\mathcal{D}$  at a finite distance from its origin.

Every location in the eigenspace of  $Q$  has a unique representation in the boundary  $\mathcal{D}$  of  $Q$  and vice versa.

Take the polar decomposition of the normal coordinate operator  $Q$  in a unitary part  $U$  and a positive operator  $N$ . The eigenspace of  $U$  is the **uni-coordinate space**. Like the unit sphere of the Hilbert space, the uni-coordinate space *is an affine space*. Besides of that also no preferred direction should exist in this unit sphere. But that is not the case!

The eigenspace of  $Q$  consists of all eigenvalues of  $Q$ . The eigenspace is not a closed set and it does not include infinity. If the eigenspace of  $Q$  was granular, then in order to be able to act as a kind of GPS the granules must have a fixed size. A dense packing of the granules would create preferred directions. It means that in that case  $Q$  is not isotropic. In contrast, the unit sphere of the Hilbert space is isotropic. This sphere contains all eigenvectors of  $Q$ . With granularity spread in a regulated order, the granularity raises preferred imaginary directions. As a consequence the size of an infinitesimal step will depend on direction. This does not generally correspond with physical reality. Only in condensed matter such conditions may occur. We can conclude that regulated spread granularity of the eigenspace of  $Q$  leads to unphysical eigenvalues. Thus, let us restrict to countability. However, this restriction prohibits the use of sets of eigenvalues as parameters in differentiation operations.

With artificial means the eigenspace of the coordinate operator may be closed by adding all limits of converging rows of eigenvalues. In this way a closed set of quaternions results. However, most members of this closed set are not eigenvalues of the coordinate operator  $Q$ . The set is eigenspace of a corresponding coordinate operator  $\tilde{Q}$  in a rigged Hilbert space  $\mathbf{H}$ . Still, the use of the separable Hilbert space  $\mathbf{H}$  coordinate operator  $Q$  will always prevent differentiation. Thus, for realistic physical conditions an alternative for this coordinate operator  $Q$  must be sought. The coordinate operator  $\tilde{Q}$  that has its

residence in the rigged Hilbert space  $\mathbf{H}$  does not suffer from preferred imaginary directions and has an eigenspace that is a continuum. For that reason we can use it as a **background coordinate operator**. In the future we will indicate the background operator  $\check{Q}$  as **the (background) GPS operator**. We will use the name **GPS like operator** for any operator that has an eigenspace that can be obtained via an invertible continuous transform or a reflection from the eigenspace of operator  $\check{Q}$ .

## Canonical conjugate

The four dimensions of the quaternions enable the split of  $Q$  into one Hermitian and three anti-Hermitian components. Via the inner product of the Hilbert space, each of these components gets a canonical conjugate. This creates a GMS-like operator.

$$\begin{aligned} \langle \varrho_\mu | \mathcal{P}_\mu \rangle &= \tilde{f}_\mu(\mathcal{P}_\mu) = \langle \mathcal{P}_\mu | \varrho_\mu \rangle^* = f_\mu^*(\varrho_\mu) \\ &= \exp(\mathbf{n}_\mu \cdot \mathcal{P}_\mu \cdot \varrho_\mu / \hbar) \end{aligned} \quad (1)$$

$\mu = 0, 1, 2, 3$  is the index of the dimension.

$\mathbf{n}_\mu = i, \mathbf{i}, \mathbf{j}, \mathbf{k}$  are imaginary base numbers.

$$\mathbf{n}_\mu^2 = -1 \quad (2)$$

The Hilbert vector  $|\varrho_\mu \rangle$  is eigenvector of operator  $Q_\mu$  and corresponds with eigenvalue  $\varrho_\mu$ .

The Hilbert vector  $|\mathcal{P}_\mu \rangle$  is eigenvector of operator  $\mathcal{P}_\mu$  and corresponds with eigenvalue  $\mathcal{P}_\mu$ .

The constant  $\hbar$  relates to the size of the granules.

For each dimension index  $\mu$  holds:

$$\langle \varrho_\mu | \mathcal{P}_\mu f \rangle \cdot \Delta \varrho_\mu = \mathbf{n}_\mu \cdot \hbar \cdot \Delta \langle \varrho_\mu | f \rangle = \mathbf{n}_\mu \cdot \hbar \cdot \Delta f(\varrho_\mu) \quad (3)$$

$$[\mathcal{P}_\mu, Q_\mu] = \mathcal{P}_\mu Q - Q_\mu \mathcal{P}_\mu = \mathbf{n}_\mu \cdot \hbar \quad (4)$$

The definitions of the four canonical conjugates also define four (decoupled) complex Fourier transforms. The granularity decouples the Fourier transforms.

## The Hilbert space GMS

The GMS operator  $\check{\mathcal{P}}$  of the rigged Hilbert space  $\mathbf{H}$  is the canonical conjugate of the rigged Hilbert GPS operator  $\check{\mathcal{Q}}$ . Both operators reside in the Gelfand triple that corresponds to the separable Hilbert space  $\mathbf{H}$ . The canonical conjugate  $\mathcal{P}$  of  $\mathcal{Q}$  is formed from the combination of the four  $\mathcal{P}_\mu$  operators. The same reasoning that is used for the  $\mathcal{Q}$  operator also holds for the  $\mathcal{P}$  operator. It means that also the  $\mathcal{P}$  operator has a countable eigenspace and it has a boundary  $\mathbb{C}$ . Both boundaries have a one to one correspondence with the unit sphere  $\Theta$  of the Hilbert space, but none of the eigenvectors of the  $\mathcal{Q}$  operator coincides with an eigenvector of the  $\mathcal{P}$  operator.

GPS stands for Global Positioning System.

GMS stands for Global Momentum System.

Like with positions, in a given static status quo not all momentum eigenvalues of the GMS operator are physical quantities. Only a countable subset deserves that qualification.

## The fourth dimension

Often time is perceived as the fourth dimension. However, Piron and Einstein prove that when space is occupying the imaginary part, our common notion of time is unfit to act as the fourth dimension of the hyper complex number space. Einstein's special relativity indicates that a Minkowski signature characterizes the common spacetime concept. It means that a rectangular triangle relation exists between the spacetime step, the space step and the coordinate time step, where the coordinate time steps acts as the hypotenuse. Thus the coordinate time step is not perpendicular to the space step as the fourth dimension in quaternion space would be.

In combination with Garret Birkhoff and John von Neumann, Constantin Piron proved that the values of inner products of Hilbert vectors must belong to a division ring. When observables must stay expressible in such

numbers, then they can maximally be quaternions. In that case, the real part of the quaternion of which the imaginary part represents a space coordinate, cannot be coordinate time.

Another argument is the following. Neither traditional quantum logic nor the corresponding separable Hilbert space  $\mathbf{H}$ , can represent dynamics. Thus, time is not an observable that fits in this separable Hilbert space  $\mathbf{H}$ . However, both traditional quantum logic and the corresponding Hilbert space may contain items that represent the precondition of change.

Due to the fact that the Hilbert space is separable, the observable quantities must be countable. In fact most physical quantities are granular. For example the granularity of space is characterized by the Planck length  $l_P$ . The fourth dimension is supposed to be granular as well.

Two quantities, other than space and time that are known to be granular are action and entropy. The granularity of action is characterized by the Planck constant. The granularity of entropy is characterized by the Boltzmann constant. Both are valid candidates for the fourth dimension. Energy is not a valid candidate, because it represents action per unit of time. Thus, it would introduce a notion of time via this backdoor. Action represents change. Entropy represents potential change. Field values represent preconditions of change.

Another possibility is to use the spacetime step as the fourth dimension. This step is perpendicular to the space step. This interpretation immediately poses the question what then the physical significance is of this spacetime step.

Until we encounter the requirement to fill it, the gap of the fourth dimension can be left open. One thing is for sure; coordinate time does not fit in that gap.

## Time and dynamics

Dynamics and its progression parameter time do not fit in a Hilbert space that can only represent a static status quo. That also means that this Hilbert space does not support the corresponding operator. However, the static representation of the preconditions of change is represented in this Hilbert space. Its interpretation is then as the precondition for the change that will follow in the next dynamical step.

For example potential displacement is characterized by momentum, which is the canonical conjugate of space. A time step is required in order to determine the actual displacement. The time step occurs between the instants of validity of subsequent Hilbert spaces. As a consequence the displacement gets its significance by comparing subsequent Hilbert spaces.

The move goes together with a reconfiguration of the fields. This on its turn may go together with an acceleration of the moving item.

Action is change. In this respect its role is similar to the role of displacement. Also the action step gets its significance by comparing subsequent Hilbert spaces. Fields represent the preconditions for the next action step.

When the smallest possible space step  $l_{pl} = \sqrt{\hbar G/c^3}$  and the smallest possible coordinate time step  $t_{pl} = \sqrt{\hbar G/c^5}$  are put into the Minkowski signature,  $\Delta t^2 = \Delta \tau^2 - \Delta q^2/c^2$  then the corresponding spacetime step  $\Delta \tau$  is zero.

The number of Planck time steps equals the number of global progression steps. The number of Planck length steps must always be lower than the number of Planck time steps. The photon never takes a non-zero spacetime step. The number of its space steps always equals the number of its time steps.

Any particle that does not travel with light speed skips some of its space steps. Any particle can take a space step in a direction that differs from the direction of a previous step.

## Hilbert functions

Coordinate operators enable the definition of a special type of functions. Take a coordinate operator  $Q$ . Next take an arbitrary Hilbert vector  $|f\rangle$ . Construct the inner products of this vector with all eigenvectors  $\{|\varphi\rangle\}$  of  $Q$ . Use the eigenvalues  $\{\varphi\}$  of  $Q$  as variable. Now we have defined **Hilbert function**  $f(\varphi)$  as

$$f(\varphi) = \langle f|\varphi\rangle; \text{ for all } \varphi \text{ in } \{\varphi\} \quad (1)$$

Now the Fourier transform  $\tilde{f}(\mathcal{P})$  of  $f(\varphi)$  is defined using the canonical conjugate  $\mathcal{P}$  of  $Q$  via:

$$\tilde{f}(\mathcal{P}) = \langle \tilde{f}|\mathcal{P}\rangle; \text{ for all } \mathcal{P} \text{ in } \{\mathcal{P}\} \quad (2)$$

This is again a Hilbert function, but it uses a different coordinate operator ( $\mathcal{P}$ ).

Hilbert functions are sampled functions. They are not differentiable. They can be approximated by a corresponding continuous function, which may be differentiable.

The continuous approximation of  $f(\varphi)$  is indicated as  $\tilde{\tilde{f}}(\tilde{\varphi})$ . Both the function and its parameters are smooth.

The sampled version of continuous function  $g(q)$  is indicated as  $\tilde{g}(\tilde{q})$ .

The components of Hilbert functions are always decoupled. The same holds for their Fourier transforms. *For Hilbert functions no divergence and no curl exists.*

The components of (quaternionic) continuous functions are always coupled. The same holds for their Fourier transforms. For continuous functions divergence and

curl may exist. However, inside a separable Hilbert space  $\mathbf{H}$  continuous functions only can act as Hilbert vectors. This is the case in  $\ell^2$  space<sup>43</sup>.

## Limitedness

### Countability

The separable Hilbert space  $\mathbf{H}$  has a countable dimension. It means that the eigenvalues of normal operators may offer a dense coverage of a connected part of the number space, but it is not a closed coverage. It does not include all limits of all convergent rows. At least the whole number space is densely covered by the set of eigenvectors. An eigenvector represents an atomic predicate that represents the corresponding attribute of the considered item. The eigenvector lies inside the subspace that represents the item. The corresponding atomic predicate states that the corresponding attribute of the item lies inside the environment that is represented by the eigenvector.

### Granularity

The fact that the separable Hilbert space  $\mathbf{H}$  has a countable number of orthonormal base vectors does not on itself render the eigenspace of every normal operator granular. We could cover the whole quaternionic number space with a countable number of rational quaternions. However, the Planck length sets a minimum difference for positions and this renders the corresponding position operator granular. The way this granularity is distributed may cause particular features. For example dense packing causes preferred directions. Preferred directions do not commonly occur in nature. Such directions occur in condensed matter. Thus, dense packing or any other kind of organized packing does not generally occur in nature. This means that the physical use of a granular coordinate operator is restricted to specific situations. However, from the  $\mathcal{Q}$  operator a corresponding background GPS operator  $\tilde{\mathcal{Q}}$  can be derived that resides in the corresponding rigged Hilbert space  $\mathbf{H}$ . The set of closed subspaces of this rigged Hilbert space  $\mathbf{H}$  is not lattice isomorph with

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<sup>43</sup> [http://en.wikipedia.org/wiki/Lp\\_space#Hilbert\\_spaces](http://en.wikipedia.org/wiki/Lp_space#Hilbert_spaces)

traditional quantum logic. Thus, it is not a proper model of that logic. Both  $\mathcal{Q}$  and  $\tilde{\mathcal{Q}}$  are not suitable as granular position operator. We must find a possible realization of a granular position operator that resides in the separable Hilbert space.

### Investigating an alternative operator

The alternative operator cannot support an eigenspace that contains multidimensional subsets otherwise it would also generate preferred directions. Still it must deliver positions as eigenvalues. Part of the solution is that this new operator relates to a background GPS coordinate system.

A corresponding continuous GPS coordinate operator that can deliver such a GPS background coordinate system can only reside in the rigged Hilbert space  $\mathbf{H}$  that corresponds to the considered separable Hilbert space.

The eigenspace of the target operator may consist of

- a set of separate points
- a set of curves

These elements are located with respect to the mentioned background coordinate system.

It must be possible to locate the current position of ALL physical particles with the eigenvalue set of the new operator.

On the other hand, the operator must exhibit the granularity of the position attribute. At the same time, the position must not be related to a fixed lattice. As a consequence: Any position difference must be equal or larger than the Planck length.

Between subsequent Hilbert spaces the position may stay stationary. However, when a difference occurs, it must again be equal or larger than the Planck length. On its turn this means that between subsequent Hilbert spaces the eigenspaces of the target operator must be related.

The solution may be given by a set of chains of granules. Each chain has a sub-chain of past granules, a current granule and a sub-chain of future granules. During the step to the next Hilbert space, the first granule in the future chain of the current Hilbert space becomes the current granule in the subsequent Hilbert space. The current granule turns into the last granule in the past chain. Another possibility is that during the step the position of the current granule does not change.

In each chain only the current granules will deliver observable values. Fields take care that in each chain sufficient smoothness exists around the current granules.

For that reason the field in the surrounds of the current granules acts like a probability amplitude distribution that regulates the position of that granule. The result of this investigation is a strand operator.

### **Schiller's strands**

Schiller's approach takes another route. In his strand theory the fluctuations of strands determine the field that surrounds the strand. Here, in our model, we let the field determine the fluctuation of the chain. However, this relation plays its major role in the neighborhood of the current granule.

Schiller does not distinguish strongly between different fields. Here we consider all fields to be constituted from the same stuff. However, the curvature (=gravitation) field is derived from the superposition of the other fields.

### **Strand operator**

The strand operator does not cause preferred directions. Thus, strands do not suffer the anomalies of the eigenspace of the Q operator. The elements in its eigenspace have a direction, but that direction is related to local physical conditions.

A strand operator  $\mathcal{Q}$  can be defined along the following steps:

- Take a chain of granules.
- All granules have the same size.
- Each granule in this chain can be given an integer ordering number.
- The background coordinate GPS operator can be used to give each granule in a strand a unique position.
- Each chain consists of a past sub-chain, a current granule and a future sub-chain. The ordering number of the current granule is zero.
- If the set of Hilbert spaces steps to the subsequent Hilbert space, then the position of the current granule stays stationary or it becomes the position of the last granule in the past sub-chain. In that case the current granule becomes the place of what was the first granule in the future sub-chain.
- A probability amplitude distribution takes care that in each chain sufficient smoothness exists around the current granules.
- Define a set of such chains. We will call them **strands**.
- These measures leave a freedom that corresponds to a fluctuation of the strands over subsequent Hilbert spaces.
- Taken over a small set of subsequent Hilbert spaces, the movement of the current granule reflects the influence of the probability amplitude distribution that controls the smoothness of the chain in the surround of the current granule.
- This distribution describes the properties of a moving, rotating and diffusing cloud of virtual information carrying quanta.
- Depending on how the distribution is viewed, the probability amplitude distribution describes the probability density of the information carried by these quanta. In any case the squared modulus of the probability amplitude distribution describes the probable position of the current granule. Taking the Fourier transform of the distribution reveals similar information about the canonical conjugated coordinate.

Further:

1. The eigenspace of the strand operator can only house a finite number of strands.
2. The eigenspace of the strand operator does not house surfaces.
3. The eigenspace of the strand operator does not house volumes.

Because the normal operator is bounded, a boundary surface exists at a finite distance from the origin of the background GPS coordinate system. We will call this boundary the **outer horizon**.

The covered space has an outer horizon, but it may also contain closed **inner horizons**. Outside the outer horizon and inside the inner horizons no strands exist.

A strand may be a closed chain. Closed chains reflect tangentially at the border of horizons.

An open chain connects horizons. This may be one and the same horizon. Thus, the eigenspace does not contain loose ends of strands.

Like the internal horizons, complex particles may act as creation and annihilation centers of chains. For example, field configurations that locally look like functions that are invariant under Fourier transformation appear to feature creation and annihilation operators.

On the other hand a second particle may scatter or even reflect a passing particle. This will be reflected in the corresponding chains.

The probability amplitude distribution that guides the current granule becomes part of the surrounding fields. It forms the basic constituent of the field. Its introduction extends the concept of separable Hilbert space. In a similar way it extends the concept of quantum logic.

These rules only define the immediate neighborhood of the current granule of the strand. If required, the local direction of a strand near other granules in the chain is also guided by the local properties of the surrounding field(s). The covering field represents the combination of these fields.

### **Strand interpretation**

In a single Hilbert space a strand may represent a piece of a potential past, present and future path of a particle. The present part of the path is formed by the direct surround of a single granule that acts as the current granule. In this single Hilbert space the granule corresponds to a Hilbert vector which is an eigenvector of the strand operator. In subsequent Hilbert spaces this potential path may differ. The path is determined by the current configuration of the field(s) that influence(s) the path. As a

consequence, when taken over a sequence of Hilbert spaces, the strands fluctuate. This gives strands a place in Hilbert space.

- A boson is represented by a single strand. Thus, a boson corresponds with a single Hilbert vector in a Hilbert space.
- A quark is represented by a pair of strands. Thus, a quark corresponds with a Hilbert vector pair in a Hilbert space.
- A lepton is represented by a triple of strands. Thus, a lepton corresponds with a triple of Hilbert vectors in a Hilbert space.

Taken over a sequence of Hilbert spaces the granules that represent the actual state of the strand represent the actual paths of the corresponding particle.

Only the current state of the strand becomes observable. An event marks the observation. The event is characterized by a set of basic values:

- The space step, measured in Planck lengths.
- The progression step, measured in Planck times.
- The change (action), measured in units of the size of the Planck constant.
- The potential change measured in units of the size of the Boltzmann constant.

This current state gets its values from the granules in the direct surround of the current granule.

Secondary information can be derived from the field in the immediate surrounds of the current granule(s).

## Fundamental measures and units

*Events* are instants of creation or annihilation of quanta. After creation the quantum becomes observable. After annihilation the quantum is no longer observable.

A *change* is the stepwise variation of the information carried by a quantum.

The *information* carried by a quantum is its position, its momentum and its chirality.

The *distance* between two items equals the number of granules that fit between them.

The *progression time* between two events equals the number of progression steps between them.

The *action* in a progression interval equals the number of steps in that interval during which a change took place.

The *entropy* of a system equals the number of steps during which a change can take place in that system.

In relation to the [covering field<sup>44</sup>](#), a probability amplitude distribution provides secondary information.

The basic measures of physics are:

$$\text{The Planck length, } l_{\text{Pl}} = \sqrt{\hbar G/c^3} = 1.6 \cdot 10^{-35} \text{ m} \quad (1)$$

$$\text{The Planck (coordinate) time, } t_{\text{Pl}} = \sqrt{\hbar G/c^5} = 5.4 \cdot 10^{-44} \text{ s} \quad (2)$$

$$\text{The unit of action is the constant of Planck, } \hbar = 1.055 \cdot 10^{-34} \quad (3)$$

$$\text{The unit of entropy is the constant of Boltzmann, } k = 1.38 \times 10^{-23} \quad (4)$$

## Numbers

### Sign selections

Four possibilities exist due to the sign selections of the quaternions. One sign selection is covered by the conjugation  $a \rightarrow a^*$ . The other is caused by the handedness (chirality)  $a \rightarrow a^\otimes$ . When both combine then the superscript

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<sup>44</sup> Functions and fields; Hilbert field; Covering field

$a \rightarrow a^\oplus$  is used. It is also possible to use the extended **quaternionic conjugation**:

$$a^* = a^{①} \quad (1)$$

$$a^\otimes = a^{②} \quad (2)$$

$$a^\oplus = a^{③} \quad (3)$$

$$a^{****} = a^{\otimes\otimes} = a^{④} = a^{④} = a \quad (4)$$

This differs from the complex conjugation:

$$a^{**} = (a^{①})^{①} = a \quad (5)$$

The effects of the quaternionic conjugation are visible in the base numbers **1, i, j, k**:

$$1^* = 1 \quad (6)$$

$$i^{①} = -i \quad (7)$$

$$i^{②} = j \quad (8)$$

$$i^{③} = -j \quad (9)$$

$$i^{④} = i \quad (10)$$

$$j^{①} = -j \quad (11)$$

$$j^{②} = i \quad (12)$$

$$j^{③} = -i \quad (13)$$

$$\mathbf{j}^{(4)} = \mathbf{j} \quad (14)$$

$$\mathbf{k}^{(1)} = -\mathbf{k} \quad (15)$$

$$\mathbf{k}^{(2)} = \mathbf{k} \quad (16)$$

$$\mathbf{k}^{(3)} = -\mathbf{k} \quad (17)$$

$$\mathbf{k}^{(4)} = \mathbf{k} \quad (18)$$

Thus  $\mathbf{k}$  follows the rules of complex conjugation. This renders its direction to a special direction. This direction is called the **longitudinal** direction. The directions of  $\mathbf{i}$  and  $\mathbf{j}$  are transverse directions. Apart from that they are mutual perpendicular and perpendicular to  $\mathbf{k}$ , they have no preferred direction.

These properties hold locally, they are related to other properties of the operators or the fields that carry these properties.

### Product rule

We use the quaternionic product rule. It has eight (16-8) versions. When either  $a$  or  $b$  is fixed, then the product has four versions:

$$ab = a^{(0)}b^{(0)} = a_0b_0 - \langle \mathbf{a}, \mathbf{b} \rangle + a_0\mathbf{b} + \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} \quad (1)$$

$$a^{(0)}b^{(1)} = a_0b_0 + \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} + \mathbf{a}b_0 - \mathbf{a} \times \mathbf{b} \quad (2)$$

$$a^{(0)}b^{(2)} = a_0b_0 - \langle \mathbf{a}, \mathbf{b} \rangle + a_0\mathbf{b} + \mathbf{a}b_0 - \mathbf{a} \times \mathbf{b} \quad (3)$$

$$a^{(0)}b^{(3)} = a_0b_0 + \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} + \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} \quad (4)$$

$$a^{(1)}b^{(0)} = a_0b_0 + \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} + \mathbf{a}b_0 - \mathbf{a} \times \mathbf{b} = a^{(0)}b^{(1)} \quad (5)$$

$$a^{(1)}b^{(1)} = a_0b_0 - \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} - \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} \quad (6)$$

$$a^{①}b^{②} = a_0b_0 + \langle \mathbf{a}, \mathbf{b} \rangle + a_0\mathbf{b} - \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} \quad (7)$$

$$a^{①}b^{③} = a_0b_0 - \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} - \mathbf{a}b_0 - \mathbf{a} \times \mathbf{b} \quad (8)$$

$$a^{②}b^{①} = a_0b_0 - \langle \mathbf{a}, \mathbf{b} \rangle + a_0\mathbf{b} + \mathbf{a}b_0 - \mathbf{a} \times \mathbf{b} = a^{①}b^{②} \quad (9)$$

$$a^{②}b^{③} = a_0b_0 + \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} + \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} = a^{①}b^{③} \quad (10)$$

$$a^{③}b^{①} = a_0b_0 - \langle \mathbf{a}, \mathbf{b} \rangle + a_0\mathbf{b} + \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} = a^{①}b^{③} \quad (11)$$

$$a^{③}b^{②} = a_0b_0 + \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} + \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} = a^{①}b^{③} \quad (12)$$

$$a^{③}b^{③} = a_0b_0 + \langle \mathbf{a}, \mathbf{b} \rangle + a_0\mathbf{b} - \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} = a^{①}b^{②} \quad (13)$$

$$a^{③}b^{①} = a_0b_0 - \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} - \mathbf{a}b_0 - \mathbf{a} \times \mathbf{b} = a^{①}b^{③} \quad (14)$$

$$a^{③}b^{②} = a_0b_0 + \langle \mathbf{a}, \mathbf{b} \rangle + a_0\mathbf{b} - \mathbf{a}b_0 - \mathbf{a} \times \mathbf{b} \quad (15)$$

$$a^{③}b^{③} = a_0b_0 - \langle \mathbf{a}, \mathbf{b} \rangle - a_0\mathbf{b} - \mathbf{a}b_0 + \mathbf{a} \times \mathbf{b} = a^{①}b^{①} \quad (16)$$

Products of the form  $aa$  have two versions.

$$aa = a^{①}b^{①} = a^{①}a^{②} = a^{②}a^{①} \quad (17)$$

$$= a^{①}a^{③} = a^{③}a^{①} = a^{①}a^{①} = a^{②}a^{②} = a^{③}a^{③}$$

$$= a_0a_0 - \langle \mathbf{a}, \mathbf{a} \rangle$$

$$a^{①}a^{①} = a^{①}a^{①} = a^{①}a^{③} = a^{③}a^{①} = a^{②}a^{③} \quad (18)$$

$$= a^{③}a^{②} = a^{①}a^{②} = a^{②}a^{①}$$

$$= a_0 a_0 + \langle \mathbf{a}, \mathbf{a} \rangle$$

### Product sub-terms

The product  $ab$  contains two particular sub-terms:

$$a_0 b_0 - \langle \mathbf{a}, \mathbf{b} \rangle + a_0 \mathbf{b} + \mathbf{a} b_0 + \mathbf{a} \times \mathbf{b} \quad (1)$$

$$\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 \quad (2)$$

$$\mathbf{a} \times \mathbf{b} = \mathbf{i}(a_2 b_3 - a_3 b_2) + \mathbf{j}(a_3 b_1 - a_1 b_3) + \mathbf{k}(a_1 b_2 - a_2 b_1) \quad (3)$$

The products  $a_0 \mathbf{b}$  and  $\mathbf{a} b_0$  have two versions.

The product  $\langle \mathbf{a}, \mathbf{b} \rangle$  has two versions.

The product  $\mathbf{a} \times \mathbf{b}$  has two versions.

The sum  $a_0 \mathbf{b} + \mathbf{a} b_0$  has four versions. If either  $a$  or  $b$  is fixed it has two versions.

### Operators

The sign selections of operator  $\nabla = (\nabla_0, \nabla)$  depend on the sign selections of position operator  $Q$ , which determines the sign selections for its eigenvalues  $q = (q_0, \mathbf{q})$ .

Normally we consider the sign selection for operators  $Q$  and  $\nabla$  fixed to operators  $Q^{(0)}$  and  $\nabla^{(0)}$ . Sometimes we chose instead operator  $\nabla^{(1)}$ .

Quaternionic conjugation is directly connected with the concepts of **parity** and **spin**.

For quaternionic functions symmetry reduces the differences that are produced by conjugation and anti-symmetry stresses the differences. The same holds for operators.

## Matrices

Another possibility is to present sign selections by [matrices](#)<sup>45</sup>.

$$\alpha_1 = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix} \quad (1)$$

$$\alpha_2 = \begin{bmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{bmatrix} \quad (2)$$

$$\alpha_3 = \begin{bmatrix} 0 & \mathbf{k} \\ -\mathbf{k} & 0 \end{bmatrix} \quad (3)$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (4)$$

## Construction

The Cayley-Dickson construction formula enables the generation of a quaternion from two complex numbers:

$$p = a_0 + a_1\mathbf{k} + \mathbf{i}(b_0 + b_1\mathbf{k}) \quad (1)$$

$$q = c_0 + c_1\mathbf{k} + \mathbf{i}(d_0 + d_1\mathbf{k}) \quad (2)$$

$$(a, b) (c, d) = (ac - db^*, a^*d + cb) \quad (3)$$

$$r = pq \quad (4)$$

$$r_0 = a_0c_0 - a_1c_1 - b_0d_0 - b_1d_1 \quad (5)$$

$$r_k = a_0c_1 - a_1c_0 - b_0d_1 + b_1d_0 \quad (6)$$

$$r_i = a_0d_0 + a_1d_1 + b_0c_0 - b_1c_1 \quad (7)$$

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<sup>45</sup> <http://www.vttoth.com/qt.htm>

$$r_j = -a_1 d_0 + a_0 d_1 + b_0 c_1 + b_1 c_0 \quad (8)$$

Quaternionic conjugation <sup>②</sup> means the exchange of  $r_i$  and  $r_j$ .

## Colors

In the following text, the consequences for the product of the sign choices of the conjugate <sup>①</sup> is indicated by blue color  $\pm$ . The consequence for the product of the choice of the handedness <sup>②</sup> of the cross product is indicated by red color  $\pm$ . The mixed conjugate <sup>③</sup> acts accordingly on both colors.

The sign selections split the ring of quaternions in four different realizations.

## Path characteristics

The Frenet-Serret frame is devised for describing curved paths of particles

Let  $\{\alpha_{qt}\}_t = \alpha(q,t)$  describe a curved path consisting of infinitesimal steps through a landscape  $\{\alpha_q\}_q = \alpha(q)$  of imaginary quaternions  $\alpha_{qt}$ , such that  $\|\dot{\alpha}(q(t))\| = 1$  for all  $t$ .

The 3D Frenet-Serret frame for the above path is given by:

$$\mathbf{T}(q(t)) := \frac{\partial \alpha(q(t))}{\partial t} = \mathbf{T}(t) = \dot{\alpha}(t) \quad (1)$$

$$\kappa(t) := \|\dot{\mathbf{T}}(t)\| \quad (2)$$

$$\kappa(t) \cdot \mathbf{N}(t) := \dot{\mathbf{T}}(t) \quad (3)$$

$$\mathbf{B}(t) := \mathbf{T}(t) \times \mathbf{N}(t) \quad (4)$$

$$\|\mathbf{T}(t)\| = \|\mathbf{N}(t)\| = \|\mathbf{B}(t)\| = 1 \quad (5)$$

$\mathbf{T}(t)$  is the **tantrix** of curve  $\alpha(q(t))$  at instance  $t$ .

$\mathbf{N}(t)$  is the **principal normal** of curve  $\alpha(q(t))$  at instance  $t$ . It is only defined when  $\kappa(t) \neq 0$ .

$\mathbf{B}(t)$  is the **binormal** of curve  $\alpha(q(t))$  at instance  $t$ .

$\mathbf{T}(t)$ ,  $\mathbf{N}(t)$  and  $\mathbf{B}(t)$  are imaginary quaternions.

$\kappa(t)$  is the curvature of curve at  $\alpha(q(t))$  at instance  $t$ .

$r(t) = 1/\kappa(t)$  is the radius of curvature at instance  $t$ .

$\tau(t)$  is the torsion of curve  $\alpha(q(t))$  at instance  $t$ .

$$\begin{bmatrix} \dot{\mathbf{T}}(t) \\ \dot{\mathbf{N}}(t) \\ \dot{\mathbf{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(t) \\ \mathbf{N}(t) \\ \mathbf{B}(t) \end{bmatrix} \quad (6)$$

The Frenet-Serret curves have particular characteristics. The path may be curved and curled. The path is completely determined by its tantrix, curvature and torsion given by functions of  $t$ . Each coordinate of the quaternionic function  $\alpha(q(t))$  has its own set of characteristics. This means that for a given quaternionic function these characteristics are quaternions rather than real numbers and they are all functions of parameter  $t$ .

## Path equations

The path equations are given by

$$\dot{\mathbf{T}}(t) = \kappa(t) \cdot \mathbf{N}(t) \quad (1)$$

$$\dot{\mathbf{N}}(t) = -\kappa(t) \cdot \mathbf{T}(t) + \tau(t) \cdot \mathbf{B}(t) = -\kappa(t) \cdot \mathbf{T}(t) + \tau(t) \cdot \mathbf{T}(t) \times \mathbf{N}(t) \quad (2)$$

$$\dot{\mathbf{B}}(t) = -\tau(t) \cdot \mathbf{N}(t) = \mathbf{T}(t) \times \dot{\mathbf{N}}(t) + \dot{\mathbf{T}}(t) \times \mathbf{N}(t) \quad (3)$$

$$= \tau(t) \cdot \mathbf{T}(t) \times \mathbf{B}(t)$$

## Curve length

The curve length  $l(a, b)$  is defined by:

(1)

$$l(a, b) = \int_{x=a}^{x=b} |\dot{\alpha}(q(x))| dx$$

The integration over the square of the modulus delivers the **action**  $S$  of the curve.

$$S(a, b) = \int_{x=a}^{x=b} |\dot{\alpha}(q(x))|^2 dx \quad (2)$$

## Reparameterization

The path characteristics  $\kappa(t)$  and  $\tau(t)$  together with the curve length and the curve action are independent of any reparameterization  $s(t)$  of the progression parameter  $t$ .

A natural reparameterization is given by  $s(t) = l(t_0, t)$ .

This turns the curve  $\alpha(q(t))$  into a **natural** curve  $\gamma(q(s))$ :

$$\gamma(q(s)) = \alpha(q(t)) \quad (1)$$

Curves on a surface which minimize length between the endpoints are called geodesics.

The natural curve corresponds to a [geodesic](#)<sup>46</sup>.

The consequence is that in three-dimensional space the corresponding movement obeys the [geodesic equation](#)<sup>47</sup>. The Lagrangian is an equivalent of this equation.

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<sup>46</sup> <http://en.wikipedia.org/wiki/Geodesic>

<sup>47</sup> Equations of motion; Lagrangian

## Functions and fields

### Distributions in quaternionic Hilbert space

Using a compact normal operator  $\mathcal{Q}$  and a second distribution operator  $\rho$  with the same eigenvectors  $\{|q\rangle_q\}$  but with eigenvalues  $\{\rho_q\}$  we can generate a Hilbert [distribution](#)<sup>48</sup>.

$$\rho(q) = \langle q | \rho | q \rangle \tag{1}$$

Operator  $\rho$  need not be a compact normal operator. Its spectrum of eigenvalues may be confined to a discrete set of points. Its eigenvectors are used. Its eigenvalues need not be used. . If they are used, the eigenvalues may consist of any kind of hyper complex number.

A Hilbert distribution is not differentiable. It can be seen as a combination of a set of Dirac delta functions that are multiplied with hyper complex numbers. If all numbers are quaternions, then it is a linear combination of Dirac delta functions that each represents a Hilbert vector.

The Hilbert space is separable. This means that the set of eigenvalues of an operator is countable. Thus a Hilbert distribution  $\rho(q)$  is always discrete:

$$\rho(q) = \sum_{i=1}^N q_{E_i} \cdot \delta(q - q_i) \tag{2}$$

The factors  $q_{E_i}$  are hyper complex  $2^n$ -ons.

Convolution of a Hilbert distribution with a blurring spread function can render the result (mostly) differentiable. In fact in the convolution the distribution is represented by a set of Dirac delta functions. Depending on the blur, the result may still be singular for example on the definition points of the Hilbert distribution. The blur may represent a probability distribution. Those blurs are well-formed.

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<sup>48</sup> [http://en.wikipedia.org/wiki/Distribution\\_\(mathematics\)](http://en.wikipedia.org/wiki/Distribution_(mathematics))

A Hilbert function is also a Hilbert distribution. (The reverse is not true). A special form of Hilbert distribution is the representation of a probability amplitude distribution as a Hilbert function.

A special kind of Hilbert distributions is formed by the **elementary Hilbert distributions**. These distributions contain a single or only a few Hilbert vectors. They form the anchor points under the private fields, which represent elementary particles. Private fields are special kinds of Hilbert fields.

As stated before, every (quaternionic) Hilbert function can be split into four decoupled components. And every Hilbert function has a Fourier transform that consists of four decoupled Fourier transforms.

As in the case of a Hilbert function, a Hilbert distribution can represent a very dense coverage. In that case the distribution may become quasi differentiable.

## Hilbert field

By blurring the Hilbert distribution with a suitable spread function, the distribution can be transformed into a mostly continuous function. When the blur is the same for every element of the Hilbert distribution, then this converts the Hilbert distribution  $\rho(q)$  into a [skew Hilbert field](#)<sup>49</sup>  $\phi(q)$  via the convolution:

$$\phi(q) = f(q) \circ \rho(q) \tag{1}$$

With a given Hilbert distribution  $\rho(q)$ , each blurring function  $f(q)$  causes in this way a corresponding Hilbert field that is characterized by the blurring function  $f(q)$ .

Mathematically this convolution is impossible in a separable Hilbert space, but the corresponding rigged Hilbert space  $\mathbf{H}$  is a suitable alternative. The vectors in an orthonormal base consisting of eigenvectors of the normal operator  $\mathcal{Q}$  that resides in the separable Hilbert space  $\mathbf{H}$  are represented in

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<sup>49</sup> [http://en.wikipedia.org/wiki/Skew\\_field](http://en.wikipedia.org/wiki/Skew_field)

the rigged Hilbert space  $\mathbf{H}$  by corresponding Dirac delta functions. We only use the vectors that belong to the Hilbert distribution  $\rho(q)$ . The values of the result of the convolution can be attached to the same orthonormal base vectors of the separable Hilbert space  $\mathbf{H}$ . This procedure attaches the field onto the separable Hilbert space  $\mathbf{H}$ . In principle the field covers all unit sized vectors of the separable Hilbert field.

When the blurs differ per element of the Hilbert distribution, then the Hilbert field can still be interpreted as the superposition of the contributing blurs, but it can no longer be considered as a convolution. Like with the convolution, the location of the blur must be reckoned in this superposition.

When there are only a few types of blurs, then each type constitutes via convolution with a corresponding Hilbert distribution a type specific Hilbert field. The covering Hilbert field can then be interpreted as the superposition of the (blur) type specific Hilbert fields.

The blurs are not hanging as a loose substance in the separable Hilbert space  $\mathbf{H}$ . The blurs are spread over the Hilbert vectors. Each Hilbert vector in the domain of a blur touches this blur and carries the local value of that blur.

Hilbert fields that correspond to the same Hilbert distribution form a Hilbert field set.

A Hilbert field or type specific subfield can be categorized according to its:

- Symmetries
- Conjugation
- Corresponding blur function
- Corresponding Hilbert distribution

Hilbert fields are differentiable. The dimension related components of a Hilbert field are coupled. The differential of a symmetric field or field part is anti-symmetric.

The differential of an anti-symmetric field or field part is symmetric.

### Sampled Hilbert field

A **sampled Hilbert field** consists of its values attached to the eigenvectors of a normal operator, whose eigenspace acts as a coordinate system. A sampled Hilbert field is NOT differentiable. Its dimension related components are

decoupled. It closely approximates a corresponding Hilbert field. The granular eigenspace of the normal operator closely matches the corresponding continuous eigenspace of a coordinate operator that resides in rigged Hilbert space  $\mathbf{H}$ .

It can be interpreted as the distributed superposition of a number of Hilbert functions.

### Blur function

The blur is a spread function. It is the reason of the significant similarity between optics and quantum physics. On the other hand, the blur is a probability distribution. This is the source of quantum noise. The probability distribution can be a probability amplitude distribution or its squared modulus, which is a probability density distribution. A quaternionic probability amplitude distribution has the advantage that its squared modulus can specify the probability and the parameter can specify the full location, while the resulting factor represents related data in the form of a unitary quaternion. This quaternion can also carry its sign selection data, which includes its chirality and its parity. Compared to a complex amplitude distribution, this is a wealth of extra information. The shape of the blur contains secondary information. For example the Fourier transform of the blur offers momentum related data and the rotation of the blur represents angular momentum related data.

The simplest kind of blur that belongs to a particle relates to its [ground state](#)<sup>50</sup>.

Hermite functions, which are eigenfunctions of the Fourier transformation, have a Gaussian blur. [Coherent states](#)<sup>51</sup>, which are eigenfunctions of creation or annihilation operators have a Poissonian blur.

[Quantum shot noise](#)<sup>52</sup> produces a Poisson distribution. When large numbers of quanta are produced the distribution approaches a Gaussian distribution. A binomial process that follow a noise generating Poisson process can be combined with that binomial process into a generalized Poisson process with a lower efficiency. The binomial process represents a weakening effect.

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<sup>50</sup> Functions and fields; Quaternionic transform split; Ground state

<sup>51</sup> Functions and fields; Quaternionic transform split; Coherent state

<sup>52</sup> [http://en.wikipedia.org/wiki/Quantum\\_noise](http://en.wikipedia.org/wiki/Quantum_noise)

Spatial blur can be interpreted as a binomial process. This is because it represents a spatial diffusion effect. In the static model the Poisson processes only represent a lateral distribution. Taken over a sequence of Hilbert spaces the Poisson processes represent an additional temporal distribution. The efficiency of the detection of quanta is characterized by the [detective quantum efficiency<sup>53</sup>](#) (DQE) of the detector. Together with the Fourier transform of the spatial spread function this determines the signal to noise ratio in the information stream. The spread has an integrating (smoothing) effect. A sharper spread improves the signal, but also increases the noise. Any temporal integration reduces the noise. The effect of the lateral spread can be characterized by the Optical Transfer Function (OTF).

When the quanta are given a direction, then the blur becomes the equivalent of a probability amplitude distribution. In strand theory the observable values of crossing switches of strands form probability amplitude distributions. See: <http://www.motionmountain.net/research.html> .

The blur plays a role when canonical conjugate operators occur together or in sequence. An extra blur is caused by the inaccuracy of the combination of these operators.

The blur has many functions and interpretations:

- Convolution with a smooth spread function makes a Hilbert distribution differentiable.
- The spread ensures the compactness of corresponding operators. It also reduces the frequency range that is covered by its Fourier transform.
- The spread function represents a probability density distribution or more in detail a quaternionic probability amplitude distribution.
- Each elementary blur has a Fourier transform
- The probability distribution is characteristic for the inaccuracy of the expectation value of a category of operators, such as the GPS operator, the GMS operator, the Fourier transform, the creation/annihilation operators, the ladder operators and the number and ladder operators.
- The ground state is characterized by a typical spread function.

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<sup>53</sup> [http://en.wikipedia.org/wiki/Detective\\_quantum\\_efficiency](http://en.wikipedia.org/wiki/Detective_quantum_efficiency)

- The spread represents the probability that virtual items exchange roles with actual items.
- The virtual items represent subspaces of virtual Hilbert spaces that are ready to exchange roles with the currently valid Hilbert space.
- The virtual items represent virtual quantum logical propositions that may exchange roles with currently actual propositions.
- The virtual quantum logical propositions are elements of a virtual traditional quantum logic that is ready to exchange roles with the currently actual traditional quantum logic.
- The blur can be interpreted as a spatial quantum noise distribution.
- The blur can be interpreted as a spatial distribution of crossing switches of strands.
- The blur can be interpreted as a spatial distribution of generations or annihilations of quanta. The annihilation of a quant is equivalent to the generation of the corresponding anti-quant.
- The blur works as storage of past, present and future conditions.
- The blur can be squeezed in order to reflect the importance of momentum versus position.
- A basic (ground) blur has in each direction a symmetric cut. An odd-times differentiated blur has in one direction an asymmetric cut. An even-times differentiated blur has in each direction a symmetric cut.
- The blur represents the sticky resistance of the universe against unordered changes (= changes of uniform movement in a geodesic). This is proved by the existence of [inertia](#)<sup>54</sup>.
- The blur represents the sticky resistance of the collection of all propositions against unordered redefinitions.
- Blurs can be categorized according to the corresponding particle type.
- The superposition of blurs forms a field.
- A particle can be interpreted as the local excitation of this field.
- During a progression step the blur may get distorted.

In short: Without blur (quantum) physics is impossible!

## Bypassing granularity

The fact that the Hilbert space is separable means that normal operators have a countable number of eigenvalues. That may still be an infinite

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<sup>54</sup> Influence; Inertia

number, but it means that the eigenspace of these operators is a countable set. It is not a continuum. It is possible to define a procedure that attaches an increasing natural number to each eigenvector and to each eigenvalue. It means that functions that are defined using such eigenvalues as parameters or as function values cannot be differentiated.

This does not mean that differentiable functions cannot exist in Hilbert space. For example,  $\ell^2$  is isomorphic with a separable Hilbert space  $\mathbf{H}$ , and consists of integrable and differentiable functions, but, as with any separable Hilbert space, the eigenvalues of operators in  $\ell^2$  do not form a closed set. The mentioned functions act as Hilbert vectors. They are NOT Hilbert functions.

It is possible to use a trick that enables differentiation of fields that are defined as functions with eigenvalues of a normal operator as their parameter values. The trick consists of blurring all or a subset of the corresponding eigenvectors. When the blur is differentiable, then the field becomes differentiable as well. Still, if the blur extends wide enough, all members of an orthonormal base of Hilbert vectors touch a value of this field.

The blur does not fit IN the considered separable Hilbert space  $\mathbf{H}$ . It anchors onto a vector of this separable Hilbert space  $\mathbf{H}$ . In addition, all members of an orthonormal base of the Hilbert space touch a value of the blur.

The fact that differentiable quaternionic functions have an isotropic multi-dimensional parameter space (in the imaginary part of the quaternions) means that in contrast to the eigenspaces of coordinate operators in separable Hilbert space  $\mathbf{H}$ , this parameter space is coherent. All its dimension related components are coupled. Instead in the canonical conjugated coordinate space a decoupling exists along not necessarily straight radial lines that decompose rotation free and divergence free parts of the quaternionic functions.

In our model the real part of quaternions that are applied as parameters appears to play a rather minor or at least a quite different role. For that reason, in most cases the results of differential geometry are more applicable than the theory of regular quaternionic functions.

Differential geometry also decomposes local space into three independent coordinate directions. These dimensions are selected according to the divergence and rotation properties of the analyzed functionality. This is similar to the approach in the Helmholtz or Hodge decomposition theorem. For example, the [Frenet-Serret frame](#)<sup>55</sup> features three mutually perpendicular directions.

### The basic constituent and private field

There is only one basic type of constituent to Hilbert fields. That constituent type is a probability amplitude distribution. A small subset of Hilbert vectors forms an [elementary Hilbert distribution](#)<sup>56</sup>. It is possible that a corresponding **private field** is formed from the convolution of a single basic constituent with this elementary Hilbert distribution. In that case the Hilbert vectors in this distribution are represented by Dirac delta functions. It is also possible that for each of the Hilbert vectors the basic constituent differs. In that case the private field is the superposition of the basic constituents. In this superposition the separate anchor points of the basic constituents must be reckoned. In both cases the basic constituents anchor on the Hilbert vectors.

The basic constituent covers the whole Hilbert space. This means that every member of an orthonormal base of the Hilbert space touches a value of the constituent. Via the anchor points and via the touching values the private fields are embedded in separable Hilbert space  $\mathbf{H}$ . The private field represents an elementary particle and the physical fields that belong

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<sup>55</sup> Path characteristics

<sup>56</sup> Functions and fields; Elementary Hilbert distribution

to that particle. The anchor points are eigenvectors of a strand operator. The corresponding eigenvalues are taken from a background coordinate system, which is in fact the eigenspace of a GPS-like operator that resides in the rigged Hilbert space that belongs to the separable Hilbert space. The strand operator resides in separable Hilbert space and has an equivalent in rigged Hilbert space. In this rigged Hilbert space the eigenvectors of the strand operator get their GPS-value. The eigenspace of the strand operator contains a set of chains of granules. In each chain one granule is singled out and represents the current granule. It forms the anchor point of the chain's basic constituent. Depending on its type each elementary particle owns one or more of these anchor points.

The strand operator possesses an outer horizon. This guarantees its compactness.

The private field is a continuous function with parameters that are taken from a selected coordinate system. *The field itself is independent of the selection of this coordinate system.* Thus, only when the field is taken as a function of the coordinates it depends on the coordinate selection. The selected coordinate system is related to a corresponding orthonormal base of the Hilbert space. That base consists of eigenvectors of a normal operator that resides in separable Hilbert space. Its eigenvalues are spread dense in the background coordinate system. That background coordinate system corresponds to the eigenspace of a GPS-like operator, which resides in rigged Hilbert space. This eigenspace is a continuum.

We assume that in the context of this e-paper all private fields are differentiable. Due to the fact that the basic constituent has divergence and curl, its dimension related components are intermixed. The field can be decomposed in rotation free part and a divergence free part. As a consequence it has intermixed dimension related Fourier transforms. However by redistributing the eigenvalues of the coordinate system a new coordinate system can be established for which the decomposition runs along straight coordinate lines. An appropriate reorientation of this

coordinate system puts the decomposition in the canonical conjugated coordinate system along straight radial lines. In this coordinate system the ideal form of the Fourier transform can be applied to the considered configuration of the field.

When subsequent Hilbert spaces are considered, the private fields [move](#)<sup>57</sup> together with the corresponding elementary Hilbert distribution. Apart from a linear movement the private fields may rotate. When a given Hilbert distribution contains just one Hilbert vector, then the constituent can rotate free around that point. If it contains two vectors, then one axis is fixed with respect to the anchor points. If it contains three independent vectors, then the basic constituent can only rotate together with these anchor points. The movements are stochastic and have average characteristics such as position, speed, rotation axis, rotation phase and chirality. At each position within the private field these data may differ. Also the relative position of the carrying Hilbert vectors with respect to each other may change. The basic constituent can be interpreted as the probability amplitude distribution whose squared modulus describes the probable location of the carrying Hilbert vector. The private field does that for all its anchor points.

The granularity of the eigenspace of the position operator determines the minimal distance that can exist between the carrying vectors. It also describes the maximal change in average position that can occur during a single progression step.

It is sensible to select the coordinate system such that the members of the elementary Hilbert distribution are eigenvectors of the corresponding position operator. When a Fourier transform is taken, then this can no longer be valid. In that case the members of the elementary Hilbert distribution must be linear combinations of the eigenvectors of the canonical conjugate of the original coordinate operator.

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<sup>57</sup> Dynamics

The Hilbert vectors that are member of a given elementary Hilbert distribution can be interpreted as eigenvectors of a strand operator. The corresponding eigenvalue is the value of a granule that is the current granule of the chain.

## Covering field

Physical fields are not identifiable physical items. In our model, physical fields are represented by [Hilbert fields](#)<sup>58</sup>. For each Hilbert field, every member of an orthonormal base of the Hilbert space corresponds to a value of the field. If for this base the set of eigenvectors of a normal operator is selected, then in this way this field can be coupled to a coordinate system that is formed by the corresponding eigenvalues.

These coordinates are not necessarily physical quantities. The physical coordinates of identifiable physical items are granular. They can only change with steps that are equal to a Planck length. Coordinates that are eigenvalues of normal operators in separable Hilbert space are countable. The set of rational quaternions is countable, but this set is not granular. In rigged Hilbert space the eigenspace of a normal operator may be uncountable. It means that this space forms a continuum. The set of all imaginary quaternions forms a continuum. In a given static status quo, only a countable and granular subset of these eigenvalues can be physical quantities.

Each elementary particle corresponds to a **private field**. A **covering field** is formed by the superposition of these private fields. Each private field that belongs to an elementary particle is characterized by a **blur function** whose characteristics are typical for the particle type. That blur function can be interpreted as a *probability amplitude distribution*. This blur function anchors on a **small number of Hilbert vectors**, which together span a Hilbert subspace. These vectors form an elementary Hilbert distribution. The number of Hilbert vectors on which this elementary Hilbert distribution is based depends on the type of the elementary particle. The anchor points correspond to current granules of chains that reside in the eigenspace of the strand operator.

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<sup>58</sup>Distributions and fields; Hilbert fields

The blurs of elementary particles are smooth and fade out at long distances. As a consequence the covering field is smooth as well and its squared modulus can be considered as a single - very wide spread - probability distribution. The simplest form of blur is a Poisson distribution.

Depending on the type of the constituting particles the covering field can be divided in subfields. Each type has its own subfield.

Depending on the coordinate operator that is selected for the background coordinate system, the parameters of the probability distributions are GPS related or GMS related.

## Decomposition

The imaginary part of a Hilbert field can be decomposed in a rotation free part and a divergence free part.

The Helmholtz decomposition splits the static vector field  $\mathbf{F}$  in a (transversal) divergence free part  $\mathbf{F}_t$  and a (one dimensional longitudinal) rotation free part  $\mathbf{F}_l$ .

$$\mathbf{F} = \mathbf{F}_t + \mathbf{F}_l = \nabla \times \mathbf{A} - \nabla \varphi \quad (1)$$

Here  $\varphi$  is a scalar field and  $\mathbf{A}$  is a vector field. In quaternionic terms  $\varphi$  and  $\mathbf{A}$  are the real and the imaginary part of a quaternionic field.  $\mathbf{F}$  is an imaginary quaternion.<sup>59</sup>

The significance of the terms “longitudinal” and “transversal” can be understood by computing the three-dimensional Fourier transform of the vector field  $\mathbf{F}$ , which we call  $\tilde{\mathbf{F}}$ . Next decompose this field, at each point  $\mathbf{k}$ , into two components, one of which points longitudinally, i.e. parallel to  $\mathbf{k}$ , the other of which points in the transverse direction, i.e. perpendicular to  $\mathbf{k}$ .

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<sup>59</sup> See next paragraph

$$\tilde{\mathbf{F}}(\mathbf{k}) = \tilde{\mathbf{F}}_l(\mathbf{k}) + \tilde{\mathbf{F}}_t(\mathbf{k}) \quad (2)$$

$$\langle \mathbf{k}, \tilde{\mathbf{F}}_t(\mathbf{k}) \rangle = 0 \quad (3)$$

$$\mathbf{k} \times \tilde{\mathbf{F}}_l(\mathbf{k}) = \mathbf{0} \quad (4)$$

The Fourier transform converts gradient into multiplication and vice versa. Due to these properties the inverse Fourier transform gives:

$$\mathbf{F} = \mathbf{F}_l + \mathbf{F}_t \quad (5)$$

$$\langle \nabla, \mathbf{F}_t \rangle = 0 \quad (6)$$

$$\nabla \times \mathbf{F}_l = \mathbf{0} \quad (7)$$

so this split indeed conforms to the Helmholtz decomposition.

If we take the covering field as the subject, then the above picture no longer fits. The covering field is a superposition of a very large number of constituents that each bear on their own anchor point. These anchor points disturb the ideal picture. As a result the  $\mathbf{k}$  lines are no longer straight lines but they get curved in the neighborhood of anchor points. The curvature of these lines can be used to define a local curvature value.

The decomposition depends on the choice of the selected coordinate system. In general such decomposition runs along curved lines. However, for a fixed field configuration it is possible to select a coordinate system for which the decomposition runs along straight lines. For this coordinate system it is possible to define a globally valid multidimensional Fourier transform. With respect to this coordinate system, other coordinate systems possess a locally defined curvature.

## Decomposition and quaternionic Fourier transform

The above relations are the consequence of the properties of the quaternionic Fourier transform with respect to differentiation. The quaternionic differentiation of a quaternionic field runs;

$$\begin{aligned} \mathbf{g}(q) &= \nabla f(q) \\ &= \nabla_0 f_0(q) \mp \langle \nabla, \mathbf{f}(q) \rangle \pm \nabla_0 \mathbf{f}(q) + \nabla f_0(q) \\ &\quad \pm (\pm \nabla \times \mathbf{f}(q)) \end{aligned} \quad (1)$$

The colored  $\mp$  and  $\pm$  signs refer to the sign selections of quaternionic multiplication.

In Fourier space differentiation becomes multiplication with the canonical conjugate coordinate and therefore the equivalent equation becomes:

$$\begin{aligned} \tilde{\mathbf{g}}(k) &= k \tilde{f}(k) \\ &= k_0 \tilde{f}_0(k) \mp \langle \mathbf{k}, \tilde{\mathbf{f}}(k) \rangle \pm k_0 \tilde{\mathbf{f}}(k) + \mathbf{k} \tilde{f}_0(k) \\ &\quad \pm (\pm \mathbf{k} \times \tilde{\mathbf{f}}(k)) \end{aligned} \quad (2)$$

For the imaginary parts holds:

$$\mathbf{g}(q) = \pm \nabla_0 \mathbf{f}(q) + \nabla f_0(q) \pm (\pm \nabla \times \mathbf{f}(q)) \quad (3)$$

$$\tilde{\mathbf{g}}(k) = \pm k_0 \tilde{\mathbf{f}}(k) + \mathbf{k} \tilde{f}_0(k) \pm (\pm \mathbf{k} \times \tilde{\mathbf{f}}(k)) \quad (4)$$

For the static part ( $\nabla_0 f(q) = 0$ ) holds:

$$\mathbf{g}(q) = \nabla f_0(q) \pm (\pm \nabla \times \mathbf{f}(q)) \quad (5)$$

$$\tilde{\mathbf{g}}(k) = \mathbf{k} \tilde{f}_0(k) \pm (\pm \mathbf{k} \times \tilde{\mathbf{f}}(k)) \quad (6)$$

Since

$$\nabla \times \nabla f_0(q) = \mathbf{0} \quad (7)$$

and

$$\langle \nabla, \nabla \times f(q) \rangle = 0 \tag{8}$$

this conforms to the previous paragraph<sup>60</sup>.

## Curvature field

The decomposition properties of the covering field determine the curvature of a secondary coordinate system with respect to the original GPS coordinate system. That curvature on its turn defines a secondary field which we will call **curvature field**. In this view the curvature field is derived from the covering field, which is built via superposition from the private fields of the separate particles.

The curvature that is caused by the blur of an elementary distribution represents a **private curvature field** and is independent of any electric charge (or color charge) of the elementary distribution. The curvature of the private curvature field is non-negative. It can be thought of being distributed over the domain of the private curvature field or its equivalent “charge” value being located at a center point. This “charge” is called mass and the center point is the center of mass.

At each location the local curvature can also act as a guide for the local direction of strands in that environment.

## About the field concept

It is common practice to treat the EM fields and the gravitation field as different and independent subjects. In this interpretation, the gravitation field generates the curvature of the coordinate system in which the other fields must operate.

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<sup>60</sup> [http://www.plasma.uu.se/CED/Book/EMFT\\_Book.pdf](http://www.plasma.uu.se/CED/Book/EMFT_Book.pdf) ;Formulas:F.104, F.105

This paper takes a different approach. It puts the reason for the curvature of the coordinates in the properties and configuration of the covering field. The curvature that exists in this way is used to derive the curvature field. The wave function is also interpreted as a constituent of the covering field. In this way it also contributes to the curvature field. This picture unifies all fields.

## Functions in quaternionic Hilbert space

Due to their definition the Hilbert functions are only defined for an infinite but countable number of parameter values that lay dense in quaternion space. The Hilbert functions are close to a corresponding differentiable function that resides in the corresponding rigged Hilbert space  $\mathbf{H}$ . In contrast to what holds for the Hilbert function, the dimension related components of the differentiable function are coupled.

A locatable probability distribution can be described by the convolution of a Dirac delta function that corresponds to the Hilbert vector, which represents the location of the weighted center of that distribution and a function  $f(q)$  that describes the distribution relative to that location. In this way a blurred Hilbert vector is defined. This means that a blurred Hilbert vector can be closely approximated by a Hilbert function that is defined by the combination of a sharp *locator* Hilbert vector and a sharp *shape* Hilbert vector. We will use the addition "Hilbert" to the name of a continuous function for the Hilbert function that closely approximates that continuous function.

Thus, in Hilbert space the representative of the blurred locator Hilbert vector by a Hilbert function is a **Hilbert blur** or more specifically a **Hilbert probability amplitude distribution**.

It is also possible to use an [elementary Hilbert distribution](#)<sup>61</sup> as the anchor of the continuous probability amplitude distribution. This construct may

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<sup>61</sup> Functions and fields; Elementary Hilbert distribution

represent an elementary particle. It is closely approximated by a private Hilbert field that is formed by the superposition of the Hilbert functions that are formed by a small set of locator Hilbert vectors and a single shape Hilbert vector.

[Pure states](#)<sup>62</sup> are characterized by blurred elementary Hilbert distributions.

### **Elementary Hilbert distribution**

An elementary Hilbert distribution is a discrete distribution in which a single or a small number of Hilbert vectors participate. Together these Hilbert vectors span a closed subspace that represents an un-blurred elementary item.

A unary distribution uses only one Hilbert vector that is eigenvector of the position operator.

A binary distribution uses two Hilbert vectors that are different eigenvectors of the position operator.

A ternary distribution uses three Hilbert vectors that are different eigenvectors of the position operator.

If the eigenvectors are selected such that they belong to mutually perpendicular imaginary (base) eigenvalues, then the elementary distributions are restricted to the mentioned three classes.

When the Hilbert vectors in an elementary distribution are blurred with the same spread function then the resulting subfield gets a typical elementary spread function.

### **Characteristic functions**

Now the position is connected to eigenvectors of the strand operator. The physical item is connected to a subspace rather than to a single vector. This subspace is spanned by the eigenvectors. So we can use a localizer

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<sup>62</sup> States; State definition; Pure states

that represents the (weighted) average position as a more precise indicator of the position of the physical item. On the other hand physical items are characterized by a state.

A state is either a [wave function](#)<sup>63</sup> or a probability density operator. Both use background coordinate position as their parameter. The wave function is a probability amplitude distribution. Each wave function can be approximated by a Hilbert function. The squared modulus of the wave function indicates the probability of finding the position of the localizer.

The probability density operator is a weighted projection operator that is related both to the subspace that represents the item and to the position operator. It represents the probability that after measuring the position the parameter of the density distribution is found as the result.

## Differentiation

Let  $\check{Q}$  be the selected coordinate operator.

The nabla operator  $\nabla$  is directly related to operator  $\check{Q}$ . Thus, the sign selections for  $\check{Q}$  transfer to the sign selections for operator  $\nabla$ . Due to sign selection four nabla operations exist when the field is held fixed. With a fixed nabla operator there may exist four results.

$$\begin{aligned} g(q) = \nabla f(q) & \\ &= \nabla_0 f_0(q) \mp \langle \nabla, f(q) \rangle \pm \nabla_0 f(q) + \nabla f_0(q) \\ &\pm (\pm \nabla \times f(q)) \end{aligned} \tag{1}$$

$$\begin{aligned} \bar{g}(q) = \nabla^{\textcircled{1}} f(q) & \\ &= \nabla_0 f_0(q) \pm \langle \nabla, f(q) \rangle \pm \nabla_0 f(q) - \nabla f_0(q) \\ &\pm (\mp \nabla \times f(q)) \end{aligned} \tag{2}$$

$\nabla$  turns a symmetric field  $f(q)$  into an anti-symmetric field  $\nabla f(q)$  and an anti-symmetric field into a symmetric field.

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<sup>63</sup> States

The fact that  $\nabla f(q) = 0$  means that  $f(q)$  is constant or that at location  $q$  function  $f(q)$  is in a maximum, a minimum, a saddle point or an asymmetric plateau. The consequence of this restriction is:

$$\nabla_0 f_0(q) = \mp \langle \nabla, \mathbf{f}(q) \rangle \pm \nabla_0 \mathbf{f}(q) + \nabla f_0(q) = \mp (\pm \nabla \times \mathbf{f}(q)) \quad (3)$$

The fact that  $\nabla^{\textcircled{1}} f(q) = 0$  leads to different equations.

$$\nabla_0 f_0(q) = \pm \langle \nabla, \mathbf{f}(q) \rangle \pm \nabla_0 \mathbf{f}(q) - \nabla f_0(q) = \pm (\pm \nabla \times \mathbf{f}(q)) \quad (4)$$

The quaternionic Laplace operator  $\Delta$  is defined by

$$h(q) = \Delta f(q) = \nabla^{\textcircled{1}} \nabla f(q) = \nabla \nabla^{\textcircled{1}} f(q) = \nabla_0^2 f(q) + \nabla^2 f(q) \quad (5)$$

A quaternionic function that fulfills  $\Delta f(q) = 0$  is a harmonic function. A quaternionic function that fulfills  $\nabla^2 f(q) = 0$  is a spatial harmonic function.

## Canonical conjugate

Remember that the operator  $\check{Q}$  that resides in rigged Hilbert space  $\mathbf{H}$  is defined such that the decomposition of the covering field runs along straight lines. At least we suppose that for the environment that we investigate an operator exists that does this with sufficient accuracy. First we restrict to a selected longitudinal direction. This restricts to a complex subspace of the full quaternionic number space.

The canonical conjugate of the operator  $\check{Q}$  is the operator  $\check{P}$ . It is defined by using a complex subspace of a quaternionic number space that is used to specify inner products. It is defined by specifying the function that defines the inner products of the eigenvectors  $|q\rangle$  of  $\check{Q}$  and  $|p\rangle$  of  $\check{P}$  with real eigenvalues  $q$  and  $p$ .

$$\langle q|p \rangle = \tilde{f}(p) = \langle p|q \rangle^* = f^*(q) = \exp(\mathbf{k} \cdot \mathbf{p} \cdot q/\hbar) \quad (1)$$

The constant  $\hbar$  is Planck's constant and relates to the granularity of the eigenspaces. The imaginary base number  $\mathbf{k}$  belongs to a complex subspace of the quaternionic number space.

Due to its specification, the canonical conjugate operator  $\check{P}$  can be interpreted as a generator of displacement of the eigenvalues of  $\check{Q}$ . For this purpose the considered function  $f(q)$  must be differentiable.

$$\check{P} = \mathbf{k} \cdot \hbar \cdot \frac{\partial}{\partial q} \quad (2)$$

$$\langle q|\check{P} f \rangle = \mathbf{k} \cdot \hbar \cdot \frac{\partial}{\partial q} f(q) \quad (3)$$

**This interpretation of the operator  $\check{P}$  shows that the complex canonical conjugate shown here corresponds with the imaginary direction in which the differentiated function  $f(q)$  is rotation free.**

The definition leads to the commutator:

$$[\check{P}, \check{Q}] = \check{P}\check{Q} - \check{Q}\check{P} = \mathbf{k} \cdot \hbar \quad (4)$$

The sign selections of  $\check{P}$  depend on the sign selections of  $\check{Q}$ .

### Complex Fourier transform

The specification of the complex canonical conjugate also defines a **complex** Fourier transform.

**Also here the imaginary direction must correspond to the direction in which the analyzed function is rotation free.**

Let  $|f\rangle$  be the generator of a quaternionic function that is generated with the help of the eigenvectors and eigenfunctions of operator  $\check{Q}$  with canonical conjugate  $\check{P}$ .

The Fourier transform  $F_{\mathbf{i}} = U_{qp}$  converts the base  $\{|q\rangle\}_q$  into the base  $\{|p\rangle\}_p$ . The inverse Fourier transform  $U_{pq}$  does the reverse. These transforms reside in rigged Hilbert space  $\mathbf{H}$ .

$$\langle q|f \rangle = \langle f|q \rangle^* = f^*(q) = \sum_p (\langle q|p \rangle \cdot \langle p|f \rangle) \quad (1)$$

$$= \sum_p \langle p (\langle p|q \rangle) |f \rangle$$

$$= \sum_p \langle p U_{pq} |f \rangle$$

$$= \sum_p \langle p | U_{qp} f \rangle$$

$$\langle p|f \rangle = \sum_q (\langle p|q \rangle \cdot \langle q|f \rangle) \quad (2)$$

$$= \sum_q \langle q | U_{pq} f \rangle$$

The complex Fourier transform of a symmetric (complex) function is a cosine transform. It is a real function.

The complex Fourier transform of an anti-symmetric (complex) function is a sine transform. It is an imaginary function.

Through complex Fourier transformation the operators  $\check{P}$  and  $\check{Q}$  exchange roles.

The Hilbert function  $\tilde{f}(p) = \langle f | p \rangle$  denotes the Fourier transform of the Hilbert function  $f(q) = \langle f | q \rangle$

### Heisenberg's uncertainty

The Heisenberg's uncertainty principle is a consequence of the definition of the combination of the canonical conjugate and the definition of the Hilbert field. It means that a small spread of  $q$  values goes together with a large spread of  $p$  values and vice versa.

$$\Delta q \cdot \Delta p \geq \hbar/2 \tag{1}$$

A **squeezed coherent state**<sup>64</sup> is any state such that the uncertainty principle is saturated. That is:

$$\Delta q \cdot \Delta p = \hbar/2 \tag{2}$$

See: [http://en.wikipedia.org/wiki/Squeezed\\_coherent\\_state](http://en.wikipedia.org/wiki/Squeezed_coherent_state).

For animations: <http://gerdbreitenbach.de/gallery/>.

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<sup>64</sup> Functions and fields: Quaternionic Fourier transform split: Functions invariant under Fourier transform: Coherent states

## The quaternionic displacement generator

The formula that defines  $\check{P}$  as a complex displacement generator:

$$\langle q | \check{P} f \rangle = \mathbf{k} \cdot \hbar \cdot \frac{\partial}{\partial q} f(q) \quad (1)$$

can more generally be written as a quaternionic displacement generator for eigenvalues of operator  $\check{Q}$ .

$$\langle q | \check{P} f \rangle = \hbar \cdot \nabla_q \langle q | f \rangle = \hbar \cdot \nabla_q f^*(q) \quad (2)$$

$$= \hbar \int_p \nabla_q \langle q | p \rangle \cdot \langle p | f \rangle = \int_p \langle q | p \rangle p \langle p | f \rangle$$

This means that for all  $q$  and its canonical conjugated  $p$  holds:

$$\nabla_q \langle q | p \rangle = \langle q | p \rangle p \quad (3)$$

Here  $|q\rangle$  is the eigenvector belonging to eigenvalue  $q$  and  $|p\rangle$  is the eigenvector belonging to eigenvalue  $p$ .  $\nabla_q$  represents quaternionic differentiation with respect to eigenvalues of operator  $\check{Q}$ .

It is [shown](#)<sup>65</sup> that locally the operator  $\nabla_q$  splits field  $f^*(q)$  in a longitudinal rotation free part and a transverse divergence free part.

### Idealized field conditions

Only in a complex subspace of the quaternionic number space the relation (3) between the canonical conjugates  $p$  and  $q$  can be simplified to:

$$\langle q | p \rangle = \exp\left(\mathbf{k} \cdot p \cdot \frac{q}{\hbar}\right) \quad (1)$$

The longitudinal direction runs in  $p$  space. The above simplification can only be valid when the longitudinal direction runs along straight radial

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<sup>65</sup> Decomposition: Decomposition and quaternionic Fourier transform

lines. This simplification also enables the specification of a complex Fourier transform that is based on this formula (1).

It behaves as if the analyzed function is constant in dimensions that belong to other quaternionic imaginary directions. It is well-known that the Fourier transform of a constant delivers a Dirac delta function. Thus, the complex longitudinal Fourier transform equals a cut through the quaternionic Fourier transform of the full 3D imaginary quaternionic function or field.

This is similar to the cut through the 2D optical transfer function that is obtained when the Fourier analysis of the imaging device is confined to the image of a thin slit.

The configuration of the analyzed field determines whether the conditions are sufficiently ideal. Otherwise the field configuration induces at every location a local curvature of the actual background coordinate system that is defined using operator  $\check{Q}$ . The eigenspace of the actual operator  $\check{Q}$  is curved with respect to the eigenspace of the idealized operator  $\check{Q}$ .

The position operator  $\check{Q}$  is defined such that when the analyzed function or field is specified with  $\check{Q}$  coordinates the longitudinal direction is stationary. It runs along straight radial lines. The construction of such an idealized position operator is possible for a given configuration of the analyzed field. When the analyzed field is the static covering field, then it holds for that field and not for a part of this field or the covering field that belongs to another static status quo.

In this paper, when nothing else is indicated, we confine Fourier analysis to the ideal quaternionic Fourier transform. When nothing is indicated we presume  $\check{Q}$  coordinates and analysis of the covering field.

Formula (3) of the previous paragraph specifies the relation between canonical conjugated coordinates when field conditions are not idealized.

### Affine space

The eigenvectors of a normal operator form an orthonormal base of the separable Hilbert space  $\mathbf{H}$ . This orthonormal base defines an affine unit sphere. Apparently, the correspondence with a  $\check{Q}$  type GPS operator that is equipped with an origin in its eigenspace is not natural thing for this affine Hilbert unit sphere. On the other hand, like this Hilbert unit sphere, the imaginary eigenspace of the  $\check{Q}$  type GPS operator has no preferred direction.

If a field covers all vectors of an orthonormal base, then it covers all of Hilbert space. The orientation along the longitudinal direction of the (covering) field is not natural for the Hilbert space, but it is natural for the combination of the field and a position operator that keeps the longitudinal lines straight. Apart from a shift of the origin, the position operator  $\check{Q}$  is fully determined by the properties of the field.

The origin of the eigenspace of the  $\check{Q}$  operator may be interpreted as the position of the observer. That selection would consume the last freedom for this operator.

There exists a point to point relation between an arbitrary  $\check{Q}$  type GPS operator and the  $\check{Q}$  operator. This point to point relation defines the curvature field.

### Quaternionic Fourier transform split

The longitudinal Fourier transform represents only part of the full quaternionic Fourier transform. It depends on the selection of a radial line  $\mathbf{k}(q)$  in  $p$  space that under ideal conditions runs along a straight line.

$$\mathcal{F}_{\mathbf{k}}(g(q)) = \mathcal{F}(g(q), \mathbf{k}(q)) \quad (1)$$

Or

$$\mathcal{F}_{\parallel}(g(q)) \stackrel{\text{def}}{=} \mathcal{F}(g_{\parallel}(q)) \quad (2)$$

It relates to the full quaternionic Fourier transform  $\mathcal{F}$

$$\mathcal{F}(g(q)) = \tilde{g}(p) \quad (3)$$

The inverse Fourier transform runs:

$$\mathcal{F}^{-1}(\tilde{g}(p)) = g(q) \quad (4)$$

The split in longitudinal and transverse Fourier transforms corresponds to a corresponding split in the multi-dimensional Dirac delta function.

We consider a field  $g(q)$  that equals the quaternionic differentiation of another field  $f$  with respect to a selected coordinate system  $Q$ .

$$g(q) = \nabla_q f \quad (5)$$

We use the results of the paragraph on decomposition. We only use the static and imaginary version of field  $g(q)$ .

For the static imaginary part  $\mathbf{g}(q)$  holds:

$$\mathbf{g}(q) = \nabla f_0(q) \pm (\pm \nabla \times \mathbf{f}(q)) = \mathbf{g}_l(q) + \mathbf{g}_t(q) \quad (6)$$

In Fourier space differentiation becomes multiplication with the canonical conjugate coordinate  $\mathbf{p}$  and therefore the equivalent equation becomes:

$$\tilde{\mathbf{g}}(p) = \mathbf{p}\tilde{f}_0(p) \pm (\pm \mathbf{p} \times \tilde{\mathbf{f}}(p)) = \tilde{\mathbf{g}}_l(p) + \tilde{\mathbf{g}}_t(p) \quad (7)$$

Since

$$\nabla \times \nabla f_0(q) = \mathbf{0} \rightarrow \nabla \times \mathbf{g}_l(q) = \mathbf{0} \quad (8)$$

and

$$\langle \nabla, \nabla \times \mathbf{f}(q) \rangle = 0 \rightarrow \langle \nabla, \mathbf{g}_t(q) \rangle = 0 \quad (9)$$

Now we take

$$\begin{aligned} \langle q | \check{P} f \rangle &= \hbar \cdot \nabla_q \langle q | f \rangle = \hbar \cdot \nabla_q f^*(q) = \mathbf{g}(q) \\ &= \int_{\mathbf{p}} \langle q | p \rangle \cdot \langle p | g \rangle \end{aligned} \quad (10)$$

The static imaginary part is

$$\begin{aligned} \langle q | \check{P} f \rangle &= \hbar \cdot \nabla_q \langle q | f \rangle = \hbar \cdot \nabla_q f^*(q) = \mathbf{g}(q) \\ &= \text{Im} \left( \int_{\mathbf{p}} \langle q | p \rangle \cdot \langle p | \mathbf{g} \rangle \right) = \int_{\mathbf{p}} \text{Im}(\langle q | p \rangle \cdot \langle p | \mathbf{g} \rangle) \\ &= \int_{\mathbf{p}} \text{Im}(\langle q | p \rangle \cdot \langle p | \mathbf{g}_l \rangle) + \int_{\mathbf{p}} \text{Im}(\langle q | p \rangle \cdot \langle p | \mathbf{g}_t \rangle) \\ &= \int_{\mathbf{p}} \text{Im}(\langle q | p \rangle \cdot \tilde{\mathbf{g}}_l(p)) + \int_{\mathbf{p}} \text{Im}(\langle q | p \rangle \cdot \tilde{\mathbf{g}}_t(p)) \end{aligned} \quad (11)$$

The left part is the longitudinal inverse Fourier transform of field  $\tilde{\mathbf{g}}(p)$ .  
The right part is the transverse inverse Fourier transform of field  $\tilde{\mathbf{g}}(p)$ .  
For the Fourier transform of  $\mathbf{g}(q)$  holds the split:

$$(12)$$

$$\begin{aligned}\tilde{\mathbf{g}}(p) &= \int_q \text{Im}(\langle p|q \rangle \cdot \mathbf{g}_l(q)) + \int_p \text{Im}(\langle p|q \rangle \cdot \mathbf{g}_t(q)) \\ &= \int_q \text{Im}(\langle p|q \rangle \cdot \mathbf{g}(q))\end{aligned}$$

The longitudinal direction is a one dimensional (radial) space. The corresponding transverse direction is tangent to a sphere in 3D. Its direction depends on the field  $\mathbf{g}(q)$  or alternatively on the combination of field  $f$  and the selected coordinate system  $Q$ .

### Alternative transverse plane

The Cayley-Dickson construction, as well as Warren Smith's construction formula shows that the transverse part can be considered as a complex number space multiplied with a fixed imaginary quaternionic base number. The selection of the imaginary base number  $i$  is arbitrary as long as it is perpendicular to  $k$ . The resulting plane is spanned by axes  $i$  and  $ik$ . When base number  $i$  is divided away, then a normal complex number space results.

Also here a complex Fourier transform can be defined in a way that is similar to the longitudinal Fourier transform. It must be reckoned that the sign selections for these directions differ.

### Alternative approach to Fourier transform

The following draws from the work of [S. Thangavelu](#)<sup>66</sup>.

Let us take the non-abelian group  $\mathbb{H}_1$  which is  $\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}$  with the group law

$$(x, y, t)(x_0, y_0, t_0) = (x + x_0; y + y_0; t + t_0 + x y_0) \quad (1)$$

Then it is clear that  $\mathbb{H}_1$  is non-abelian and the Lebesgue measure  $dx dy dt$  is both left and right invariant Haar measure on  $\mathbb{H}_1$ . With this measure we

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<sup>66</sup> <http://www.math.iitb.ac.in/atm/faha1/veluma.pdf>

can form the Hilbert space  $L^2(\mathbb{H}_1)$ . Let  $\Gamma = \mathbb{Z} \otimes \mathbb{Z} \otimes \mathbb{Z}$ . Then it is easy to check that  $\Gamma$  is a subgroup of  $\mathbb{H}_1$  so that we can form the quotient  $M = \Gamma/\mathbb{H}_1$  consisting of all right cosets of  $\Gamma$ . Functions on  $M$  are naturally identified with left  $\Gamma$ -invariant functions on  $\mathbb{H}_1$ . As the Lebesgue measure  $dx dy dt$  is left  $\Gamma$ -invariant we can form  $L_2(M)$  using the Lebesgue measure restricted to  $M$ . As a set we can identify  $M$  with  $[0, 1)^3$  and we just think of  $L^2(M)$  as  $L^2([0, 1)^3)$ .

Fourier expansion in the last variable allows us to decompose  $L^2(M)$  into a direct sum of orthogonal subspaces. Simply define  $\mathcal{H}_k$  to be the set of all  $f \in L^2(M)$  which satisfy the condition

$$f(x, y, t + s) = \exp(2 \pi i k s) f(x, y, t) \quad (2)$$

Then  $\mathcal{H}_k$  is orthogonal to  $\mathcal{H}_j$  whenever  $k \neq j$  and any  $f \in L^2(M)$  has the unique expansion

$$f = \sum_{k=-\infty}^{\infty} f_k; f_k \in \mathcal{H}_k \quad (3)$$

In quaternionic terms, the split sees  $ik$  as imaginary quaternion  $\mathbf{k}$  and the quaternionic Hilbert space is split in components according to the imaginary direction of  $\mathbf{k}$ , where the choice is between three mutually perpendicular directions.

For the moment, we are mainly interested in  $\mathcal{H}_1$  which is a Hilbert space in its own right. It is interesting to note that functions in  $\mathcal{H}_1$  are also invariant under the left action of  $\Gamma$ .

Our next example of a unitary operator is the following. Consider the map  $J : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  given by

$$J(x, y, t) = (-x, y, t - xy) \quad (4)$$

$$J^\dagger(x, y, t) = (x, -y, t - xy) \quad (5)$$

$$J^\dagger = J^{-1} \quad (6)$$

$$J^2(x, y, t) = J(-x, \cdot y; t - xy) = (-x, -y; t) \quad (7)$$

$$J^4 = I \quad (8)$$

$$J(0, 0, t) = (0, 0, t) \quad (9)$$

$$Jf(x, y, t) = f(J(x, y; t)) = f(-x, \cdot y, t - xy) \quad (10)$$

### Weil-Brezin transform

Next consider the Weil-Brezin transform V:

$$Vf(x, y, t) = \exp(2\pi \mathbf{k} t) \sum_n f(x + n) \exp(2\pi \mathbf{k} n y) \quad (11)$$

$$\int_{y=0}^1 |Vf(x, y, t)|^2 dy = \int_{x=0}^1 \sum_{n=-\infty}^{n=\infty} |f(x + n)|^2 dx \quad (12)$$

$$\iiint_0^1 |Vf(x, y, t)|^2 dx dy dt = \int_0^1 |f(x)|^2 dx \quad (13)$$

V is unitary.

See also [Zak transform](#)

### Fourier transform

We define the Fourier transform  $\mathcal{F}$  by:

$$\mathcal{F} = V^\dagger J V \quad (14)$$

- $\mathcal{F}^4 f = f$ ; for every  $f \in L^2(\mathbb{R})$
- $\mathcal{F}^2 f(x) = f(-x)$ ; for almost every  $x \in \mathbb{R}$
- $\|\mathcal{F} f\|^2 = \|f\|^2$

For  $f \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$  the Fourier transform is given by

$$\mathcal{F}f(\xi) = \int_{x \in \mathbb{R}} f(x) \cdot \exp(2 \pi i x \xi) dx \quad (15)$$

If we further assume that  $\mathcal{F}f \in L^1(\mathbb{R})$  then for almost every  $x$  we have

$$f(x) = \int_{\xi \in \mathbb{R}} \mathcal{F}f(\xi) \exp(2 \pi i x \xi) d\xi \quad (16)$$

## Functions invariant under Fourier transform

In this section we confine to a complex part of the Hilbert space.

See [http://en.wikipedia.org/wiki/Hermite\\_polynomials](http://en.wikipedia.org/wiki/Hermite_polynomials).

There exist two types of Hermite polynomials: (1, 2)

1. The probabilist's Hermite polynomials:

$$H_n^{prob}(z) = (-1)^n \exp(1/2 z^2) \frac{d^n}{dz^n} \exp(-1/2 z^2).$$

2. The physicist's Hermite polynomials

$$\begin{aligned} H_n^{phys}(z) &= (-1)^n \exp(z^2) \frac{d^n}{dx^n} \exp(-z^2) \\ &= \exp(1/2 z^2) \left( z - \frac{d}{dz} \right) \exp(-1/2 z^2) \end{aligned}$$

These two definitions are *not* exactly equivalent; either is a rescaling of the other:

$$H_n^{phys}(z) = 2^{n/2} H_n^{prob}(z\sqrt{2}) \quad (3)$$

In the following we focus on the physicist's Hermite polynomials.

The Gaussian function  $\phi(z)$  defined by

$$\varphi(x) = \exp(-\pi z^2) \quad (4)$$

is an eigenfunction of  $\mathcal{F}$ . It means that its Fourier transform has the same form.

As  $\mathcal{F}^4 = I$  any  $\lambda$  in its spectrum  $\sigma(\mathcal{F})$  satisfies  $\lambda^4 = 1$ : Hence,

$$\sigma(\mathcal{F}) = \{1; -1; i; -i\}. \quad (5)$$

We take the Fourier transform of the expansion:

$$\exp(-\frac{1}{2} z^2 + 2 z c - c^2) = \sum_{n=0}^{\infty} \exp(-\frac{1}{2} z^2) H_n(z) c^n / n! \quad (6)$$

First we take the Fourier transform of the left hand side:

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} \exp(-\mathbf{k} z p_z) \exp(-\frac{1}{2} z^2 + 2 z c - c^2) dz & \quad (7) \\ & = \exp(-\frac{1}{2} p_z^2 - 2 \mathbf{k} p_z c + c^2) \\ & = \sum_{n=0}^{\infty} \exp(-\frac{1}{2} p_z^2) H_n(p_z) (-\mathbf{k} c)^n / n! \end{aligned}$$

The Fourier transform of the right hand side is given by

$$\frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \int_{z=-\infty}^{\infty} \exp(-\mathbf{k} z p_z) \cdot \exp(-\frac{1}{2} z^2) H_n(z) c^n / n! dz \quad (8)$$

Equating like powers of  $c$  in the transformed versions of the left- and right-hand sides gives

$$\begin{aligned} & \frac{1}{\sqrt{2\pi}} \int_{z=-\infty}^{\infty} \exp(-\mathbf{k} z p_z) \cdot \exp(-1/2 z^2) H_n(z) c^n/n! dz \\ & = (-\mathbf{k})^n \cdot \exp(-1/2 p_z^2) H_n(p_z) \frac{c^n}{n!} \end{aligned} \quad (9)$$

Let us define the Hermite functions  $\psi_n(z)$

$$\psi_n(z) \stackrel{\text{def}}{=} \langle z | \psi_n \rangle = c_n \exp(-1/2 z^2) H_n(z) \quad (10)$$

$$|\mathcal{F} \psi_n \rangle = |\psi_n \rangle (-\mathbf{k})^n \quad (11)$$

with suitably chosen  $c_n$  so as to make

$$\|\psi_n\|^2 = 1 \quad (12)$$

$$c_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} \quad (13)$$

The importance of the Hermite functions lie in the following theorem.

“The Hermite functions  $\psi_n; n \in \mathbb{N}$  form an orthonormal basis for  $L^2(\mathbb{R})$ ”

Consider the operator

$$H = -1/2 \frac{d^2}{dz^2} + 1/2 z^2 \quad (14)$$

Apply this to  $\psi_n(z)$ :

$$H \cdot \psi_n(z) = (1/2 + n) \psi_n(z) \quad (15)$$

Thus,  $\psi_n$  is an eigenfunction of  $H$ .

Let  $f = \psi_{4k+j}$  be any of the Hermite functions. Then we have

$$\sum_{n=-\infty}^{\infty} f(y + n) \cdot \exp(-2 \pi \mathbf{k} x (y + n)) \quad (16)$$

$$= (-\mathbf{k})^j \sum_{n=-\infty}^{\infty} f(x + n) \exp(2 \pi \mathbf{k} n y)$$

Proof: As

$$\mathcal{F} = V^\dagger J V \quad (17)$$

the equation

$$\mathcal{F}f = (-\mathbf{k})^j f \quad (18)$$

translates into

$$JVf(x; y; t) = (-\mathbf{k})^j Vf(x; y; t) \quad (19)$$

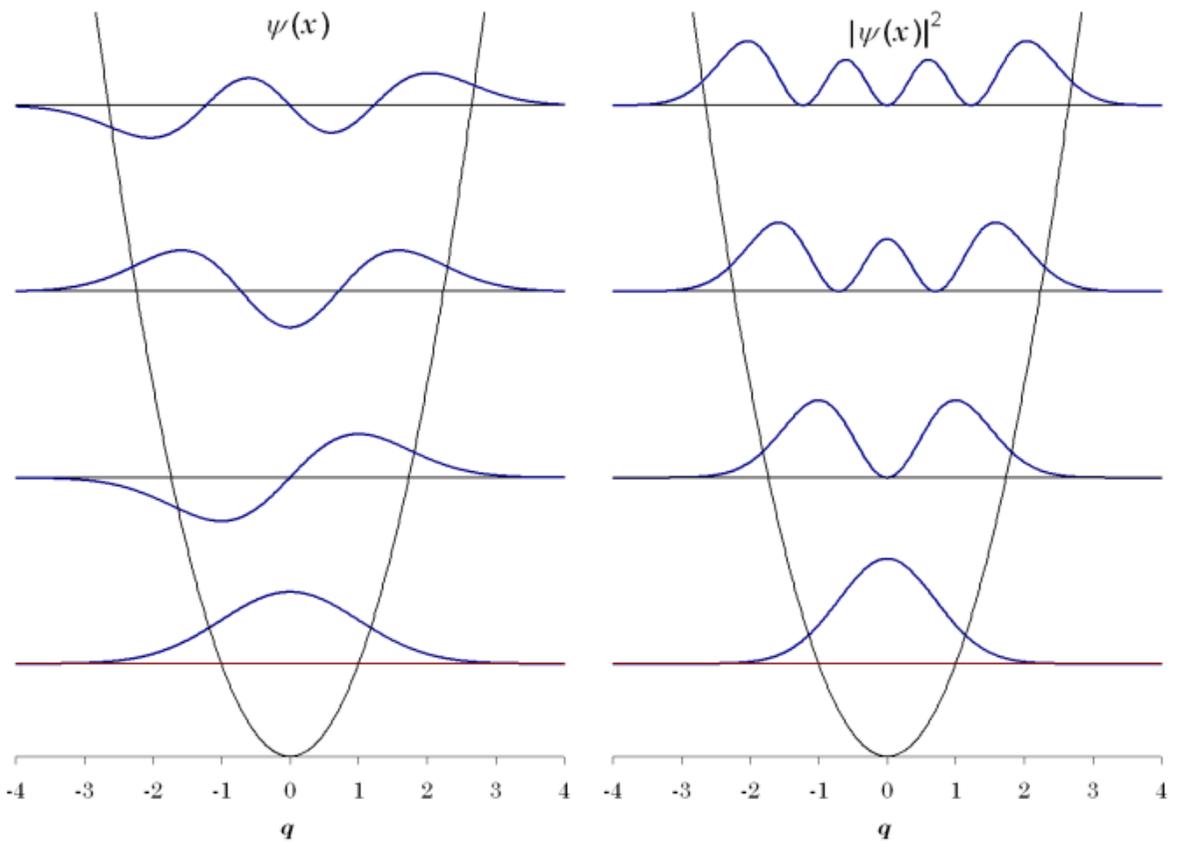
With the definition of  $V$  and  $t = xy$ :

$$V f(x, y, t) = \exp(2 \pi \mathbf{k} t) \sum_n f(x + n) \exp(2 \pi \mathbf{k} n y) \quad (20)$$

QED.

The vectors  $|\psi_n\rangle$  are eigenvectors of the Fourier transform operator with eigenvalues  $(-\mathbf{k})^n$ . The eigenfunctions  $\psi_n(x)$  represent eigenvectors  $|\psi_n\rangle$  that span the complex Hilbert space  $\mathbf{H}_k$ .

For higher  $n$  the central parts of  $\psi_n(x)$  and  $|\psi_n(x)|^2$  become a sinusoidal form.



A [coherent state](#)<sup>67</sup> is a specific kind of [state](#)<sup>68</sup> of the quantum harmonic oscillator whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system. The ground state is a [squeezed coherent state](#)<sup>69</sup>.

<sup>67</sup> [http://en.wikipedia.org/wiki/Coherent\\_state](http://en.wikipedia.org/wiki/Coherent_state)

<sup>68</sup> States

<sup>69</sup> Canonical conjugate: Heisenberg's uncertainty

## Ladder operator

The Hermite functions  $\psi_n$  represent [Fock states](#)<sup>70</sup>.

Boson ladder operators are characterized by

$$\mathcal{A}|\psi_n\rangle = \sqrt{n} |\psi_{n-1}\rangle \quad (1)$$

$$\mathcal{A}^\dagger|\psi_n\rangle = \sqrt{n+1} |\psi_{n+1}\rangle \quad (2)$$

$$\mathcal{A} = \frac{1}{\sqrt{2}} \left( c_1 \frac{d}{dq} + c_2 q \right) = -k \check{P} \sqrt{\frac{1}{2 \hbar m \omega}} + \check{Q} \sqrt{\frac{m \omega}{2 \hbar}}$$

$$\mathcal{A}^\dagger = \frac{1}{\sqrt{2}} \left( -c_1 \frac{d}{dq} + c_2 q \right) = k \check{P} \sqrt{\frac{1}{2 \hbar m \omega}} + \check{Q} \sqrt{\frac{m \omega}{2 \hbar}}$$

In the Heisenberg picture, the operators have the following time dependence:

$$\mathcal{A}(t) = \mathcal{A}(t_0) \exp(-k \omega (t - t_0)) \quad (3)$$

$$\mathcal{A}^\dagger(t) = \mathcal{A}^\dagger(t_0) \exp(k \omega (t - t_0)) \quad (4)$$

We can also define an enumeration operator  $N$  which has the following property:

$$N = \mathcal{A}^\dagger \mathcal{A} \quad (5)$$

$$N|\psi_n\rangle = |\psi_n\rangle n \quad (6)$$

In deriving the form of  $\mathcal{A}^\dagger$ , we have used the fact that the operators  $X$  and  $P_x$ , which represent observables, are Hermitian. These observable

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<sup>70</sup> [http://en.wikipedia.org/wiki/Fock\\_state](http://en.wikipedia.org/wiki/Fock_state)

operators can be expressed as a linear combination of the ladder operators as

$$\check{Q}(t) = \sqrt{\frac{\hbar}{2 m \omega}} (\mathcal{A}^\dagger(t) + \mathcal{A}(t)) \quad (7)$$

$$\check{P}(t) = k \sqrt{\frac{1}{2} \hbar m \omega} (\mathcal{A}^\dagger(t) - \mathcal{A}(t)) \quad (8)$$

The  $\check{Q}$  and  $\check{P}$  operators obey the following identity, known as the canonical commutation relation:

$$[\check{Q}, \check{P}] = k \hbar \quad (9)$$

Using the above, we can prove the identities

$$H = \hbar \omega (\mathcal{A}^\dagger \mathcal{A} + \frac{1}{2}) = \hbar \omega (N + \frac{1}{2}) \quad (10)$$

$$[\mathcal{A}^\dagger, \mathcal{A}] = 1 \quad (11)$$

Now, let  $|\psi_E\rangle$  denote an energy eigenstate with energy  $E$ . The inner product of any ket with itself must be non-negative, so

$$\langle \psi_E | \mathcal{A} \mathcal{A} | \psi_E \rangle = \langle \psi_E | \mathcal{A}^\dagger \mathcal{A} | \psi_E \rangle \geq 0 \quad (12)$$

Expressing  $\mathcal{A}^\dagger \mathcal{A}$  in terms of the Hamiltonian  $H$ :

$$\langle \psi_E | (H/(\hbar \omega) - \frac{1}{2}) | \psi_E \rangle = (E/(\hbar \omega) - \frac{1}{2}) \geq 0 \quad (13)$$

so that

$$E \geq \frac{1}{2} \hbar \omega. \quad (14)$$

Note that when  $|\mathcal{A} \psi_E \rangle = |0 \rangle$  (is the zero ket i.e. a ket with length zero), the inequality is saturated, so that

$$E = \frac{1}{2} \hbar \omega \quad (15)$$

It is straightforward to check that there exists a state satisfying this condition; it is the ground state

$$|\psi_{ground}\rangle = |\psi_{E_n}\rangle; (n = 0) \quad (16)$$

Using the above identities, we can now show that the commutation relations of  $\mathcal{A}$  and  $\mathcal{A}^\dagger$  with  $H$  are:

$$[H, \mathcal{A}] = -\hbar \omega \mathcal{A} \quad (17)$$

$$[H, \mathcal{A}^\dagger] = \hbar \omega \mathcal{A}^\dagger \quad (18)$$

Thus, provided  $|\mathcal{A} \psi_E\rangle$  is not the zero ket,

$$\begin{aligned} |H \mathcal{A} \psi_E\rangle &= |[H, \mathcal{A}] + \mathcal{A} H \psi_E\rangle \\ &= |-\hbar \omega \mathcal{A} + \mathcal{A} E \psi_E\rangle \\ &= |-\hbar \omega \mathcal{A} + \mathcal{A} E \psi_E\rangle \\ &= (E - \hbar \omega) |\mathcal{A} \psi_E\rangle \end{aligned} \quad (19)$$

Similarly, we can show that

$$|H \mathcal{A}^\dagger \psi_E\rangle = (E + \hbar \omega) |\mathcal{A}^\dagger \psi_E\rangle \quad (20)$$

In other words,  $\mathcal{A}$  acts on an eigenstate of energy  $E$  to produce, up to a multiplicative constant, another eigenstate of energy  $E - \hbar \omega$ , and  $\mathcal{A}^\dagger$  acts on an eigenstate of energy  $E$  to produce an eigenstate of energy  $E + \hbar \omega$ . For this reason,  $\mathcal{A}$  is called a "lowering operator", and  $\mathcal{A}^\dagger$  "raising operator". The two operators together are called ladder operators. In quantum field theory,  $\mathcal{A}$  and  $\mathcal{A}^\dagger$  are alternatively called "annihilation"

and "creation" operators because they destroy and create particles, which correspond to our quanta of energy.

Given any energy eigenstate, we can act on it with the lowering operator  $\mathcal{A}$ , to produce another eigenstate with  $\hbar \omega$ -less energy. By repeated application of the lowering operator, it seems that we can produce energy eigenstates down to  $E = -\infty$ . However, this would contradict our earlier requirement that  $E \geq \hbar \omega/2$ .

### Ground state

Therefore, there must be a ground-state energy eigenstate, which we label  $|\psi_{\text{ground}}\rangle$ , such that

$$|\mathcal{A} \psi_{\text{ground}}\rangle = |0\rangle; \text{ (zero ket).} \quad (1)$$

In this case, subsequent applications of the lowering operator will just produce zero kets, instead of additional energy eigenstates. Furthermore, we have shown above that

$$|H \psi_{\text{ground}}\rangle = (\frac{1}{2} \hbar \omega) |\psi_{\text{ground}}\rangle \quad (2)$$

Finally, by acting on  $|\psi_{\text{ground}}\rangle$  with the raising operator and multiplying by suitable normalization factors, we can produce an infinite set of energy eigenstates

$$\{|\psi_{\text{ground}}\rangle, |\psi_{E_1}\rangle, |\psi_{E_2}\rangle, \dots, |\psi_{E_n}\rangle\}, \quad (3)$$

such that

$$|H \psi_{E_n}\rangle = \hbar \omega (n + \frac{1}{2}) |\psi_{E_n}\rangle \quad (4)$$

which matches the energy spectrum.

This method can also be used to quickly find the ground state wave function of the quantum harmonic oscillator.

Indeed

$$|\mathcal{A} \psi_{ground} \rangle = |0 \rangle \quad (5)$$

becomes

$$\psi_{ground}(x) = \psi_0(x) \stackrel{\text{def}}{=} \langle x | \psi_{ground} \rangle = -\frac{\hbar}{m \omega} \frac{d}{dx} \psi_0(x) \quad (6)$$

so that

$$d \psi_0(x) = \psi_0(x) \frac{\hbar}{m \omega} x dx \Rightarrow \ln(\psi_0(x)) = \frac{m \omega}{2\hbar} x^2 + const \quad (7)$$

After normalization this leads to the following position space representation of the ground state wave function.

$$\psi_0(x) = \sqrt[4]{\frac{m \omega}{\pi \hbar}} e^{-\frac{m \omega}{2\hbar} x^2} \quad (8)$$

### Coherent state

A **coherent state** is a specific kind of [state](#)<sup>71</sup> of the [quantum harmonic oscillator](#)<sup>72</sup> whose dynamics most closely resemble the oscillating behavior of a classical harmonic oscillator system.

The coherent state  $|\alpha\rangle$  is defined to be the 'right' eigenstate of the annihilation operator  $\mathcal{A}$ . Formally, this reads:

$$|\mathcal{A} \alpha \rangle = \alpha |\alpha \rangle \quad (1)$$

Since  $\mathcal{A}$  is not Hermitian,  $\alpha$  is a hyper complex number that is not necessarily real, and can be represented as

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<sup>71</sup>States

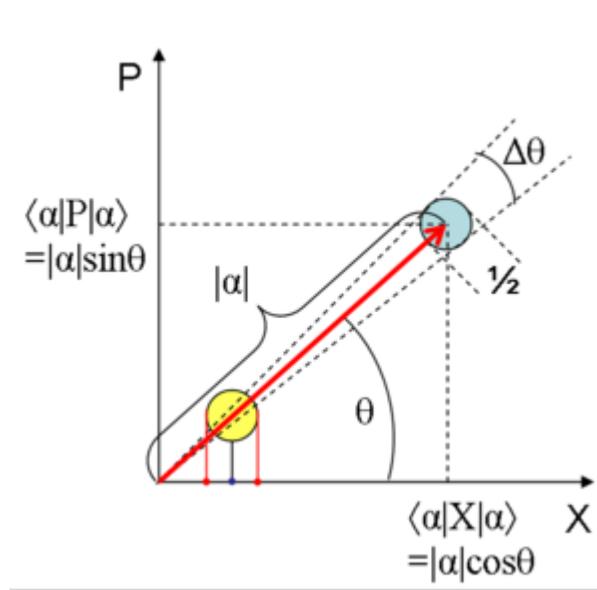
<sup>72</sup> Functions invariant under Fourier transform

$$\alpha = |\alpha| \exp(i\theta) \quad (2)$$

where  $\theta$  is a real number.  $|\alpha|$  is the amplitude and  $\theta$  is the phase of state  $|\alpha\rangle$ .

This formula means that a coherent state is left unchanged by the annihilation or the creation of a particle. The eigenstate of the annihilation operator has a [Poissonian](#)<sup>73</sup> number distribution. A Poisson distribution is a necessary and sufficient condition that all annihilations are statistically independent.

The coherent state's location in the complex plane ([phase space](#)<sup>74</sup>) is centered at the position and momentum of a classical oscillator of the same phase  $\theta$  and amplitude. As the phase increases the coherent state circles the origin and the corresponding disk neither distorts nor spreads. The disk represents Heisenberg's uncertainty. This is the most similar a quantum state can be to a single point in phase space.



<sup>73</sup> <http://en.wikipedia.org/wiki/Poissonian>

<sup>74</sup> [http://en.wikipedia.org/wiki/Phase\\_space](http://en.wikipedia.org/wiki/Phase_space)

Phase space plot of a coherent state. This shows that the uncertainty (blur) in a coherent state is equally distributed in all directions. The horizontal and vertical axes are the X and P quadratures of the field, respectively. Oscillations that are said to be *in quadrature*, if they are separated in phase by  $\pi/2$  radians. The red dots on the x-axis trace out the boundaries of the quantum noise. Further from the origin the relative contribution of the quantum noise becomes less important.

The representation of the coherent state in the basis of Fock states is:

$$|\alpha\rangle = \exp(-\frac{1}{2}|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle = \exp(-\frac{1}{2}|\alpha|^2) \exp(\alpha \mathcal{A}^\dagger) |0\rangle \quad (3)$$

where  $|n\rangle$  are Hermite functions (eigenvectors of the Hamiltonian). This is a Poissonian distribution. The probability of detecting  $n$  photons is:

$$\mathcal{P}(n) = \exp(-\langle n \rangle) \frac{\langle n \rangle^n}{n!} \quad (4)$$

Similarly, the average photon number in a coherent state is

$$\langle n \rangle = \langle \mathcal{A}^\dagger \mathcal{A} \rangle = |\alpha|^2 \quad (5)$$

and the variance is

$$(\Delta n)^2 = \text{Var}(\mathcal{A}^\dagger \mathcal{A}) = |\alpha|^2 \quad (6)$$

### Squeezing

The [squeezing operator](#) can squeeze a state more or less in the direction of either P or Q. The operator is defined as:

$$S_q(z) = \exp\left(\frac{1}{2}(z^* \mathcal{A} + z \mathcal{A}^\dagger)\right) \quad (1)$$

$$z = r \exp(i\theta) \quad (2)$$

The ground state is a saturated squeezed coherent state where

$$\Delta p = \Delta q \text{ and } \Delta q \cdot \Delta p = \hbar/2 \quad (3)$$

### Base transforms

Now we have discovered the following base transforms:

Position $\Leftrightarrow$ momentum:

$$\langle q|p \rangle = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(\frac{k q p}{\hbar}\right) \quad (1)$$

Position $\Leftrightarrow$ Fock state:

$$\langle q|n \rangle = \frac{1}{\sqrt{\pi\hbar}} \frac{1}{\sqrt{2^n n!}} \exp\left(-\frac{m\omega}{2\hbar} q^2\right) H_n\left(q\sqrt{\frac{m\omega}{\hbar}}\right) \quad (2)$$

Fock state $\Leftrightarrow$ coherent state:

$$\langle n|z \rangle = \frac{1}{\sqrt{n!}} z^n \exp(-\frac{1}{2}|z|^2) \quad (3)$$

### Harmonic oscillating Hilbert field

Take the ingredients of the complex harmonic oscillator and interpret these as similar ingredients of a harmonic oscillating Hilbert field that is based on a Gaussian blur. The blur delivers the conditions of the ground state.

$$\psi_0(r) = \sqrt{\frac{m\omega}{\pi\hbar}} e^{-\frac{m\omega}{2\hbar} r^2} \quad (1)$$

This means that the ground state corresponds with a Gaussian charge distribution. Higher states correspond to a blurred current. We indicate this current as vector potential  $\boldsymbol{\varphi}$ . Its time derivative  $\dot{\boldsymbol{\varphi}}$  is perpendicular to  $\boldsymbol{\varphi}$ . The other ingredients are  $P$ ,  $Q$ ,  $\mathcal{A}$  and  $\mathcal{A}^\dagger$ .

$$Q \Leftrightarrow \boldsymbol{\varphi} = \varphi_x = \sqrt{\frac{\hbar}{2m\omega}} (\mathcal{A}_x + \mathcal{A}_x^\dagger) \quad (2)$$

$$\mathcal{A}_x - \mathcal{A}_x^\dagger = 0 \quad (3)$$

$$P \Leftrightarrow m\dot{\boldsymbol{\varphi}} = m\dot{\varphi}_y = \sqrt{\frac{m\omega\hbar}{2}} (-\mathcal{A}_y + \mathcal{A}_y^\dagger) \quad (4)$$

$$\mathcal{A}_y + \mathcal{A}_y^\dagger = 0 \quad (5)$$

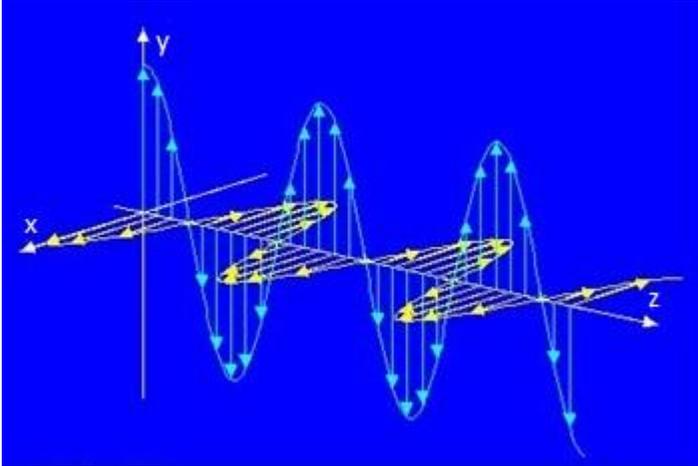
$$\mathcal{A} \Leftrightarrow \mathcal{A} = i\mathcal{A}_x - ik\mathcal{A}_y = \sqrt{\frac{m\omega}{2\hbar}} \left( \boldsymbol{\varphi} + \frac{\dot{\boldsymbol{\varphi}}}{\omega} \right) = \sqrt{\frac{m\omega}{2\hbar}} \left( i\varphi_x + ik\frac{\dot{\varphi}_y}{\omega} \right) \quad (6)$$

$$\mathcal{A}^\dagger \Leftrightarrow \mathcal{A}^\dagger = i\mathcal{A}_x^\dagger + ik\mathcal{A}_y^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( \boldsymbol{\varphi} - \frac{\dot{\boldsymbol{\varphi}}}{\omega} \right) = \sqrt{\frac{m\omega}{2\hbar}} \left( i\varphi_x - ik\frac{\dot{\varphi}_y}{\omega} \right) \quad (7)$$

The  $\boldsymbol{\varphi}$  field and the  $\dot{\boldsymbol{\varphi}}$  field are mutually perpendicular. If both fields are subjected to a synchronized quantum harmonic oscillation, then an oscillating wave results. We take the same ground state for each of the fields. These ground states correspond to a spherical symmetric Gaussian blur.

When bounds of the cavity are removed or relaxed, then the higher order modes may differ in a phase shift. The sign selections set the eigenvalues of the spin operator. The result is an elliptically polarized wave that moves in directions along  $\boldsymbol{\varphi} \times \dot{\boldsymbol{\varphi}}$ .

$\boldsymbol{\varphi}$  no longer stands for a single position, but instead for a Gaussian distribution of positions. Similarly  $\dot{\boldsymbol{\varphi}}$  does not stand for a single moving particle, but for a moving Gaussian cloud of virtual particles.



### **Annihilator and creator**

The annihilator  $\mathcal{A}$  and the creator  $\mathcal{A}^\dagger$  are examples of boson operators. This is a consequence of their commutation relations.

$$\mathcal{A} + \mathcal{A}^\dagger = \alpha \varphi \tag{1}$$

$$\mathcal{A} - \mathcal{A}^\dagger = \beta \dot{\varphi} \tag{2}$$

$$\mathcal{A} = \frac{1}{2}\alpha \varphi + \frac{1}{2}\beta \dot{\varphi} \tag{3}$$

$$\mathcal{A}^\dagger = \frac{1}{2}\alpha \varphi - \frac{1}{2}\beta \dot{\varphi} \tag{4}$$

$$[\mathcal{A}(f), \mathcal{A}^\dagger(g)] = \langle f|g \rangle \tag{5}$$

$$[\mathcal{A}(f), \mathcal{A}(g)] = 0 \tag{6}$$

$$[\mathcal{A}^\dagger(f), \mathcal{A}^\dagger(g)] = 0 \tag{7}$$

The corresponding fermion operators are:

$$\{\mathcal{B}(f), \mathcal{B}^\dagger(g)\} = \langle f|g \rangle \tag{8}$$

$$\{\mathcal{B}(f), \mathcal{B}(g)\} = 0 \quad (9)$$

$$\{\mathcal{B}^\dagger(f), \mathcal{B}^\dagger(g)\} = 0 \quad (10)$$

The fermion operators can be represented by imaginary quaternionic base numbers:

$$\mathcal{B} + \mathcal{B}^\dagger = i \quad (11)$$

$$\mathcal{B} - \mathcal{B}^\dagger = j \quad (12)$$

$$\mathcal{B} = \frac{1}{2}(i + j) \quad (13)$$

$$\mathcal{B}^\dagger = \frac{1}{2}(i - j) \quad (14)$$

$$\begin{aligned} (\mathcal{B} + \mathcal{B}^\dagger)(\mathcal{B} - \mathcal{B}^\dagger) &= \mathcal{B}\mathcal{B} - \mathcal{B}\mathcal{B}^\dagger + \mathcal{B}^\dagger\mathcal{B} - \mathcal{B}^\dagger\mathcal{B}^\dagger \\ &= \mathcal{B}^\dagger\mathcal{B} - \mathcal{B}\mathcal{B}^\dagger = i j \end{aligned} \quad (15)$$

### Rotational symmetry

In case of rotational symmetry in the imaginary part of quaternion space, the exponential function must be replaced by a Bessel function. The corresponding Fourier transform then becomes a [Hankel transform](#)<sup>75</sup>.

The spherical harmonics are eigenfunctions of the square of the [orbital angular momentum](#) operator  $-i\hbar\mathbf{r} \times \nabla$  and therefore they represent the different [quantized](#) configurations of [atomic orbitals](#).

### Spherical harmonics

The following draws from the work of [S. Thangavelu](#)<sup>76</sup>.

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<sup>75</sup> [http://en.wikipedia.org/wiki/Hankel\\_transform](http://en.wikipedia.org/wiki/Hankel_transform)

<sup>76</sup> <http://www.math.iitb.ac.in/atm/faha1/veluma.pdf>

In this subsection we look for eigenfunctions of the Fourier transform which have spherical symmetry. As in the one dimensional case we consider functions of the form

$$f(x) = p(x) \exp(-\pi|x|^2) \quad (1)$$

This will be an eigenfunction of  $\mathcal{F}$  if and only  $p$  satisfies

$$\int_{\mathbb{R}^n} p(x - iy) \exp(-\pi \cdot |x|^2) dx = \lambda p(y) \quad (2)$$

Here in quaternion terms  $x$  and  $iy$  represent two mutually perpendicular imaginary numbers while  $x$  and  $y$  are parallel. Thangavelu uses complex numbers. We keep as close as is possible to his text.

If (2) is true for all  $y \in \mathbb{R}^n$  then we should also have

$$\int_{\mathbb{R}^n} p(x + y) \exp(-\pi \cdot |x|^2) dx = \lambda p(iy) \quad (3)$$

Integrating in polar coordinates the integral on the left is

$$\int_{r=0}^{\infty} |\mathbb{S}^{n-1}| \left( \int_{\mathbb{S}^{n-1}} p(y + r \omega) d\sigma(\omega) \right) \exp(-\pi \cdot r^2) \cdot r^{n-1} dr \quad (4)$$

where  $d\sigma(\omega)$  is the normalised surface measure on the unit sphere  $\mathbb{S}^{n-1}$ .

If  $p$  is homogeneous of degree  $m$  then

$$p(iy) = i^m p(y) \quad (5)$$

and hence for such polynomials the equation

$$\int_{\mathbb{R}^n} p(x + y) \exp(-\pi \cdot |x|^2) dx = \lambda i^m p(iy) \quad (6)$$

will be satisfied for

$$\lambda = (-i)^m \tag{7}$$

if  $p$  has the mean value property

$$\int_{\mathbb{S}^{n-1}} p(y + r \omega) d\sigma(\omega) = p(y) \tag{8}$$

Such functions are precisely the harmonic functions satisfying

$$\Delta u = 0 \tag{9}$$

Thus we have proved:

Let

$$f(x) = p(x) \exp(-\pi |x|^2) \tag{10}$$

where  $p$  is homogeneous of degree  $m$  and harmonic. Then

$$\mathcal{F} f = (-i)^m f \tag{11}$$

Let  $\mathcal{P}^m$  stand for the finite dimensional space of homogeneous harmonic polynomials of degree  $m$ :

The above theorem says that the finite dimensional subspace of  $L^2(\mathbb{R}^n)$  consisting of functions of the form

$$p(x) \exp(-\pi |x|^2); p \in \mathcal{P}^m \tag{12}$$

is invariant under the Fourier transform.

We claim that the following extension is true.

Let

$$f \in L^2(\mathbb{R}^n) \tag{13}$$

be of the form

$$f(x) = p(x)g(|x|); p \in \mathcal{P}^m \tag{14}$$

Then

$$\mathcal{F} f(\xi) = p(\xi)g(|\xi|) \tag{15}$$

Thus the subspace of functions of the form

$$f(x) = p(x)g(|x|); p \in \mathcal{P}^m \tag{16}$$

is invariant under the Fourier transform.

Let

$$f \in L^2(\mathbb{R}^n) \tag{17}$$

be of the form

$$f(x) = p(x)g(|x|); p \in \mathcal{P}^m \tag{18}$$

Then

$$\mathcal{F}_n (f) = (-i)^m p \mathcal{F}_{n+2m} g \tag{19}$$

The above result is known as the Hecke-Bochner formula for the Fourier transform.

We conclude our discussion on invariant subspaces with the following result which shows that the Fourier transform of a radial function reduces to an integral transform whose kernel is a Bessel function. This relates to [the Hankel transform](#).

Let  $J_\alpha$  stand for the Bessel function of type  $\alpha > -1$   
 If

$$f(x) = g(|x|) \tag{20}$$

is radial and integrable then

$$\mathcal{F}_n(f)(\xi) = c_n \int_0^\infty g(r) \cdot \frac{J_{\frac{n-1}{2}}(2\pi r |\xi|)}{(2\pi r |\xi|)^{\frac{n-1}{2}}} \cdot r^{n-1} dr \tag{21}$$

### Spherical harmonics eigenvalues

See: [http://en.wikipedia.org/wiki/Spherical\\_harmonics](http://en.wikipedia.org/wiki/Spherical_harmonics) for more details.

Spherical harmonics are best presented in polar coordinates. There exists a corresponding polar Fourier transform. This Fourier transform also has invariant functions. Like in the rectangular case, they form the basis for spherical harmonics.

Laplace's equation in spherical coordinates is:

$$\nabla^2 f = r^{-2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 f}{\partial \varphi^2} = 0 \tag{1}$$

Try to find solutions in the form of the eigenfunctions of the Fourier transform.

By separation of variables, two differential equations result by imposing Laplace's equation:

$$f(r, \theta, \varphi) = R(r) \cdot Y(\theta, \varphi) \tag{2}$$

$$R^{-1} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \lambda \tag{3}$$

$$\frac{1}{Y \sin(\theta)} \frac{\partial}{\partial \theta} \left( \sin(\theta) \frac{\partial Y}{\partial \theta} \right) + \frac{1}{Y \sin^2(\theta)} \frac{\partial^2 Y}{\partial \varphi^2} = -\lambda \tag{4}$$

The second equation can be simplified under the assumption that  $Y$  has the form

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi) \quad (5)$$

Applying separation of variables again to the second equation gives way to the pair of differential equations

$$\frac{1}{\Phi(\varphi)} \frac{d^2\Phi(\varphi)}{d\varphi^2} = -m^2 \quad (6)$$

$$\lambda \sin^2(\theta) + \frac{\sin(\theta)}{\Theta(\theta)} \frac{d}{d\theta} \left[ \sin(\theta) \frac{d\Theta}{d\theta} \right] = m^2 \quad (7)$$

for some number  $m$ . A priori,  $m$  is a complex constant, but because  $\Phi$  must be a periodic function whose period evenly divides  $2\pi$ ,  $m$  is necessarily an integer and  $\Phi$  is a linear combination of the complex exponentials  $\exp(\pm i m \theta)$ . The solution function  $Y(\theta, \varphi)$  is regular at the poles of the sphere, where  $\theta = 0, \pi$ . Imposing this regularity in the solution  $\Theta$  of the second equation at the boundary points of the domain is a [Sturm–Liouville problem](#)<sup>77</sup> that forces the parameter  $\lambda$  to be of the form  $\lambda = \ell(\ell + 1)$  for some non-negative integer with  $\ell \geq |m|$ ; this is also explained below in terms of the orbital angular momentum. Furthermore, a change of variables  $t = \cos\theta$  transforms this equation into the Legendre equation, whose solution is a multiple of the [associated Legendre function](#)<sup>78</sup>.  $P_\ell^m(\cos(\theta))$ . Finally, the equation for  $R$  has solutions of the form  $R(r) = A r^\ell + B r^{-\ell-1}$ ; requiring the solution to be regular throughout  $\mathbb{R}^3$  forces  $B = 0$ .

Here the solution was assumed to have the special form

$$Y(\theta, \varphi) = \Theta(\theta) \Phi(\varphi) \quad (8)$$

<sup>77</sup> [http://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville\\_problem](http://en.wikipedia.org/wiki/Sturm%E2%80%93Liouville_problem)

<sup>78</sup> [http://en.wikipedia.org/wiki/Associated\\_Legendre\\_function](http://en.wikipedia.org/wiki/Associated_Legendre_function)

For a given value of  $\ell$ , there are  $2\ell + 1$  independent solutions of this form, one for each integer  $m$  with  $-\ell \leq m \leq \ell$ . These angular solutions are a product of trigonometric functions, here represented as a complex exponential, and associated Legendre functions:

$$Y_l^m(\theta, \varphi) = N \exp(i m \varphi) P_l^m(\cos(\theta)) \quad (9)$$

which fulfill

$$r^2 \nabla^2 Y_l^m(\theta, \varphi) = -l(l + 1) Y_l^m(\theta, \varphi) \quad (10)$$

Here  $Y_l^m$  is called a spherical harmonic function of degree  $\ell$  and order  $m$ ,  $P_l^m$  is an associated Legendre function,  $N$  is a normalization constant,  $\theta$  represents the colatitude and  $\phi$  represents the longitude. In particular, the [colatitude](#)<sup>79</sup>  $\theta$ , or polar angle, ranges from 0 at the North Pole to  $\pi$  at the South Pole, assuming the value of  $\pi/2$  at the Equator, and the [longitude](#)<sup>80</sup>  $\varphi$ , or [azimuth](#)<sup>81</sup>, may assume all values with  $0 \leq \varphi < 2\pi$ . For a fixed integer  $\ell$ , every solution  $Y(\theta, \varphi)$  of the eigenvalue problem

$$r^2 \nabla^2 Y = -l(l + 1) Y \quad (11)$$

is a linear combination of  $Y_l^m$ . In fact, for any such solution,  $r^\ell Y(\theta, \varphi)$  is the expression in spherical coordinates of a homogeneous polynomial that is harmonic, and so counting dimensions shows that there are  $2\ell + 1$  linearly independent of such polynomials.

The general solution to Laplace's equation in a ball centered at the origin is a linear combination of the spherical harmonic functions multiplied by the appropriate scale factor  $r^l$ ,

(12)

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<sup>79</sup> <http://en.wikipedia.org/wiki/Colatitude>

<sup>80</sup> <http://en.wikipedia.org/wiki/Longitude>

<sup>81</sup> <http://en.wikipedia.org/wiki/Azimuth>

$$f(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_l^m r^l Y_l^m(\theta, \varphi)$$

where the  $f_l^m$  are constants and the factors  $r^l Y_l^m$  are known as [solid harmonics](#)<sup>82</sup>. Such an expansion is valid in the ball

$$r < R = 1 / \limsup_{l \rightarrow \infty} |f_l^m|^{1/l} \quad (13)$$

### Orbital angular momentum

In quantum mechanics, Laplace's spherical harmonics are understood in terms of the [orbital angular momentum](#)<sup>83</sup>

$$\mathbf{L} = -i \hbar \mathbf{x} \times \nabla = L_x \mathbf{i} + L_y \mathbf{j} + L_z \mathbf{k} \quad (1)$$

The spherical harmonics are eigenfunctions of the square of the orbital angular momentum

$$\begin{aligned} L^2 &= -r^2 \nabla^2 + \left( r \frac{\partial}{\partial r} + 1 \right) r \frac{\partial}{\partial r} \\ &= \frac{1}{\sin(\theta)} \frac{\partial}{\partial \theta} \sin(\theta) \frac{\partial}{\partial \theta} - \frac{1}{\sin^2(\theta)} \cdot \frac{\partial^2}{\partial \varphi^2} \end{aligned} \quad (2)$$

Laplace's spherical harmonics are the joint eigenfunctions of the square of the orbital angular momentum and the generator of rotations about the azimuthal axis:

$$L_z = -i \cdot \hbar \cdot \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = i \hbar \frac{\partial}{\partial \varphi} \quad (3)$$

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<sup>82</sup> [http://en.wikipedia.org/wiki/Solid\\_harmonics](http://en.wikipedia.org/wiki/Solid_harmonics)

<sup>83</sup> [http://en.wikipedia.org/wiki/Orbital\\_angular\\_momentum](http://en.wikipedia.org/wiki/Orbital_angular_momentum)

These operators commute, and are densely defined self-adjoint operators on the Hilbert space of functions  $f$  square-integrable with respect to the normal distribution on  $\mathbb{R}^3$ :

$$(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} |f(x)|^2 \exp(-|x|^2/2) < \infty \quad (4)$$

Furthermore,  $\mathbf{L}^2$  is a positive operator.

If  $Y$  is a joint eigenfunction of  $\mathbf{L}^2$  and  $L_z$ , then by definition

$$\mathbf{L}^2 Y = \lambda Y \quad (5)$$

$$L_z Y = m Y \quad (6)$$

for some real numbers  $m$  and  $\lambda$ . Here  $m$  must in fact be an integer, for  $Y$  must be periodic in the coordinate  $\varphi$  with period a number that evenly divides  $2\pi$ . Furthermore, since

$$\mathbf{L}^2 = L_x^2 + L_y^2 + L_z^2 \quad (7)$$

and each of  $L_x, L_y, L_z$  are self-adjoint, it follows that  $\lambda \geq m^2$ .

Denote this joint eigenspace by  $E_{\lambda, m}$ , and define the raising and lowering operators by

$$L_+ = L_x + i L_y \quad (8)$$

$$L_- = L_x - i L_y \quad (9)$$

Then  $L_+$  and  $L_-$  commute with  $\mathbf{L}^2$ , and the Lie algebra generated by  $L_+, L_-, L_z$  is the special linear Lie algebra, with commutation relations

$$[L_z, L_+] = L_+ \quad (10)$$

$$[L_z, L_-] = -L_- \quad (11)$$

$$[L_+, L_-] = 2 L_z \quad (12)$$

Thus  $L_+ : E_{\lambda,m} \rightarrow E_{\lambda,m+1}$  (it is a "raising operator") and  $L_- : E_{\lambda,m} \rightarrow E_{\lambda,m-1}$  (it is a "lowering operator"). In particular,  $L_+^k : E_{\lambda,m} \rightarrow E_{\lambda,m+k}$  must be zero for  $k$  sufficiently large, because the inequality  $\lambda \geq m^2$  must hold in each of the nontrivial joint eigenspaces. Let  $Y \in E_{\lambda,m}$  be a nonzero joint eigenfunction, and let  $k$  be the least integer such that

$$L_+^k Y = 0 \quad (13)$$

Then, since

$$L_- L_+ = \mathbf{L}^2 - L_z^2 - L_z \quad (14)$$

it follows that

$$0 = L_- L_+^k Y = (\lambda - (m+k)^2 - (m+k)) Y \quad (15)$$

$$\text{Thus } \lambda = \ell(\ell+1) \text{ for the positive integer } \ell = m+k. \quad (16)$$

### Spherical harmonics expansion

The Laplace spherical harmonics form a complete set of orthonormal functions and thus form an orthonormal basis of the Hilbert space of square-integrable functions. On the unit sphere, any square-integrable function can thus be expanded as a linear combination of these:

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_l^m \Upsilon_l^m(\theta, \varphi) \quad (1)$$

This expansion holds in the sense of mean-square convergence — convergence in  $\mathbf{L}^2$  of the sphere — which is to say that

$$\lim_{N \rightarrow \infty} \int_0^{2\pi} \int_0^\pi \left| f(\theta, \varphi) - \sum_{l=0}^N \sum_{m=-l}^l f_l^m \Upsilon_l^m(\theta, \varphi) \right|^2 \sin(\theta) d\theta d\varphi = 0 \quad (2)$$

The expansion coefficients are the analogs of Fourier coefficients, and can be obtained by multiplying the above equation by the complex conjugate of a spherical harmonic, integrating over the solid angle  $\Omega$ , and utilizing the above orthogonality relationships. This is justified rigorously by basic Hilbert space theory. For the case of orthonormalized harmonics, this gives:

$$\begin{aligned} f_l^m &= \int_{\Omega} f(\theta, \varphi) Y_l^{m*}(\theta, \varphi) d\Omega \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \sin(\theta) f(\theta, \varphi) Y_l^{m*}(\theta, \varphi) \end{aligned} \quad (3)$$

If the coefficients decay in  $\ell$  sufficiently rapidly — for instance, exponentially — then the series also converges uniformly to  $f$ . A real square-integrable function  $f$  can be expanded in terms of the real harmonics  $Y_{\ell m}$  above as a sum

$$f(\theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l f_l^m Y_l^m(\theta, \varphi) \quad (4)$$

Convergence of the series holds again in the same sense.

### **Spin weighted spherical harmonics**

Regard the sphere  $S^2$  as embedded into the three-dimensional imaginary part of the quaternionic number field. At a point  $\mathbf{x}$  on the sphere, a positively oriented orthonormal basis of tangent vectors at  $\mathbf{x}$  is a pair  $\mathbf{a}$ ,  $\mathbf{b}$  of vectors such that

$$(\mathbf{x}, \mathbf{a}) = (\mathbf{x}, \mathbf{b}) = (\mathbf{a}, \mathbf{b}) = 0 \quad (1)$$

$$(\mathbf{a}, \mathbf{a}) = (\mathbf{b}, \mathbf{b}) = 1 \quad (2)$$

$$\langle \mathbf{x}, \mathbf{a} \times \mathbf{b} \rangle > 0 \quad (3)$$

where the first pair of equations states that  $\mathbf{a}$  and  $\mathbf{b}$  are tangent at  $\mathbf{x}$ , the second pair states that  $\mathbf{a}$  and  $\mathbf{b}$  are unit vectors,  $\mathbf{a}$  and  $\mathbf{b}$  are orthogonal, and the  $\{\mathbf{x}, \mathbf{a}, \mathbf{b}\}$  is a right-handed basis of  $\mathbb{R}^3$ .

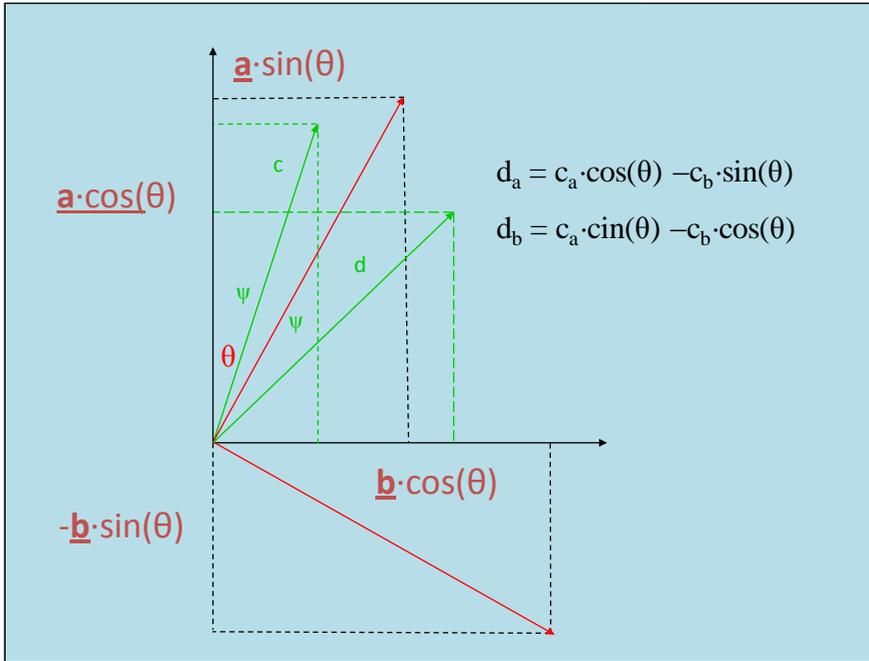


Figure 3:  $\theta$  and the parameters  $\mathbf{a}$  and  $\mathbf{b}$  of the spin-weight function  $f$ .

A spin-weight  $s$  function  $f$  is a function accepting as input a point  $\mathbf{x}$  of  $S^2$  and a positively oriented orthonormal basis of tangent vectors at  $\mathbf{x}$ , such that

$$f(\mathbf{x}, \mathbf{a} \cos(\theta) - \mathbf{b} \sin(\theta), \mathbf{a} \sin(\theta) + \mathbf{b} \cos(\theta)) = \exp(i s \theta) f(\mathbf{x}, \mathbf{a}, \mathbf{b}) \quad (4)$$

for every rotation angle  $\theta$ .

Following Eastwood & Tod (1982), denote the collection of all spin-weight  $s$  functions by  $\mathbf{B}(s)$ . Concretely, these are understood as functions  $f$  on  $\mathcal{C}^2 \setminus \{0\}$  satisfying the following homogeneity law under complex scaling

$$f(\lambda \mathbf{z}, \bar{\lambda} \bar{\mathbf{z}}) = \left(\frac{\bar{\lambda}}{\lambda}\right)^s f(\mathbf{z}, \bar{\mathbf{z}}) \tag{5}$$

This makes sense provided  $s$  is a half-integer.

Abstractly,  $\mathbf{B}(s)$  is isomorphic to the smooth vector bundle underlying the antiholomorphic vector bundle  $O^*(2 \cdot s)$  of the Serre twist on the complex projective line  $\mathbf{C}P^1$ . A section of the latter bundle is a function  $g$  on  $\mathbf{C}^2 \setminus \{0\}$  satisfying

$$g(\lambda \mathbf{z}, \bar{\lambda} \bar{\mathbf{z}}) = (\bar{\lambda})^{2s} g(\mathbf{z}, \bar{\mathbf{z}}) \tag{6}$$

Given such a  $g$ , we may produce a spin-weight  $s$  function by multiplying by a suitable power of the Hermitian form

$$P(\mathbf{z}, \bar{\mathbf{z}}) = \mathbf{z} \bar{\mathbf{z}} \tag{7}$$

Specifically,  $f = P^{-s} g$  is a spin-weight  $s$  function. The association of a spin-weighted function to an ordinary homogeneous function is an isomorphism.

### **Eth**

The spin weight bundles  $\mathbf{B}(s)$  are equipped with a differential operator  $\delta$  (eth). This operator is essentially the [Dolbeault operator](http://en.wikipedia.org/wiki/Dolbeault_operator)<sup>84</sup>,

$$\delta = \partial + \partial^* \tag{1}$$

Thus for  $f \in \mathbf{B}(s)$ ,

$$\delta f \equiv P^{-s+1} \partial(P^s f) \tag{2}$$

defines a function of spin-weight  $s + 1$ .

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<sup>84</sup> [http://en.wikipedia.org/wiki/Dolbeault\\_operator](http://en.wikipedia.org/wiki/Dolbeault_operator)

### Spin-weighted harmonic functions

See [http://en.wikipedia.org/wiki/Spin-weighted\\_spherical\\_harmonics](http://en.wikipedia.org/wiki/Spin-weighted_spherical_harmonics) for more details.

Just as conventional spherical harmonics are the eigenfunctions of the Laplace-Beltrami operator on the sphere, the spin-weight  $s$  harmonics are the eigensections for the Laplace-Beltrami operator acting on the bundles  $\mathcal{E}(s)$  of spin-weight  $s$  functions.

The spin-weighted harmonics can be represented as functions on a sphere once a point on the sphere has been selected to serve as the North Pole. By definition, a function  $\eta$  with *spin weight*  $s$  transforms under rotation about the pole via

$$\eta \rightarrow \exp(i s \psi) \eta \quad (1)$$

Working in standard spherical coordinates, we can define a particular operator  $\delta$  acting on a function  $\eta$  as:

$$\delta\eta = -\sin^s(\theta) \left\{ \frac{\partial}{\partial\theta} + \frac{i}{\sin(\theta)} \frac{\partial}{\partial\varphi} \right\} [\sin^{-s}(\theta) \eta] \quad (2)$$

This gives us another function of  $\theta$  and  $\varphi$ . [The operator  $\delta$  is effectively a covariant derivative operator in the sphere.]

An important property of the new function  $\delta\eta$  is that if  $\eta$  had spin weight  $s$ ,  $\delta\eta$  has spin weight  $s + 1$ . Thus, the operator raises the spin weight of a function by 1. Similarly, we can define an operator which will lower the spin weight of a function by 1:

$$\bar{\delta}\eta = -\sin^{-s}(\theta) \left\{ \frac{\partial}{\partial\theta} - \frac{i}{\sin(\theta)} \frac{\partial}{\partial\varphi} \right\} [(\sin^s(\theta) \eta)] \quad (3)$$

We extend the function  $Y_l^m$  to  ${}_s Y_l^m$  according to

$${}_0 Y_l^m(\theta, \varphi) = Y_l^m(\theta, \varphi) \quad (4)$$

$$l = 0, 1, 2, \dots; m = -l, \dots, 0, \dots, l \quad (5)$$

The spin-weighted spherical harmonics are then defined in terms of the usual spherical harmonics as:

$${}_s Y_l^m = \sqrt{\frac{(l-s)!}{(l+s)!}} \delta^s Y_l^m; 0 \leq s \leq l \quad (6)$$

$${}_s Y_l^m = \sqrt{\frac{(l+s)!}{(l-s)!}} (-1)^s \delta^s Y_l^m; -l \leq s \leq 0 \quad (7)$$

$${}_s Y_l^m = 0; l < |s|; \quad (8)$$

The functions  ${}_s Y_l^m$  then have the property of transforming with spin weight  $s$ .

Other important properties include the following:

$$\delta({}_s Y_l^m) = +\sqrt{(l-s)(l+s+1)} {}_{s+1} Y_l^m \quad (9)$$

$$\delta({}_s Y_l^m) = -\sqrt{(l+s)(l-s+1)} {}_{s-1} Y_l^m \quad (10)$$

### Special Fourier transform pairs

Functions that keep the same form through Fourier transformation are:

$$f(q) = \exp(-|q|^2) \quad (1)$$

$$f(q) = \frac{1}{|q|} \quad (2)$$

$$f(q) = \text{comb}(q) \quad (3)$$

The comb function consists of a set of equidistant Dirac delta functions.

Other examples of functions that are invariant under Fourier transformation are the linear and spherical harmonic oscillators and the solutions of the Laplace equation.

## Complex Fourier transform invariance properties

Each even function  $f(q) \Leftrightarrow \tilde{f}(p)$  induces a Fourier invariant:

$$h(q) = \sqrt{2\pi} f(q) + \tilde{f}(q). \quad (1)$$

$$\tilde{h}(q) = \sqrt{2\pi} h(q) \quad (2)$$

Each odd function  $f(q) \Leftrightarrow \tilde{f}(p)$  induces a Fourier invariant:

$$h(q) = \sqrt{2\pi} f(q) - \tilde{f}(q). \quad (3)$$

A function  $f(q)$  is invariant under Fourier transformation *if and only if* the function  $f$  satisfies the differential equation

$$\frac{\partial^2 f(q)}{\partial q^2} - t^2 f(q) = \alpha f(q), \text{ for some scalar } \alpha \in \mathcal{C}. \quad (4)$$

The Fourier transform invariant functions are fixed apart from a scale factor. That scale factor can be 1,  $k$ , -1 or  $-k$ .  $k$  is an imaginary base number in the longitudinal direction.

Fourier-invariant functions show iso-resolution, that is,  $\Delta_p = \Delta_q$  in the Heisenberg's uncertainty relation.

For proves see: [http://www2.ee.ufpe.br/codec/isoresolution\\_vf.pdf](http://www2.ee.ufpe.br/codec/isoresolution_vf.pdf).

## Fourier transform properties

### Parseval's theorem

Parseval's theorem runs:

$$\int f^*(q) \cdot g(q) \cdot dV_q = \int \tilde{f}^*(p) \cdot \tilde{g}(p) \cdot dV_p \quad (1)$$

This leads to

$$\int |f(q)|^2 \cdot dV_q = \int |\tilde{f}(p)|^2 \cdot dV_p \quad (1)$$

### Convolution

Through Fourier transformation a convolution changes into a simple product and vice versa.

$$\mathcal{F}(f(q) \circ g(q)) = \tilde{f}(p) \cdot \tilde{g}(p) \quad (1)$$

### Differentiation

Fourier transformation converts differentiation into multiplication with the canonical conjugated coordinate.

$$g(q) = \nabla f(q) \quad (1)$$

$$\tilde{g}(p) = p\tilde{f}(p) \quad (2)$$

### Vacuum expectation value

The **vacuum expectation value** (also called **condensate** or simply VEV) of an operator is its average, expected value in the [vacuum](#)<sup>85</sup>. The vacuum corresponds to a ground state. The vacuum expectation value of an operator  $O$  is usually denoted by  $\langle O \rangle$ .

### Hilbert field equations

Despite the obvious similarity, Hilbert field equations are not Maxwell field equations. First of all, the Hilbert field is a skew field and it carries the properties of the quaternions and the properties of the quaternionic Fourier transform. Next Hilbert fields are mathematical (skew) fields, while Maxwell fields are physical fields in a 3D geometry. Finally the Hilbert fields are constituted from a collection of member fields.

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<sup>85</sup> [http://en.wikipedia.org/wiki/Vacuum\\_state](http://en.wikipedia.org/wiki/Vacuum_state)

## Statics and dynamics

In this section we mix statics and dynamics. When we confine to the static status quo we get the two sets of equations. The first set concerns the rotation free part of the field. The second set concerns the divergence free part. When dynamics is supported as well, then these parts get coupled. A single separable Hilbert space  $\mathbf{H}_i$  only meets the static parts of the Hilbert field. This means that in this Hilbert space the two parts do not couple. Coupling only takes place during the step from one Hilbert space to the next member in the sequence. Continuity equations describe the coupling between the parts.

The parameter that controls dynamics in these differential equations is the progression parameter. This parameter stands for the counter of the progression steps. For mathematical convenience we take an approach that is similar to the move from a separable Hilbert space  $\mathbf{H}_i$  to a rigged Hilbert space  $\mathbf{H}$ . We move from a fundamentally granular progression step counter to a continuous progression parameter. This progression parameter is not our common notion of time. The derivative of the field  $f$  for this parameter is defined as  $\nabla_0 f$ .

In order to make the step from the integer progression step to the continuous progression parameter possible there must be a mechanism that reduces change, such that no violent steps are taken. On the other hand the mechanism must not be so strong that only a few steps are taken after which the universe is put to an eternal hold. How this in practice is regulated is shown by the phenomenon [inertia](#)<sup>86</sup>. Inertia is installed by the community of all particles. Locally this community generates an enormous potential. This potential works the same in all directions, so when nothing happens it has no influence on a local particle. A uniform movement of a local particle corresponds with the existence of a local vector potential. Also this vector potential does not apply any action. However, when the particle accelerates, then this goes together with the existence of a vector field that counteracts the acceleration. Thus, inertia

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<sup>86</sup> Influence:Inertia

does not counteract uniform movement. This uniform movement causes redistribution of the particles and with it a reconfiguration of the field. This disturbance of the static status quo is the motor that keeps dynamics going. The tolerance with respect to uniform movement is the reason that the movement does not get killed.

### The quaternionic nabla

According to the definition of quaternionic differentiation the split of quaternionic nabla operator in a real part and an imaginary part is defined by:

$$g(q) = \nabla f(q) = g_0(q) + \mathbf{g}(q) \quad (1)$$

$$g_0(q) = \nabla_0 f_0(q) \mp \langle \nabla, \mathbf{f}(q) \rangle \quad (2)$$

$$\begin{aligned} \mathbf{g}(q) &= \pm \nabla_0 \mathbf{f}(q) + \nabla f_0(q) \pm (\pm \nabla \times \mathbf{f}(q)) \\ &= \pm \nabla_0 \mathbf{f}(q) + \mathbf{g}_l(q) + \mathbf{g}_t(q) \end{aligned} \quad (3)$$

The second term on the right treats imaginary divergence. The last term treats the rotation. The first term is raised due to the dynamic coupling of the static fields.

In Fourier space the equivalent equation are:

$$\tilde{g}(p) = p f(q) = \tilde{g}_0(p) + \tilde{\mathbf{g}}(p) \quad (4)$$

$$\tilde{g}_0(p) = p_0 \tilde{f}_0(p) \mp \langle \mathbf{p}, \tilde{\mathbf{f}}(p) \rangle \quad (5)$$

$$\begin{aligned} \tilde{\mathbf{g}}(q) &= \pm p_0 \tilde{\mathbf{f}}(p) + \mathbf{p} \tilde{f}_0(p) \pm (\pm \mathbf{p} \times \tilde{\mathbf{f}}(p)) \\ &= \pm p_0 \tilde{\mathbf{f}}(p) + \tilde{\mathbf{g}}_l(p) + \tilde{\mathbf{g}}_t(p) \end{aligned} \quad (6)$$

### Blurring the charges

We may represent the members of the Hilbert distribution with Dirac delta functions. These Dirac delta functions can be multiplied with a hyper complex number. Such a distribution raises problems with the nabla operator.

However, since these members represent anchor points and since each anchor point attaches to a probability amplitude distribution, it has more sense to start directly with these blurred points. We introduce the quaternionic function  $\rho(q)$  that represent the presence of one or more blurred anchor points in its real part and represent the flow of these blurred anchor points in its imaginary part. This leads to integral and differential continuity equations.

Not all anchor points must be equal. When this is true, it is better to categorize them and treat each category separately. Each member of such a category represents a charge that is typical for that category.

The probability amplitude distribution  $\rho(q)$  can be interpreted as the combination of a scalar potential  $\rho_0(q)$  and a vector potential  $\boldsymbol{\rho}(q)$ .

### Continuity equation for charges

When  $\rho_0(q)$  is interpreted as a probability density distribution, then the conservation of the corresponding charge is given by the continuity equation:

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \quad (1)$$

$$\frac{d}{dt} \int_V \rho_0 dV = \oint_S \hat{\mathbf{n}} \rho_0 \mathbf{v} dS + \int_V s_0 dV \quad (2)$$

$$\int_V \nabla_0 \rho_0 dV = \int_V \langle \nabla, \boldsymbol{\rho} \rangle dV + \int_V s_0 dV \quad (3)$$

Here  $\hat{\mathbf{n}}$  is the normal vector pointing outward the surrounding surface  $S$ ,  $\mathbf{v}(q)$  is the velocity at which the charge density  $\rho_0(q)$  enters volume  $V$  and  $s_0$  is the source density inside  $V$ .  $\boldsymbol{\rho}$  stands for  $\rho_0\mathbf{v}$ . The conversion from formula (2) to formula (3) uses the Gauss theorem<sup>87</sup>. This results in the law of charge conservation

$$\begin{aligned} s_0(q) &= \nabla_0\rho_0(q) \pm \langle \nabla, (\rho_0(q)\mathbf{v}(q) + \nabla \times \mathbf{a}(q)) \rangle \\ &= \nabla_0\rho_0(q) \mp \langle \nabla, \boldsymbol{\rho}(q) \rangle \end{aligned} \quad (4)$$

The blue colored  $\pm$  indicates quaternionic sign selection through conjugation <sup>①</sup> or <sup>③</sup>.

In Fourier space this becomes:

$$\tilde{s}_0(p) = p_0\tilde{\rho}_0(p) \mp \langle \mathbf{p}, \tilde{\boldsymbol{\rho}}(q) \rangle \quad (5)$$

This equation represents a balance equation for charge (or mass) density. Here  $\rho_0(q)$  is the charge distribution,  $\boldsymbol{\rho}(q)$  is the current density and  $\mathbf{a}(q)$  is an arbitrary differentiable vector function.

The law of charge conservation (4) resembles equation (2) of the paragraph on the quaternionic nabla. This only treats the real part of the full equation. The full equation runs:

$$s(q) = \nabla\rho(q) = s_0(q) + \mathbf{s}(q) \quad (6)$$

$$\begin{aligned} s_0(q) &= \nabla_0\rho_0(q) \pm \langle \nabla, (\rho_0(q)\mathbf{v}(q) + \nabla \times \mathbf{a}(q)) \rangle \\ &= \nabla_0\rho_0(q) \mp \langle \nabla, \boldsymbol{\rho}(q) \rangle \end{aligned} \quad (7)$$

$$\mathbf{s}(q) = \pm \nabla_0\boldsymbol{\rho}(q) + \nabla\rho_0(q) \pm (\pm \nabla \times \boldsymbol{\rho}(q)) \quad (8)$$

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<sup>87</sup> [http://en.wikipedia.org/wiki/Divergence\\_theorem](http://en.wikipedia.org/wiki/Divergence_theorem)

### Another interpretation of $\rho$

The probability amplitude distribution  $\rho(q)$  can be used to define a charge probability density and probability current density. See <http://www.vttoth.com/qt.htm>.

$$\rho(q) = \alpha \rho_0(q) \quad (1)$$

By using in  $\alpha$  the matrix<sup>88</sup> form of the sign selections<sup>89</sup> we get the Dirac equation:

$$\alpha_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \quad (2)$$

$$\alpha_2 = \begin{bmatrix} 0 & j \\ -j & 0 \end{bmatrix} \quad (3)$$

$$\alpha_3 = \begin{bmatrix} 0 & k \\ -k & 0 \end{bmatrix} \quad (4)$$

$$\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (5)$$

Compare Dirac<sup>90</sup>:

$$\nabla_0 \psi = (-\alpha i \nabla \pm \beta m) \psi \quad (6)$$

Where  $\rho_0$  corresponds to  $\psi \psi^*$ . See <http://www.vttoth.com/qt.htm>.

There exist also a relation between  $\alpha_1, \alpha_2, \alpha_3$  and the Pauli<sup>91</sup> matrices  $\sigma_1, \sigma_2, \sigma_3$ :

$$\sigma = \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \quad (7)$$

$$1 \mapsto I, \quad i \mapsto i \sigma_1, \quad j \mapsto i \sigma_2, \quad k \mapsto i \sigma_3 \quad (17)$$

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<sup>88</sup> Numbers; Sign selections; Matrices

<sup>89</sup> Numbers; Sign selection

<sup>90</sup> Equations of motion: Dirac equation

<sup>91</sup> Equations of motion; Pauli equation

$$\nabla s_{\phi}(q) = \nabla_{\phi} \nabla \rho_{\phi}(q) + \nabla \langle \nabla, \rho(q) \rangle \quad (18)$$

### Example potential

Spatial [Harmonic functions](#)<sup>92</sup> are suitable spread functions.

For a harmonic function  $f(q)$  holds:

$$\Delta f(q) = \nabla \nabla^{\circledast} f(q) = 0 \quad (1)$$

If there is a static spherically symmetric Gaussian charge density  $\rho(r)$ :

$$\rho(q) = \frac{Q}{\sqrt{2\pi\sigma^2}^3} \exp(-|q|^2/(2\sigma^2)) \quad (2)$$

where  $Q$  is the total charge, then the solution  $\phi(r)$  of [Poisson's equation](#)<sup>93</sup>,

$$\nabla^2 \phi(q) = -\frac{\rho(q)}{\varepsilon} \quad (3)$$

is given by

$$\phi(q) = \frac{Q}{4\pi\varepsilon|q|} \operatorname{erf}\left(\frac{|q|}{\sqrt{2}\sigma}\right) \quad (4)$$

where  $\operatorname{erf}(x)$  is the error function.

In fact the quaternionic Poisson's equation represents two separate equations:

$$(\nabla_0^2 - \nabla^2)\phi_0(q) = -\frac{\rho_0(q)}{\varepsilon} \quad (5)$$

$$(\nabla_0^2 - \nabla^2)\phi(q) = -\frac{\rho(q)}{\varepsilon} \quad (6)$$

<sup>92</sup> [http://en.wikipedia.org/wiki/Harmonic\\_function](http://en.wikipedia.org/wiki/Harmonic_function)

<sup>93</sup> [http://en.wikipedia.org/wiki/Poisson%27s\\_equation](http://en.wikipedia.org/wiki/Poisson%27s_equation)

Note that, for  $|q|$  much greater than  $\sigma$ , the erf function approaches unity and the potential  $\phi(r)$  approaches the point charge potential  $\frac{q}{4\pi\epsilon|q|}$ , as one would expect. Furthermore the erf function approaches 1 extremely quickly as its argument increases; in practice for  $|q| > 3\sigma$  the relative error is smaller than one part in a thousand<sup>94</sup>.

The definition of the quaternionic potential  $\phi(q)$  is based on the convolution of a quaternionic distribution  $\rho(q)$  with the real function  $\varphi(q)$ . See Newton potential and Bertrand's theorem in Wikipedia. The real part  $\rho_0(q)$  of the distribution  $\rho(q)$  can be interpreted as a charge distribution. The imaginary part  $\boldsymbol{\rho}(q)$  can be interpreted as a current distribution. The convolution blurs the distribution such that the result becomes differentiable.

In configuration space holds:

$$\phi(q) = \rho(q) \circ \frac{1}{|q|}. \tag{7}$$

Reversely, according to Poisson's equation:

$$\rho(q) = -\Delta\phi(q) \tag{8}$$

The real part of  $\phi(q)$  presents a scalar potential. The imaginary part presents a vector potential.

$$\phi(q) = \phi_0(q) + \boldsymbol{\phi}(q) \tag{9}$$

The scalar potential is a blurred charge distribution.  
 The vector potential is a blurred current distribution.  
 Current is moving charge.  
 Mass is a form of charge.

<sup>94</sup> [http://en.wikipedia.org/wiki/Poisson's\\_equation#Potential\\_of\\_a\\_Gaussian\\_charge\\_density](http://en.wikipedia.org/wiki/Poisson's_equation#Potential_of_a_Gaussian_charge_density)

(The selected blurring function has striking resemblance with the [ground state of the quantum harmonic oscillator](#)<sup>95</sup>).

In Fourier space holds:

$$\tilde{\phi}(p) = \tilde{\rho}(p) \cdot \frac{1}{|p|} = \tilde{\phi}_0(p) + \tilde{\Phi}(p) \quad (10)$$

In Fourier space the frequency spectrum of the Hilbert distribution is multiplied with the Fourier transform of the blurring function. When this falls off when the frequencies go to infinity, then as a consequence the frequency spectrum of the potential is bounded. This is valid independent of the fact that the frequency spectrum of the Hilbert distribution might be unbounded.

The equation for the conservation of charge:

$$s_0(q) = \nabla_0 \rho_0(q) \mp \langle \nabla, \rho(q) \rangle \quad (11)$$

Translates in the source free case  $s_0(q) = 0$  into:

$$\nabla_0 \rho_0(q) = \pm \langle \nabla, \rho(q) \rangle \quad (12)$$

And in the Lorentz Gauge condition<sup>96</sup>:

$$\nabla_0 \phi_0(q) = \pm \langle \nabla, \phi(q) \rangle \quad (13)$$

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla \phi(q) \quad (14)$$

$$\mathfrak{F}_0(q) = \nabla_0 \phi_0(q) \mp \langle \nabla, \phi(q) \rangle = 0$$

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<sup>95</sup> Functions and fields: Functions invariant under Fourier transformation: Ladder operator: Ground state

<sup>96</sup> [http://en.wikipedia.org/wiki/Lorentz\\_gauge\\_condition](http://en.wikipedia.org/wiki/Lorentz_gauge_condition)

In the source divergence free case  $\nabla s_0(q) = 0$  this means:

$$\nabla_0 \nabla \rho_0(q) = \pm \nabla \langle \nabla, \rho(q) \rangle \quad (15)$$

$$\nabla_0 \nabla \phi_0(q) = \pm \nabla \langle \nabla, \phi(q) \rangle \quad (16)$$

$$\nabla \langle \nabla, \phi(q) \rangle = \nabla \times \nabla \times \phi(q) + \nabla^2 \phi(q) \quad (17)$$

Due to the fact that there are other charges present, the divergence of the scalar potential need be in the direction of the current  $\rho(q)$ , which for a spherical symmetric blur is also in the direction of the vector potential  $\phi(q)$ . However, a tendency exists to minimize that difference. Thus  $\nabla_0 \nabla \phi_0(q)$  is parallel to  $\phi(q)$ . With other words:

$$\phi(q) \times \nabla \langle \nabla, \phi(q) \rangle = 0 \quad (18)$$

Reckoning the sign selections for the sign  $\pm$  of the conjugation and the handedness  $\pm$  of the cross product will provide four different sets of equations. This will provide four different Hilbert fields.

### Discrete distribution

If  $\rho(q)$  is discrete, such that

$$\rho(q) = \sum_i q_{E_i} \cdot \delta(q - q_i) \quad (1)$$

where  $q'_E$  is a point charge at location  $q'$ , then the contribution to the field  $E(q)$  that is generated by a point charge at location  $q_i$  is given by:

$$dE(q) = q_{E_i} \cdot \frac{q_i - q}{|q_i - q|^3} = -q_{E_i} \cdot \nabla \cdot \frac{1}{|q_i - q|} \quad (2)$$

### Differential potential equations

The gradient and curl of  $\phi(q)$  are related. In configuration space holds:

$$\nabla \phi(q) = \nabla_0 \phi_0(q) \mp \langle \nabla, \phi(q) \rangle \pm \nabla_0 \phi(q) \pm \nabla \phi_0(q) \pm \nabla \times \phi(q) \quad (1)$$

$$\mathfrak{E}(q) \stackrel{\text{def}}{=} -\nabla\phi_0(q) \quad (2)$$

$$\mathfrak{B}(q) \stackrel{\text{def}}{=} \nabla \times \phi(q) \quad (3)$$

$$\mathfrak{F}(q) \stackrel{\text{def}}{=} \nabla\phi(q) = \mathfrak{F}_0(q) \mp \mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\phi(q) \quad (4)$$

$$\mathfrak{F}_0(q) = \nabla_0\phi_0(q) \mp \langle \nabla, \phi(q) \rangle \quad (5)$$

$$\mathfrak{F}(q) = \mp \mathfrak{E}(q) \pm \mathfrak{B}(q) \pm \nabla_0\phi(q) \quad (6)$$

$$\langle \nabla, \mathfrak{E}(q) \rangle = -\nabla^2\phi_0(q) = \rho_0(q) \quad (7)$$

$$\nabla \times \mathfrak{E}(q) = 0; \text{ Rotation free field} \quad (8)$$

$$\langle \nabla, \mathfrak{B}(q) \rangle = 0; \text{ Divergence free } B \text{ field} \quad (9)$$

$$\nabla \times \mathfrak{B}(q) = \nabla \langle \nabla, \phi(q) \rangle - \nabla^2\phi(q) = \nabla \langle \nabla, \phi(q) \rangle + \rho(q) + \nabla_0^2\phi(q) \quad (10)$$

$$\nabla \times \mathfrak{B}(q) = \pm \nabla_0 \nabla \phi_0(q) + \rho(q) + \nabla_0^2\phi(q) \quad (11)$$

$$= \pm \nabla_0 \mathfrak{E}(q) + \rho(q) + \nabla_0^2\phi(q)$$

Since  $\nabla_0\phi(q)$  is supposed to be parallel to  $\nabla\phi_0(q)$ , it is sensible to define  $\mathbf{E}(q)$  as the total field in longitudinal direction:

$$\mathbf{E}(q) = -\nabla\phi_0(q) - \nabla_0\phi(q) = \mathfrak{E}(q) - \nabla_0\phi(q) \quad (12)$$

And

$$\mathbf{B}(q) = \mathfrak{B}(q) \quad (13)$$

With this definition:

$$\nabla \times \mathbf{E}(q) = -\nabla_0\mathbf{B}(q) \quad (14)$$

$$\langle \nabla, \mathbf{B}(q) \rangle = 0 \quad (15)$$

$$\nabla \times \mathbf{B}(q) = \boldsymbol{\rho}(q) + \nabla_0 \mathbf{E}(q) \quad (16)$$

### In Fourier space

In Fourier space holds:

$$\tilde{\mathfrak{F}}(p) = p_0 \tilde{\phi}_0(p) - \langle \mathbf{p}, \tilde{\boldsymbol{\phi}}(p) \rangle \pm p_0 \tilde{\boldsymbol{\phi}}(p) \pm \mathbf{p} \tilde{\phi}_0(p) \pm \mathbf{p} \times \tilde{\boldsymbol{\phi}}(p) \quad (1)$$

$$\tilde{\mathfrak{F}}(p) = p \tilde{\phi}(p) = \mathfrak{F}_0(p) \mp \tilde{\mathfrak{C}}(p) \pm \tilde{\mathfrak{B}}(p) \pm p_0 \tilde{\boldsymbol{\phi}}(p) \quad (2)$$

$$\tilde{\mathfrak{F}}_0(p) = p_0 \tilde{\phi}_0(p) - \langle \mathbf{p}, \tilde{\boldsymbol{\phi}}(p) \rangle \quad (3)$$

$$\tilde{\mathfrak{C}}(p) = -\mathbf{p} \tilde{\phi}_0(p) \quad (4)$$

$$\tilde{\mathfrak{E}}(p) = -\mathbf{p} \tilde{\phi}_0(p) \pm p_0 \tilde{\boldsymbol{\phi}}(p)$$

$$\tilde{\mathfrak{B}}(p) = \mathbf{p} \times \tilde{\boldsymbol{\phi}}(p) \quad (5)$$

$$\tilde{\mathfrak{F}}(p) = \mp \tilde{\mathfrak{C}}(p) \pm \tilde{\mathfrak{B}}(p) \pm p_0 \tilde{\boldsymbol{\phi}}(p) \quad (6)$$

$$\langle \mathbf{p}, \tilde{\mathfrak{C}}(p) \rangle = -p^2 \tilde{\phi}_0(p) = \tilde{\rho}_0(p) \quad (7)$$

$$\mathbf{p} \times \tilde{\mathfrak{C}}(p) = 0; \text{ Rotation free field} \quad (8)$$

$$\langle \mathbf{p}, \tilde{\mathfrak{B}}(p) \rangle = 0; \text{ Divergence free } B \text{ field} \quad (9)$$

$$\mathbf{p} \times \tilde{\mathfrak{B}}(p) = \mathbf{p} \langle \mathbf{p}, \tilde{\boldsymbol{\phi}}(q) \rangle - p^2 \tilde{\boldsymbol{\phi}}(q) = \mathbf{p} \langle \mathbf{p}, \tilde{\boldsymbol{\phi}}(p) \rangle + \tilde{\boldsymbol{\rho}}(p) \quad (10)$$

$$\mathbf{p} \times \tilde{\mathfrak{B}}(p) = \pm p_0 \mathbf{p} \tilde{\phi}_0(p) + \tilde{\boldsymbol{\rho}}(p) = \pm p_0 \tilde{\mathfrak{C}}(p) + \tilde{\boldsymbol{\rho}}(p) \quad (11)$$

If the distribution  $\rho(q)$  is differentiable, then the same equations that hold for fields  $\phi(q)$  and  $\tilde{\phi}(p)$  hold for the non-blurred distributions  $\rho(q)$  and  $\tilde{\rho}(q)$ .

### Maxwell equations

First it must be noted that the above derived field equations hold for general quaternionic fields.

The resemblance with physical fields holds for electromagnetic fields as well as for gravitational fields and for any fields whose blurring function approximates

$$f(q) \approx \frac{1}{|q|}.$$

In Maxwell equations,  $E(\mathbf{r})$  is defined as:

$$\mathbf{E}(\mathbf{r}, t) \equiv -\nabla\phi_0(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t} = \mathfrak{E}(\mathbf{r}, t) - \frac{\partial\mathbf{A}(\mathbf{r}, t)}{\partial t}$$

Further:

$$\begin{aligned} \langle \nabla, \mathbf{E}(\mathbf{r}, t) \rangle &= -\nabla^2\phi_0(\mathbf{r}, t) - \frac{\partial\langle \nabla, \mathbf{A}(\mathbf{r}, t) \rangle}{\partial t} \\ &= \frac{\rho_0(\mathbf{r}, t)}{\varepsilon_0} - \frac{\partial\langle \nabla, \mathbf{A}(\mathbf{r}, t) \rangle}{\partial t} \end{aligned}$$

In Maxwell equations,  $\mathbf{B}(\mathbf{r})$  is defined as:

$$\mathbf{B}(\mathbf{r}, t) \equiv \nabla \times \mathbf{A}(\mathbf{r}, t) = \mathfrak{B}(\mathbf{r}, t)$$

Further:

$$\nabla \times \mathbf{E}(\mathbf{r}, t) = - \frac{\partial\mathbf{B}(\mathbf{r}, t)}{\partial t}$$

$$\langle \nabla, \mathbf{B}(\mathbf{r}, t) \rangle = 0$$

$$\nabla \times \mathbf{B}(\mathbf{r}, t) = \mu_0 \left( \mathbf{j} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

### **Differentiable distribution**

If the distribution  $\rho(q)$  is differentiable, then the same equations that hold for fields  $\phi(q)$  and  $\tilde{\phi}(p)$  hold for the non-blurred distributions  $\rho(q)$  and  $\tilde{\rho}(q)$ .

Using:

$$\mathbf{B} = \nabla \times \boldsymbol{\phi} = \mathbf{i}(\nabla_2 \phi_{\parallel} - \nabla_{\parallel} \phi_2) + \mathbf{j}(\nabla_{\parallel} \phi_1 - \nabla_1 \phi_{\parallel}) + \mathbf{k}(\nabla_1 \phi_2 - \nabla_2 \phi_1) \quad (1)$$

gives

$$\nabla_0 \phi_{\parallel}(q) = \mp \nabla_{\parallel} \phi_0(q) \quad (2)$$

$$\nabla_0 \phi_1(q) = \mp \left( \nabla_2 \phi_{\parallel}(q) - \nabla_{\parallel} \phi_2(q) \right) \quad (3)$$

$$\nabla_0 \phi_2(q) = \pm \left( \nabla_1 \phi_{\parallel}(q) - \nabla_{\parallel} \phi_1(q) \right) \quad (4)$$

$$\nabla_0 \phi_0(q) = \langle \nabla, \boldsymbol{\phi}(q) \rangle = \nabla_{\parallel} \phi_{\parallel}(q) + \nabla_1 \phi_1(q) + \nabla_2 \phi_2(q) \quad (5)$$

And correspondingly in Fourier space

$$p_0 \tilde{\phi}_{\parallel}(p) = \mp p_{\parallel} \tilde{\phi}_0(p) \quad (6)$$

$$p_0 \tilde{\phi}_1(p) = \pm \left( p_{\parallel} \tilde{\phi}_2(p) - p_2 \tilde{\phi}_{\parallel}(p) \right) \quad (7)$$

$$p_0 \tilde{\phi}_2(p) = \mp \left( p_{\parallel} \tilde{\phi}_1(p) - p_1 \tilde{\phi}_{\parallel}(p) \right) \quad (8)$$

$$p_0 \tilde{\phi}_0(p) = \langle \mathbf{p}, \tilde{\boldsymbol{\phi}}(p) \rangle = p_{\parallel} \tilde{\phi}_{\parallel}(p) + p_1 \tilde{\phi}_1(p) + p_2 \tilde{\phi}_2(p) \quad (9)$$

## Conservation laws

### Flux vector

The longitudinal direction  $\mathbf{k}$  of  $\mathbf{E}(q)$  and the direction  $\mathbf{i}$  of  $\mathbf{B}(q)$  fix two mutual perpendicular directions. This generates curiosity to the significance of the direction  $\mathbf{k} \times \mathbf{i}$ . With other words what happens with  $\mathbf{E}(q) \times \mathbf{B}(q)$ .

The **flux vector**  $\mathfrak{S}(q)$  is defined as:

$$\mathfrak{S}(q) \stackrel{\text{def}}{=} \mathbf{E}(q) \times \mathbf{B}(q) \quad (1)$$

### Conservation of energy

Field energy density

$$\begin{aligned} \langle \nabla, \mathfrak{S}(q) \rangle &= \langle \mathbf{B}(q), \nabla \times \mathbf{E}(q) \rangle - \langle \mathbf{E}(q), \nabla \times \mathbf{B}(q) \rangle \\ &= -\langle \mathbf{B}(q), \nabla_0 \mathbf{B}(q) \rangle - \langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle - \langle \mathbf{E}(q), \nabla_0 \mathbf{E}(q) \rangle \\ &= -\frac{1}{2} \nabla_0 (\langle \mathbf{B}(q), \mathbf{B}(q) \rangle + \langle \mathbf{E}(q), \mathbf{E}(q) \rangle) - \langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle \end{aligned} \quad (1)$$

The **field energy density** is defined as:

$$u_{field}(q) = \frac{1}{2} (\langle \mathbf{B}(q), \mathbf{B}(q) \rangle + \langle \mathbf{E}(q), \mathbf{E}(q) \rangle) = u_B(q) + u_E(q) \quad (2)$$

$\mathfrak{S}(q)$  can be interpreted as the **field energy current density**.

The continuity equation for field energy density is given by:

$$\nabla_0 u_{field}(q) + \langle \nabla, \mathfrak{S}(q) \rangle = -\langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle = -\phi_0(q) \langle \mathbf{E}(q), \mathbf{v}(q) \rangle \quad (3)$$

This means that  $\langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle$  can be interpreted as a source term.

$\phi_0(q) \mathbf{E}(q)$  represents **force** per unit volume.

$\phi_0(q) \langle \mathbf{E}(q), \mathbf{v}(q) \rangle$  represents **work** per unit volume, or, in other words, the power density. It is known as the Lorentz power density and is

equivalent to the time rate of change of the mechanical energy density of the charged particles that form the current  $\boldsymbol{\phi}(q)$ .

$$\nabla_0 u_{field}(q) + \langle \nabla, \boldsymbol{\mathfrak{S}}(q) \rangle = -\nabla_0 u_{mechanical}(q) \quad (4)$$

$$\nabla_0 u_{mechanical} = \langle \mathbf{E}(q), \boldsymbol{\phi}(q) \rangle = \phi_0(q) \langle \mathbf{E}(q), \mathbf{v}(q) \rangle \quad (5)$$

$$\nabla_0 \left( u_{field}(q) + u_{mechanical}(q) \right) = -\langle \nabla, \boldsymbol{\mathfrak{S}}(q) \rangle \quad (6)$$

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \quad (7)$$

$$u(q) = u_{field}(q) + u_{mechanical}(q) = u_B(q) + u_E(q) + u_{mechanical}(q) \quad (8)$$

$$U = U_{field} + U_{mechanical} = U_B + U_E + U_{mechanical} = \int_V u \, dV \quad (9)$$

$$\frac{d}{dt} \int_V u \, dV = \oint_S \langle \hat{\mathbf{n}}, \boldsymbol{\mathfrak{S}} \rangle dS + \int_V s_0 \, dV \quad (10)$$

Here the source  $s_0$  is zero.

### How to interpret $U_{mechanical}$

$U_{mechanical}$  is the energy of the private field (wave function) of the involved particle(s).

### Conservation of linear momentum

Field linear momentum

$\boldsymbol{\mathfrak{S}}(q)$  can also be interpreted as the **field linear momentum density**. The time rate change of the field linear momentum density is:

$$\nabla_0 \boldsymbol{\mathfrak{S}}(q) = \mathbf{g}_{field}(q) = \nabla_0 \mathbf{E}(q) \times \mathbf{B}(q) + \mathbf{E}(q) \times \nabla_0 \mathbf{B}(q) \quad (1)$$

$$= (\nabla \times \mathbf{B}(q) - \boldsymbol{\rho}(q)) \times \mathbf{B}(q) - \mathbf{E}(q) \times \nabla \times \mathbf{E}(q) \quad (2)$$

$$\mathbf{G}(\mathbf{E}) = \mathbf{E} \times (\nabla \times \mathbf{E}) = \langle \nabla \mathbf{E}, \mathbf{E} \rangle - \langle \mathbf{E}, \mathbf{E} \rangle = \frac{1}{2} \nabla \langle \mathbf{E}, \mathbf{E} \rangle - \langle \mathbf{E}, \mathbf{E} \rangle \quad (3)$$

$$= -\nabla(\mathbf{E}\mathbf{E}) + \frac{1}{2} \nabla \langle \mathbf{E}, \mathbf{E} \rangle + \langle \nabla, \mathbf{E} \rangle \mathbf{E}$$

$$= -\nabla(\mathbf{E}\mathbf{E} + \frac{1}{2} \mathbf{1}_3 \langle \mathbf{E}, \mathbf{E} \rangle) + \langle \nabla, \mathbf{E} \rangle \mathbf{E}$$

$$\mathbf{G}(\mathbf{B}) = \mathbf{B} \times (\nabla \times \mathbf{B}) = -\nabla(\mathbf{B}\mathbf{B} + \frac{1}{2} \mathbf{1}_3 \langle \mathbf{B}, \mathbf{B} \rangle) + \langle \nabla, \mathbf{B} \rangle \mathbf{B} \quad (4)$$

$$\mathbf{H}(\mathbf{B}) = -\nabla(\mathbf{B}\mathbf{B} + \frac{1}{2} \mathbf{1}_3 \langle \mathbf{B}, \mathbf{B} \rangle) \quad (5)$$

$$\nabla_0 \mathfrak{S}(q) = \mathbf{G}(\mathbf{B}) + \mathbf{G}(\mathbf{E}) - \boldsymbol{\rho}(q) \times \mathbf{B}(q) \quad (6)$$

$$= \mathbf{H}(\mathbf{E}) + \mathbf{H}(\mathbf{B}) - \boldsymbol{\rho}(q) \times \mathbf{B}(q) + \langle \nabla, \mathbf{B} \rangle \mathbf{B} + \langle \nabla, \mathbf{E} \rangle \mathbf{E}$$

$$= \mathbf{H}(\mathbf{E}) + \mathbf{H}(\mathbf{B}) - \boldsymbol{\rho}(q) \times \mathbf{B}(q) - \rho_0(q) \mathbf{E}(q)$$

$$= \mathbf{H}(\mathbf{E}) + \mathbf{H}(\mathbf{B}) - \mathbf{f}(q) = \mathcal{T}(q) - \mathbf{f}(q)$$

$\mathcal{T}(q)$  is the linear momentum flux tensor.

The linear momentum of the field contained in volume  $V$  surrounded by surface  $S$  is:

$$\mathbf{P}_{field} = \int_V \mathbf{g}_{field} dV = \int_V \rho_0 \boldsymbol{\phi} dV + \int_V \langle \nabla \boldsymbol{\phi}, \mathbf{E} \rangle dV + \oint_S \langle \hat{\mathbf{n}}, \mathbf{E}\mathbf{A} \rangle dS \quad (7)$$

$$\mathbf{f}(q) = \boldsymbol{\rho}(q) \times \mathbf{B}(q) + \rho_0(q) \mathbf{E}(q) \quad (8)$$

Physically,  $\mathbf{f}(q)$  is the Lorentz force density. It equals the time rate change of the mechanical linear momentum density  $\mathbf{g}_{mechanical}$ .

$$\mathbf{g}_{mechanical}(q) = \rho_{0m}(q) \mathbf{v}(q) \quad (9)$$

The force acted upon a single particle that is contained in a volume  $V$  is:

$$(10)$$

$$\mathbf{F} = \int_V \mathbf{f} dV = \int_V (\boldsymbol{\rho} \times \mathbf{B} + \rho_0 \mathbf{E}) dV$$

Brought together this gives:

$$\nabla_0 \left( \mathbf{g}_{field}(q) + \mathbf{g}_{mechanical}(q) \right) = -\langle \nabla, \mathcal{T}(q) \rangle \quad (11)$$

This is the continuity equation for linear momentum.

The component  $\mathcal{T}_{ij}$  is the linear momentum in the  $i$ -th direction that passes a surface element in the  $j$ -th direction per unit time, per unit area.

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \quad (12)$$

$$\mathbf{g}(q) = \mathbf{g}_{field}(q) + \mathbf{g}_{mechanical}(q) \quad (13)$$

$$\mathbf{P} = \mathbf{P}_{field} + \mathbf{P}_{mechanical} = \int_V \mathbf{g} dV \quad (14)$$

$$\frac{d}{dt} \int_V \mathbf{g} dV = \oint_S \langle \hat{\mathbf{n}}, \mathcal{T} \rangle dS + \int_V \mathbf{s}_g dV \quad (15)$$

Here the source  $\mathbf{s}_g = 0$ .

## Conservation of angular momentum

### Field angular momentum

The angular momentum relates to the linear momentum.

$$\mathbf{h}(\mathbf{q}_c) = (\mathbf{q} - \mathbf{q}_c) \times \mathbf{g}(q) \quad (1)$$

$$\mathbf{h}_{field}(\mathbf{q}_c) = (\mathbf{q} - \mathbf{q}_c) \times \mathbf{g}_{field}(q) \quad (2)$$

$$\mathbf{h}_{mechanical}(q) = (\mathbf{q} - \mathbf{q}_c) \times \mathbf{g}_{mechanical}(q) \quad (3)$$

$$\mathcal{K}(\mathbf{q}_c) = (\mathbf{q} - \mathbf{q}_c) \times \mathcal{T}(q) \quad (4)$$

This enables the balance equation for angular momentum:

$$\nabla_0 \left( \mathbf{h}_{field}(\mathbf{q}_c) + \mathbf{h}_{mechanical}(\mathbf{q}_c) \right) = -\langle \nabla, \mathcal{K}(\mathbf{q}_c) \rangle \quad (5)$$

Total change within  $V$  = flow into  $V$  + production inside  $V$

$$\mathbf{J} = \mathbf{J}_{field} + \mathbf{J}_{mechanical} = \int_V \mathbf{h} dV \quad (6)$$

$$\frac{d}{dt} \int_V \mathbf{h} dV = \oint_S \langle \hat{\mathbf{n}}, \mathcal{K} \rangle dS + \int_V \mathbf{s}_h dV \quad (7)$$

Here the source  $\mathbf{s}_h = 0$ .

For a localized charge density contained within a volume  $V$  holds for the mechanical torsion:

$$\boldsymbol{\tau}(\mathbf{q}_c) = \int_V (\mathbf{q}' - \mathbf{q}_c) \times \mathbf{f}(q') dV \quad (8)$$

$$= \int_V (\mathbf{q}' - \mathbf{q}_c) \times (\rho_0(q') \mathbf{E}(q') + \mathbf{j}(q') \times \mathbf{B}(q')) dV$$

$$= Q(\mathbf{q} - \mathbf{q}_c) \times (\mathbf{E}(q) + \mathbf{v}(q) \times \mathbf{B}(q))$$

$$\mathbf{J}_{field}(\mathbf{q}_c) = \mathbf{J}_{field}(\mathbf{0}) + \mathbf{q}_c \times \mathbf{P}(q) \quad (9)$$

Using

$$\langle \nabla \mathbf{a}, \mathbf{b} \rangle = \mathbf{n}_\nu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \quad (10)$$

$$\langle \mathbf{b}, \nabla \mathbf{a} \rangle = \mathbf{n}_\mu \frac{\partial a_\mu}{\partial q_\nu} b_\mu \quad (11)$$

holds

$$\begin{aligned} J_{field}(\mathbf{0}) &= \int_V \mathbf{q}' \times \mathfrak{S}(\mathbf{q}') dV = \int_V \mathbf{q}' \times \mathbf{E}(\mathbf{q}') \times \nabla \times \boldsymbol{\phi}(\mathbf{q}') dV \quad (12) \\ &= \int_V (\mathbf{q}' \times \langle (\nabla \boldsymbol{\phi}), \mathbf{E} \rangle - \langle \mathbf{q}' \times \mathbf{E}, (\nabla \boldsymbol{\phi}) \rangle) dV \\ &= \int_V \mathbf{q}' \times \langle (\nabla \boldsymbol{\phi}), \mathbf{E} \rangle dV \\ &\quad + \int_V \mathbf{E} \times \boldsymbol{\phi} dV - \int_V \langle \nabla, \mathbf{E} \mathbf{q}' \times \boldsymbol{\phi} \rangle dV \\ &\quad + \int_V (\mathbf{q}' \times \boldsymbol{\phi}) \langle \nabla, \mathbf{E} \rangle dV \end{aligned}$$

Spin

Define the non-local spin term, *which does not depend on  $\mathbf{q}'$*  as:

$$\boldsymbol{\Sigma}_{field} = \int_V \mathbf{E}(\mathbf{q}) \times \boldsymbol{\phi}(\mathbf{q}) dV \quad (13)$$

And

$$L_{field}(\mathbf{0}) = \int_V \mathbf{q}' \times \langle (\nabla \boldsymbol{\phi}), \mathbf{E} \rangle dV + \int_V \mathbf{q}' \times \rho_0 \boldsymbol{\phi} dV \quad (14)$$

Using Gauss:

$$\int_V \langle \nabla, \mathbf{a} \rangle dV = \oint_S \langle \hat{\mathbf{n}}, \mathbf{a} \rangle dS \quad (15)$$

And

$$\rho_0 = \langle \nabla, \mathbf{E} \rangle \quad (16)$$

Leads to:

$$\mathbf{J}_{field}(\mathbf{0}) = \boldsymbol{\Sigma}_{field} + \mathbf{L}_{field}(\mathbf{0}) + \oint_S \langle \hat{\mathbf{n}}, \mathbf{E} \mathbf{q}' \times \boldsymbol{\phi} \rangle dS \quad (17)$$

*Spin discussion*

The spin term is defined by:

$$\boldsymbol{\Sigma}_{field} = \int_V \mathbf{E}(q) \times \boldsymbol{\phi}(q) dV \quad (1)$$

In free space the charge density  $\rho$  vanishes and the scalar potential  $\phi_0$  shows no variance. Only the vector potential  $\boldsymbol{\phi}$  may vary with  $q_0$ . Thus:

$$\mathbf{E} = \nabla \phi_0 - \nabla_0 \boldsymbol{\phi} \approx -\nabla_0 \boldsymbol{\phi} \quad (2)$$

$$\boldsymbol{\Sigma}_{field} \approx \int_V (\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q) dV \quad (3)$$

If  $\frac{\boldsymbol{\phi}(q)}{|\boldsymbol{\phi}(q)|}$  can be interpreted as tantrix ( $q_0$ ) and  $\frac{\nabla_0 \boldsymbol{\phi}(q)}{|\nabla_0 \boldsymbol{\phi}(q)|}$  can be interpreted as the principle normal  $\mathbf{N}(q_0)$ , then  $\frac{(\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q)}{|(\nabla_0 \boldsymbol{\phi}(q)) \times \boldsymbol{\phi}(q)|}$  can be interpreted as the binormal  $\mathbf{B}(q_0)$ .

From these quantities the [curvature and the torsion](#)<sup>97</sup> can be derived.

$$\begin{bmatrix} \dot{\mathbf{T}}(t) \\ \dot{\mathbf{N}}(t) \\ \dot{\mathbf{B}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(t) & 0 \\ -\kappa(t) & 0 & \tau(t) \\ 0 & -\tau(t) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(t) \\ \mathbf{N}(t) \\ \mathbf{B}(t) \end{bmatrix} \quad (4)$$

---

<sup>97</sup>Path characteristics

Depending on the selected field  $\Sigma_{field}$  has two versions that differ in their sign. These versions can be combined in a single operator:

$$\Sigma_{field} = \begin{bmatrix} \Sigma^+_{field} \\ \Sigma^-_{field} \end{bmatrix} \quad (5)$$

## States

Where a unique closed Hilbert subspace *represents* a given physical item, its **state** *characterizes* the probabilistic properties of that item. In quantum physics, a quantum state is a set of mathematical variables that as far as is possible describes the corresponding physical item. For example, the set of 4 numbers  $\{n, l, m_l, m_s\}$  define part of the state of an electron within a hydrogen atom and are known as the electron's quantum numbers. The observables that determine the state are mutually compatible. The position of the electron within the atom is a hidden property. If two operators are each other's canonical conjugate, then only one of them can participate in the state, or the state must contain an account of the combination of both values. An example of such a combination is the ladder operator.

Quantum states can be either pure or mixed. **Pure states** cannot be described as a mixture of others. **Mixed states** correspond to a random process that blends pure states together. Realizations of elementary types are characterized by pure states.

This indicates that the notion of state is closely related to the basic constituents of Hilbert fields. Stated in other words:

The probability amplitude distribution that represents the private field of an elementary particle also represents the state of that particle.

In relation to the concepts defined earlier, a pure state corresponds to the blur of an elementary Hilbert distribution, while the blur itself can be interpreted as a probability amplitude distribution. The state then also

corresponds to the squared modulus of this probability amplitude distribution, which is a probability density function. The blur is defined with respect to a coordinate operator. This coordinate operator can be a [GPS operator](#) or a position operator ( $\check{Q}$  or  $\check{Q}$ ), or it is represented by the canonical conjugate, the GMS operator or the momentum operator ( $\check{P}$  or  $\check{P}$ ). With other words, several different states correspond with the same physical item.

When performing an observation on a quantum state, the result is generally described by a probability distribution, and the form that this distribution takes is completely determined by the quantum state and the operators that are related to the observation of the quantum state. The result of an observation is only determined probabilistically. Even when the observed quantity is quantized it still can take a range of values, each with a corresponding probability. In relation to the observables that determine the state, a pure state is characterized by the blur of [a single elementary Hilbert distribution](#) and that distribution corresponds in relation to these observables to a mathematical object known as a [wave function](#). If another observable concerns the canonical conjugate, then the corresponding Fourier transform of the wave function must be considered.

The result of a sharp observation equals one of the eigenvalues that corresponds to the set of eigenvectors over which the probability distribution is non-zero. It means that the observable must have the same eigenvectors as the operator that is used to define the wave function. The probability of getting this vector and the corresponding value is given by the probability density that corresponds to the wave function value. The probabilistic nature of observations reflects a core difference between classical and quantum physics. The granularity of observables that are afflicted with Planck limits forbids that differences are measured with precise accuracy.

Linear combinations (superpositions) of states can describe interference phenomena. A mixed state cannot be characterized by a single blurred elementary Hilbert distribution. Instead, it is described by its associated [density operator](#). It is still represented by a (blurred) closed Hilbert subspace, but that is no longer the subspace that is spanned by a single elementary Hilbert distribution.

Pure states can be represented by a single blurred elementary Hilbert distribution.

### State definition

A measure  $\mu$  on the closed subspaces of a Hilbert space obeys the rule:

$$\mu(B) = \sum_{i=1}^n A_i, \text{ for each set } \{A_i\} \text{ of closed subspaces} \quad (1)$$

Each Hilbert vector  $|v\rangle$  generates a measure  $\mu_v(A)$  via the projection  $P_A$  of  $|v\rangle$  on  $A$

$$\mu_v(A) = \|P_A(v)\|^2 \quad (2)$$

[Gleason's theorem](#) states:

Let  $\mu(A)$  be a measure on the closed subspaces  $A$  of a separable Hilbert space  $\mathbf{H}$  with dimension  $\geq 3$ , then there exists a positive definite self adjoint operator  $T$  of the trace class, such that

$$\mu(A) = \text{trace}(TP_A) \quad (3)$$

Given a state  $P$  on a space of dimension  $\geq 3$ , there is an Hermitian, non-negative operator  $\rho$  on  $\mathbf{H}$ , whose trace is unity, such that

$$\forall_{|x\rangle \in \mathbf{H}} \{P(x) = \langle x | \rho x \rangle\}, \text{ where } x \text{ is a ray spanned by } |x\rangle \quad (4)$$

With each compact normal operator  $Q$  corresponds an orthonormal base of eigenvectors  $\{|q\rangle\}_q$  with eigenvalues  $q$ . As a consequence a notion of

state is attached to each physical item combined with one or more mutually compatible compact normal operators  $\{Q_j\}_j$ .

In Hilbert space a state, or probability function, is a real function  $P$  on the Hilbert subspaces, with the following properties:

1.  $P(0) = 0$
2.  $\forall_{y \in \mathbf{H}} \{P(y) \geq 0\}$ ,  $y$  is a Hilbert subspace
3.  $\sum_j P(x_j) = 1$ , where  $|x_j\rangle$  form an orthonormal base of  $\mathbf{H}$  and  $x_j$  is the ray spanned by  $|x_j\rangle$
4.  $P(y) = \sum_{j=1}^r P(x_j)$  where  $x_j$  are mutually orthogonal rays spanning subspace  $y$

### Pure state

In particular, if some ray  $x_0$  satisfies  $P(x_0) = 1$ , then according to Born's rule:

$$\forall_{|x\rangle \in \mathbf{H}} \{P(x) = |\langle x_0, x \rangle|^2\} \quad (1)$$

This happens when  $|x_0\rangle$  represents an unary pure state.

### Item state

The state  $P(\{q_j\}_j)$  is connected to a **wave function**  $\psi(\{q_j\}_j)$ , where

$$P(\{q_j\}_j) = |\psi(\{q_j\}_j)|^2 \quad (1)$$

and  $\{q_j\}_j$  are the eigenvalues of eigenvector  $\{|q_j\rangle\}$  of the operators  $\{Q_j\}_j$ .

Two operators  $A$  and  $B$  are compatible when their commutator is zero:

$$[A, B] \stackrel{\text{def}}{=} AB - BA = 0 \quad (2)$$

If the state is characterized by a *set* of independent properties, then each of these properties corresponds with a corresponding operator. These operators must be normal, but they need not be compact. It must be possible to construct a spectral decomposition for each of the operators. Further, the operators that together determine the state must be mutually compatible. The wave function is then the product of the probability amplitudes that correspond to the separate operators. Thus the resulting wave function is a characteristic that represents the probability amplitudes of a set of mutually compatible observables that correspond to the normal operators that determine the state.

The squared modulus of the probability amplitude is the probability density. The wave function will also be a function of a progression parameter. Position can be a state characterizing observable. However, like the progression parameter, spacetime does not occur as an eigenvalue of a Hilbert space operator. The operators may vary. For example an operator may be replaced by its canonical conjugate. In that case, care must be taken that the operators that form the changed state are still compatible. Thus, even with the same physical item, the wave function is not unique.

For the operator  $Q$  with eigenfunctions  $|q\rangle$  and eigenvalues  $q$  the probability amplitude function  $\psi(q)$  is given by the smoothed version of  $\langle \psi|q \rangle$

$$\psi(q) \cong \langle \psi|q \rangle \tag{3}$$

When  $Q$  is a [compact normal operator](#) then the smoothed version of  $\psi(q)$  is a continuous function. In that case the eigenvalues of the corresponding operator  $\check{Q}$  that resides in rigged Hilbert space  $\mathbf{H}$  are used. Then  $\psi(q)$  has a Fourier transform  $\phi(p)$ , where the operator  $\check{P}$  with eigenvectors  $|p\rangle$  and eigenvalues  $p$  is the canonical conjugate of  $\check{Q}$ . Like  $\psi(q)$ , the function  $\phi(p)$  is also a function that characterizes the corresponding item and  $|\phi\rangle$  is a characterizing vector. The parameters  $q$  and  $p$  may be quaternionic.

$$\varphi(p) = \tilde{\psi}(q) = \langle \varphi | p \rangle \quad (4)$$

With respect to the correspondence with traditional quantum logic, it is wrong to take any characteristic vector including the locator or any function including the wave function as the *representative* for the item. It is ridiculous to expect that a single vector carries all properties of a complex physical item, such as a DNA molecule or an elephant.

In usual quantum mechanics the wave function can be interpreted as the combination of a stationary vector and a progression operator. The progression operator has the form  $A \cdot \exp(S/\hbar)$ . This was introduced by Dirac. A is Hermitian and positive. S is anti-Hermitian. Both operators are a function of parameter t. This is reflected in the Hamilton-Jacobi equation.

In contemporary quantum field theory the fields replace the wave function. Thus a field may be interpreted as the amplitude of the probability to find something at the location of the field value. For bosons that something may be interpreted as a virtual particle. For fermions that something may be interpreted as a pair of virtual particles. Each type of virtual particle has its own type of field.

There are some questions left with wave functions:

- Can it have non-zero values outside the subspace that represents the physical item?
  - Answer: Yes. The private field covers the whole Hilbert space.
- Is the wave function a regular function?
  - Answer: When universe is restricted by an outer horizon, then the wave function is regular.
- What happens to the representing subspace and to the wave function when a measurement on a particle is performed?
  - Answer:

- When the coordinate space stays the same, then both the subspace and the wave function will not be affected.
- When the coordinate space changes into the canonical conjugate, then the subspace changes to other base vectors and the wave function is Fourier transformed into a new form.
- Has a system a wave function?
  - Answer: In general a system must not have a wave function, but it has a density operator.

### Probability density

[Gleason's theorem](#)<sup>98</sup> states that a probability measure  $\mu(P)$  on the lattice  $L(\mathbf{H})$  of projections  $P$  on closed subspaces of a Hilbert space  $\mathbf{H}$  corresponds to a non-negative Hermitian operator  $\rho$  with trace 1, such that  $\mu(P) = \text{tr}(\rho P)$ . When the projections  $P_q$  correspond to the rays formed by the eigenvectors  $|q\rangle$  of operator  $Q$  and  $\mu_i(P_q)$  corresponds to the considered physical item, then  $\mu_i(P_q) = \langle q, \rho_i q \rangle$  corresponds to the square of the modulus of the wave function  $\psi_i(q)$ .  $\rho_i$  is the [probability density operator](#)<sup>99</sup> corresponding to  $\mu_i$ . The probability measure  $\mu$  is a regular function.

The [probability density function](#)<sup>100</sup>  $P(q) = |\psi(q)|^2$  of an absolutely continuous random variable  $q$  is a function that describes the relative chance for this random variable to occur at a given point in the  $Q$  observation space. The probability for a random variable to fall within a given set is given by the integral of its density over the set.

The probability [density operator](#)<sup>101</sup>  $\rho$  is positive-semi-definite ( $\forall |f\rangle \in \mathbf{H} \{ \langle f | \rho | f \rangle \geq 0 \}$ ), self-adjoint ( $\rho = \rho^\dagger$ ), and has trace one ( $\text{tr}(\rho) = 1$ ). For the operator

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<sup>98</sup> [http://en.wikipedia.org/wiki/Quantum\\_logic#Statistical\\_structure](http://en.wikipedia.org/wiki/Quantum_logic#Statistical_structure)

<sup>99</sup> Functions and fields: characteristic functions

<sup>100</sup> [http://en.wikipedia.org/wiki/Probability\\_density\\_function](http://en.wikipedia.org/wiki/Probability_density_function)

<sup>101</sup> [http://en.wikipedia.org/wiki/Density\\_operator](http://en.wikipedia.org/wiki/Density_operator)

Q with eigenfunctions  $|q\rangle$  and eigenvalues  $q$  with probability amplitude  $\psi(q)$ , the density operator  $\rho$  is given by

$$\rho = \sum_q \{ |\langle \psi|q \rangle|^2 \cdot |q \rangle \langle q| \} \quad (1)$$

[Von Neumann entropy](#)<sup>102</sup> is defined using the density operator of physical items.

The entropy  $S(\rho)$  describes the departure of the system from a pure state. In other words, it measures the degree of mixture ([entanglement](#)<sup>103</sup>) of the state  $|\psi\rangle$ .

The operator  $A$  can be decomposed

$$A = \sum_a |a \rangle \langle a| \quad (2)$$

For the state  $|\psi\rangle$  the **expectation value**  $\langle A \rangle$  for the observable  $A$  is

$$\langle A \rangle \stackrel{\text{def}}{=} \langle \psi|A|\psi \rangle = \sum_q \{ |\langle \psi|q \rangle|^2 \cdot \langle q|Aq \rangle \} = \text{tr}(\rho A) \quad (3)$$

A Hilbert field is a blurred Hilbert distribution. The blur represents a probability amplitude distribution.

### States and blurs

Apparently a state is the same stuff as the basic constituent of a Hilbert field. Both can be characterized as probability amplitude distributions. The squared modulus of a [probability amplitude distribution](#)<sup>104</sup> is a probability density distribution. The state corresponds with a wave function or with a probability density operator.

- The state of a physical item can be interpreted as the probability of finding the parameter value when an observation is done that corresponds to the corresponding coordinate operator.

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<sup>102</sup> [http://en.wikipedia.org/wiki/Von\\_Neumann\\_entropy](http://en.wikipedia.org/wiki/Von_Neumann_entropy)

<sup>103</sup> [http://en.wikipedia.org/wiki/Quantum\\_entanglement](http://en.wikipedia.org/wiki/Quantum_entanglement)

<sup>104</sup> [http://en.wikipedia.org/wiki/Probability\\_amplitude](http://en.wikipedia.org/wiki/Probability_amplitude); the quaternionic version is used.

- The squared modulus of the blur can be interpreted as the probability of detecting a quantum at the location specified by the parameter value that corresponds to the corresponding coordinate operator.

Blurs are the building stones of Hilbert fields. In a similar way wave functions must be interpretable as the building stones of fields.

Blurs are private fields of elementary Hilbert distributions. Thus, wave functions must also be related to elementary Hilbert distributions.

## Observables and field values

In Hilbert space observables are represented by [operators](#). The observed value is represented by an eigenvalue or by the expectation value of the operator that represents the observable. Scalar physical fields have numeric values. Vector and tensor fields consist of vectors and tensors that are constructed using numbers. Both the eigenvalues of operators and the values of fields may be hyper complex  $2^n$ -ons<sup>105</sup>.

## Numbers

The Hilbert space can be specified by using a number space that allows the mutual orthogonalization and the closure of subspaces. The real's, the complex numbers and the quaternions can perform that job. Horwitz showed that even the octonions with some trouble can achieve this (see: <http://arxiv.org/abs/quant-ph/9602001>). The real's, the complex numbers, the quaternions and the octonions are the only normed division algebras and they are the only alternative division algebras. In general the octonions are not associative, but the product of two octonions that belong to the same quaternionic subfield is associative. The alternative property of the octonions admits the closure of the subspace generated by (successively associated) products of the vector with octonion elements to order seven, i.e., after multiplication seven times by octonions, the subspace no longer grows.

Neither all quaternions nor all octonions commute. However, within complex subspaces the numbers commute. In general holds for  $2^n$ -ons that they behave like  $2^m$ -ons in their lower  $m$  dimensions.

We might take the following freedom. The fact that a given number space is used for specifying linear combinations of Hilbert space vectors does not mean that eigenvalues of operators must also be restricted to that same number space. In this sense a Hilbert space specified over the quaternions may allow eigenvalues of operators that are taken from the octonions or even higher  $2^n$ -ons. The problem with higher dimension  $2^n$ -

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<sup>105</sup> see <http://www.math.temple.edu/~wds/homepage/nce2.pdf> or the [appendix](#)

ons is that their number characteristics deteriorate with  $n$ . However, as long as the (full) eigenvalues are not used to construct linear combinations of vectors, or to specify the inner products of the Hilbert space, there is no problem. All higher dimensional  $2^n$ -ons contain several subspaces that are lower dimensional  $2^m$ -on number spaces. Further,  $2^n$ -ons behave like  $2^m$ -ons in their lower  $2^m$  dimensions.

In general the elements of curves or curved manifolds are themselves not numbers. So, in general they cannot be used as eigenvalues. However, locally the elements of a curved manifold may resemble numbers of a  $2^n$ -on number space. Smoothly curved trails of objects that locally resemble  $2^n$ -ons can be treated with the Frenet-Serret frame toolkit. Number spaces can be **attached** as tangent spaces to smoothly curved manifolds. In that way the elements of the curves and the manifolds obtain number characteristics in a small enough environment. Sequences or sets of operators can locally have eigenvalues that are numbers which can be considered as member of smooth curves or of the tangent space of a curved manifold at that location. In that way the elements of smooth curves or of curved manifolds can be related to the corresponding eigenvalues.  $2^n$ -ons are ideally suited for this purpose. This means that the eigenspaces of the subsequent operators in a trail need not overlap. These eigenspaces are only used locally. When curvature and bending of the operator trail diminish, the dimension of the local number space can be lower. When the curvature and the bending increase, the dimension must be higher. This will be reflected in the dimensionality of the local eigenvalues. Apart from the application as eigenvalues of operators the  $2^n$ -ons are suited as values of physical fields.

We will restrict to the  $2^n$ -ons as extensions of the quaternions. As we stated, the higher dimension numbers created with the Cayley-Dickson construction are not so well behaved. Alternatives are the use of Clifford algebras, Jordan algebras or Grassmann algebras. We will show that in the Hilbert space the  $2^n$ -ons for  $n > 1$  automatically introduce these latter algebras through their number waltz.

The **niners** are the most extensive  $2^n$ -on numbers that still keep a reasonable set of number characteristics. More precisely said the  $2^n$ -ons, even those that have a higher dimension than the octonions, keep reasonable number characteristics in the space spanned by their coordinates that have an index lower than nine. The real numbers, the complex numbers, the quaternions and the octonions completely fall within these boundaries. The above hyperlink describes exactly what characteristics the niners retain.

The subspace of the  $2^n$ -on field that is spanned by the first  $2^m$  dimensions acts as a  $2^m$ -on number space. Thus in a dynamic situation, an octionic operator acts locally as a quaternionic operator. In a smaller or more flat region it acts as a complex operator and at "nano"-locality as a real (or as an imaginary) operator.

### **2<sup>n</sup>-on construction**

The  $2^n$ -ons use the following doubling formula

$$(a, b) (c, d) = (a \cdot c - (b \cdot d^*)^*, (b^* \cdot c)^* + (b^* \cdot (a^* \cdot ((b^{-1})^* \cdot d^*)^*))^*) \quad (1)$$

Up until the 16-ons the formula can be simplified to

$$(a, b) (c, d) = (a \cdot c - b \cdot d^*, c \cdot b + (a^* \cdot b^{-1}) \cdot (b \cdot d)) \quad (2)$$

Up to the octonions the Cayley Dickson construction delivers the same as the  $2^n$ -on construction. From  $n > 3$  the  $2^n$ -ons are 'nicer' than the Cayley Dickson numbers. They keep more useful number characteristics. The  $2^{n+1}$ -ons contain the  $2^n$ -ons as the sub-algebra of elements of the form  $(a, 0)$

### **Waltz details**

The 16-ons lose the continuity of the map  $x \Rightarrow xy$ . Also, in general holds  $xy \cdot x \neq x \cdot yx$  for 16-ons. However, for all  $2^n$ -ons the base numbers fulfill

$e_i e_j e_i = e_i \cdot e_j e_i$ . All  $2^n$ -ons feature a conjugate and an inverse. The inverse only exists for non-zero numbers. The  $2^n$ -ons support the **number waltz**

$$c = a \cdot b a^{-1}. \quad (1)$$

Often the number waltz appears as a unitary number waltz

$$c = u^* \cdot b u \quad (2)$$

where  $u$  is a unit size number and  $u^*$  is its conjugate  $u \cdot u^* = 1$ .

In quaternion space the **quaternion waltz**  $a \cdot b \cdot a^{-1}$  can be written as

$$a \cdot b \cdot a^{-1} = \exp(2 \cdot \pi \cdot \tilde{\mathbf{i}} \cdot \phi) \cdot b \cdot \exp(-2 \cdot \pi \cdot \tilde{\mathbf{i}} \cdot \phi) \quad (3)$$

$$= b - \underline{\mathbf{b}}_{\perp} + \exp(2 \cdot \pi \cdot \tilde{\mathbf{i}} \cdot \phi) \cdot \underline{\mathbf{b}}_{\perp} \cdot \exp(-2 \cdot \pi \cdot \tilde{\mathbf{i}} \cdot \phi)$$

$$= b - \underline{\mathbf{b}}_{\perp} + \exp(4 \cdot \pi \cdot \tilde{\mathbf{i}} \cdot \phi) \cdot \underline{\mathbf{b}}_{\perp}$$

$$\Delta b = (\exp(4 \cdot \pi \cdot \tilde{\mathbf{i}} \cdot \phi) - 1) \cdot \underline{\mathbf{b}}_{\perp} \quad (4)$$

$$= (\cos(4 \cdot \pi \cdot \phi) + \tilde{\mathbf{i}} \cdot \sin(4 \cdot \pi \cdot \phi) - 1) \cdot \underline{\mathbf{b}}_{\perp}$$

$$= \exp(2 \cdot \pi \cdot \tilde{\mathbf{i}} \cdot \phi) \cdot 2 \cdot \tilde{\mathbf{i}} \cdot \sin(2 \cdot \pi \cdot \phi) \cdot \underline{\mathbf{b}}_{\perp}$$

$$||\Delta b|| = ||2 \cdot \sin(2 \cdot \pi \cdot \phi) \cdot \underline{\mathbf{b}}_{\perp}|| \quad (5)$$

Another way of specifying the difference is:

$$\Delta b = (a \cdot b - b \cdot a) / a = 2 \cdot (\underline{\mathbf{a}} \times \underline{\mathbf{b}}) / a \quad (6)$$

$$||\Delta b|| = 2 \cdot ||\underline{\mathbf{a}} \times \underline{\mathbf{b}}|| / ||a|| \quad (7)$$

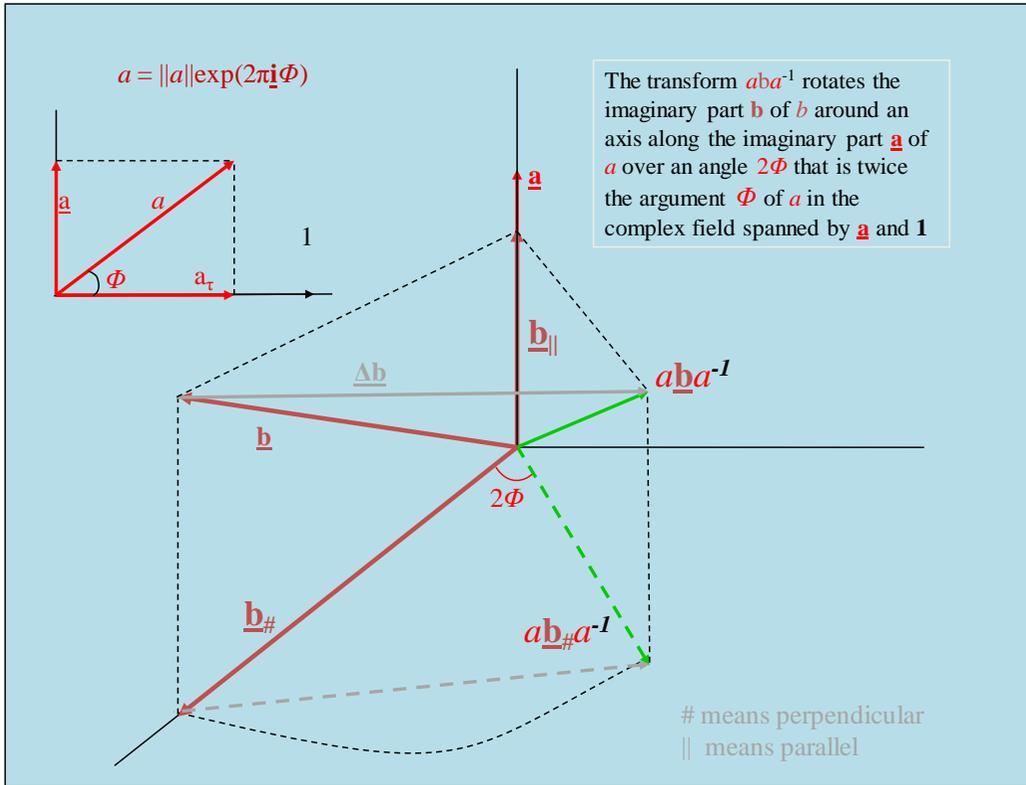


Figure 1. The rotation of a quaternion by a second quaternion.

### Infinitesimal number transformation

The number  $v$  is close to 1. Thus  $v = 1 + \Delta s$ . Let us investigate the transform  $c = v^* \cdot b \cdot v$ .

$$\begin{aligned}
 c &= (1 + \Delta s^*) \cdot b \cdot (1 + \Delta s) \\
 &= b + \Delta s^* \cdot b + b \cdot \Delta s + \Delta s^* \cdot b \cdot \Delta s \\
 &\approx b + \Delta s^* \cdot b + b \cdot \Delta s \\
 &= b + \Delta s_0 \cdot b + 2 \cdot \underline{b} \times \Delta s
 \end{aligned} \tag{1}$$

$$\Delta b = \Delta s_0 \cdot b + 2 \cdot \underline{b} \times \Delta s \tag{2}$$

This comes close to the effect of an infinitesimal number waltz, especially when  $\Delta s_0 = 0$ . In that case  $\Delta b_0 = 0$  and  $\Delta \mathbf{b}$  is perpendicular to  $\Delta \mathbf{s}$ . For  $2^n$ -ons with  $n > 1$ ,  $a \cdot b a^{-1}$  in general does not equal  $b$ . This effect stays unnoticed when quantum mechanics sticks to a complex Hilbert space.

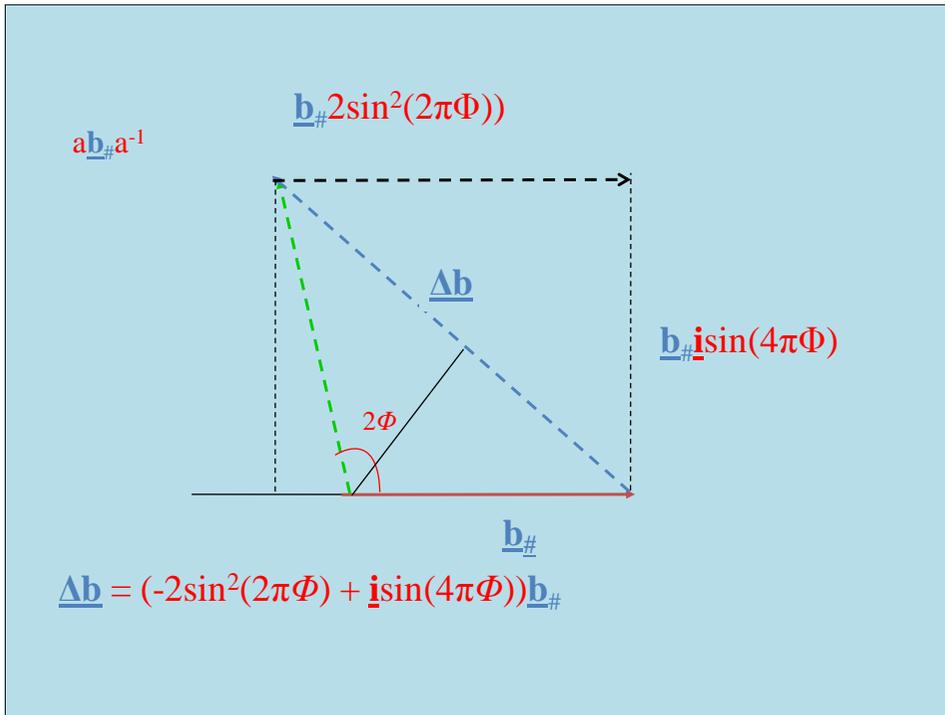


Figure 2: The difference after rotation

### Sign selections

The paper that describes  $2^n$ -ons does not describe the choice for right or left handedness of the external vector product. So, we do it here. The generally accepted convention is to let the handedness depend on the orientation of the underlying  $\mathbb{R}^n$  space. However, when numbers are constructed via the Cayley-Dickson construction or the  $2^n$ -on construction then the handedness follows from the applied construction formula. We want to get rid of these restrictions, because we want to give operators the freedom to select the handedness and other sign selections of their eigenvalues.

The  $2^n$ -ons have  $n$  independent binary base numbers and  $n$  sign selections. The real numbers do not offer a sign selection. The complex numbers offer the selection of the sign of the real or the imaginary axis. This is inherited by all higher  $2^n$ -ons. The quaternions have two independent imaginary base numbers and offer an extra sign selection that represents the handedness of its external product. The octonions have three independent imaginary base numbers and offer an extra sign selection for the handedness in external products that involve this new base number.

### Need for spinors

In the number waltz the current manipulator only needs an argument  $\alpha$  in order to turn the subject over  $2\alpha$ . This is typical behavior for spinors. Spinors also have a storage place for the handedness of rotations. By using the number waltz and the sign selections the  $2^n$ -ons can perform the same act as the spinors. Spinors are only required when quantum mechanics is restricted to complex Hilbert spaces. Spinors are the carriers of the spin phenomenon. Thus, in our model the sign selections in combination with the number waltz form the carriers of spin.

The approach taken in this paper might cause a revival of the importance of the hyper complex numbers that turned in oblivion when Gibbs introduced his vector analysis.

## Influence

The original example proposition (♠) talks about influencing the position of an item. This implies that the position of the item changes due to the mentioned influence. Thus when the influence occurs, the eigenvector that represents the position of the item is exchanged against another eigenvector. That other eigenvector corresponds to another environment inside the eigenspace of the position operator. The eigenvectors of the position operator move with respect to the vector that characterizes the subspace of the item. The movement is relative and takes place inside the Hilbert space. Another possibility is that the eigenvectors stay, but the corresponding eigenvalues change while the Hilbert subspace moves. See [Heisenberg picture versus Schrödinger picture](#)<sup>106</sup>.

Thus, there is a way to implement influence in Hilbert space. The influence causes a move of the item's subspace relative to one or more eigenvectors of the position operator. The original proposition (♠) claims that this movement is caused by other items. We must check whether this is true.

If this is true then influences are the motor behind the dynamics of the items.

## The universe of items

The original proposition (♠) states that all items influence each other's position. This includes that all items influence the considered item. Part of the items compensates each other's influences on the currently considered item. It will be shown that this holds for the largest part.

## Inertia

The influence may decrease with distance according to some function  $f(r)$  of the distance  $r$ . However the number of contributing items increases with the distance. Depending on function  $f(r)$  the most probable result is

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<sup>106</sup> Dynamics: Schrödinger or Heisenberg picture

that the strongest influence comes from the cooperative activity of the most distant items. Due to the enormous number of items in the universe, any variation of the influences of the distant items averages away. This also holds for the density distribution of the items. So there exists a fairly uniform background influence caused by the universe of items. What will happen, can be deduced from an equivalent of [Denis Sciama's analysis](#)<sup>107</sup>. We will take his analysis as a guide. Sciama's analysis uses a different setting: the (observed) 3D space and coordinate time. This setting raised critique because the approach involves instantaneous action on large distances. In Sciama's setting this is in conflict with special relativity. In our setting we do not (yet) encounter special relativity. We use the coordinate space defined by an appropriate coordinate operator and the progression parameter  $t$  that relates to the progression step counter as our setting. A location in coordinate space represents a location on the unit sphere of Hilbert space. This last location is taken by the eigenvector that corresponds to the first location. As stated before, the unit sphere of Hilbert space is an affine space. This means that we must treat position as relative data. With other words, the eigenspace of the coordinate operator has no absolute origin.

The most important aspects of the analysis are:

The total potential  $\Phi$  at the location of the influenced subject is<sup>108</sup>

$$\Phi = - \int_V \frac{\rho}{r} dV = -\rho \int_V \frac{dV}{r} \quad (1)$$

This conforms to a [Gaussian blur](#)<sup>109</sup> as a representative of the average blur function. The integral is taken over the coordinate space volume  $V$ . Indirectly, the integral is taken over the unit sphere of Hilbert space. This

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<sup>107</sup> See: <http://arxiv.org/abs/physics/0609026v4.pdf> and "On the origin of inertia", by Denis Sciama (<http://www.adsabs.harvard.edu/abs/1953MNRAS.113...34S>)

<sup>108</sup> See: [http://en.wikipedia.org/wiki/Newtonian\\_potential](http://en.wikipedia.org/wiki/Newtonian_potential)

<sup>109</sup> Hilbert field equations: example potential

is an affine space. The parameter  $r$  is the length of the vector from the actor to the location of the subject. The considered subject is located somewhere in the affine coordinate space. All other subjects have positions relative to that considered subject. At large distances, the density  $\rho$  of the contributing items can be considered to be uniformly distributed. Also any variance in strength other than the dependence on  $r$  becomes negligible because the differences are blurred and averaged away. We already assumed that the average blur of the distributed matter in universe is a Gaussian blur. We take the average of the strength as the significant parameter. We combine it with  $\rho$ . Therefore the average of  $\rho$  can be taken out of the integral. Thus, apart from its dependence on the average value of  $\rho$ ,  $\Phi$  is a huge constant. (Sciama relates  $\Phi$  to the gravitational constant). As a consequence we can consider the universe as a very large rigid body. If nothing else happens then all influences compensate each other.

In the following equations we use imaginary quaternions rather than 3D vectors. In this way we can avoid the distracting factor  $i$ .

If the considered subject moves relative to the universe with a uniform speed  $\mathbf{v}$ , then a vector potential  $\mathbf{A}$  is generated.

$$\mathbf{A} = - \int_V \frac{\mathbf{v} \cdot \rho}{c \cdot r} dV \quad (2)$$

Both  $\rho$  and  $\mathbf{v}$  are independent of  $r$ . Together with the constant  $c$  they can be taken out of the integral. Thus

$$\mathbf{A} = \Phi \cdot \mathbf{v} / c \quad (3)$$

What we have here is the reverse of the definition of the potential that goes together with a charge distribution. When we defined a [Hilbert field](#) we started in fact from a charge distribution and a current distribution and we considered the influence of these distributions on the universe.

Here we consider the influence of the universe on a local charge or current. For this purpose we use the same volume integrals!

The field that we treat in studying inertia is a curvature field rather than a covering field. The curvature field derives from the covering field by taking the curvature that is caused by the decomposition of the covering field as its charge.

The notions of charge and current correspond to equivalent notions in [Noether's theorem](#)<sup>110</sup>. Here we talk about inertia and curvature fields. Thus charge may symbolize mass.

Here the progression parameter  $t$  plays the role of "time". Be aware, this is not our usual notion of time.

According to Helmholtz theorem the Hilbert field derived from the above potentials can be split into a divergence free part and a rotation free part. The Helmholtz decomposition theorem only concerns the static versions of the derived field. It is related to the fact that the Fourier transform of a vector field can be split in a longitudinal and a transversal version. There also exists a corresponding split of the multi-dimensional Dirac delta function in a longitudinal and a transversal version. If we use the position operator  $\check{Q}$  as the coordinate operator, then the decomposition runs along straight lines. If we use the GPS operator  $\check{Q}$  then the decomposition runs along curved lines. In curved manifolds the Helmholtz decomposition theorem should be replaced by the Hodge decomposition theorem.

A variation of  $\mathbf{v}$  goes together with a variation of  $\mathbf{A}$ . On its turn this goes together with a non-zero field  $\dot{\mathbf{A}}(\mathbf{r}, t)$  which is a **dynamical** part of the derived Hilbert field.

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<sup>110</sup> [http://en.wikipedia.org/wiki/Noether%27s\\_theorem](http://en.wikipedia.org/wiki/Noether%27s_theorem)

Sciama uses a Maxwell equation to explain the relation between  $\partial\mathbf{v}/\partial t$  and  $\dot{\mathbf{A}}(\mathbf{r}, t)$ . Our setting differs, but the main reasoning is the same.

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{1}{c} \cdot \dot{\mathbf{A}}(\mathbf{r}, t) \quad (4)$$

$$\tilde{\mathbf{E}}(\mathbf{k}, \omega) = -\mathbf{k} \cdot \tilde{\Phi}(\mathbf{k}, \omega) - \frac{1}{c} \cdot \omega \tilde{\mathbf{A}}(\mathbf{k}, \omega) \quad (5)$$

If we exclude the first term because it is negligible small, we get:

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\Phi}{c^2} \cdot \frac{\partial\mathbf{v}}{\partial t} = G \cdot \frac{\partial\mathbf{v}}{\partial t} \quad (6)$$

Remark: As soon as we turn to the dynamic version (4) an extra component  $\dot{\mathbf{A}}$  of field  $\mathbf{E}$  appears that corresponds to acceleration  $\partial\mathbf{v}/\partial t$ . (See for derivation of Maxwell equations e.g. the online book <http://www.plasma.uu.se/CED/Book>; formula 3.25 or the section on [Hilbert field equations](#) in this e-paper)

As already claimed, in our setting the component  $\nabla\Phi$  of the field  $\mathbf{E}$  is negligible. With respect to this component the items compensate each other's influence. This means that if the influenced subject moves with uniform speed  $\mathbf{v}$ , then  $\mathbf{E} \approx 0$ . However, a vector potential  $\mathbf{A}$  is present due to the movement of the considered item. Any acceleration of the considered item goes together with an extra non-zero  $\mathbf{E}$  field. In this way the universe of items causes inertia in the form of a force that acts upon the accelerating item's charge.

We have used the coordinate space as a playground to implement an equivalent of Sciama's analysis. The analysis uses the fact that every item in universe causes an influence and that this influence reduces according to  $f = -k/r$ . (Compare this with [Bertrand's theorem](#)<sup>111</sup> in Wikipedia)

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<sup>111</sup> [http://en.wikipedia.org/wiki/Bertrand's\\_theorem](http://en.wikipedia.org/wiki/Bertrand's_theorem)

A uniform movement in Hilbert space does not on itself generate a reaction of the universe of items. Any alteration of that uniform movement will cause as reaction a **field**. The physical name for this reaction is **action**. It usually gets the symbol **S**. When the path of the item coincides with a **geodesic**, then it can be travelled field free.

Uniform movements do cause displacement of charges. On its turn it changes the configuration of the local field. Thus, indirectly the field will also act on uniform displacements. As we see from inertia, any field change goes together with a corresponding acceleration.

It must be noticed that the original analysis of Sciama uses observable position space rather than Hilbert space, coordinate space or action space and it uses a different notion of time. However, the general conclusion stays the same. Sciama's analysis is criticized because it uses infinite speed of information transfer. Since we do not work in observable position space, we do not encounter coordinate time. So for us, this criticism is misplaced.

[Coordinate time<sup>112</sup>](#) relates to observations of position. It is a local player in the game, where the progression parameter is a global player.

The situation with electromagnetic fields is different, because with this field positive and negative charges compensate each other's long range influence. For that reason there exists no electromagnetic background influence or it is far less important. The masses of the gravitational and inertial fields only compensate each other's long range influences through geometrical circumstances. Still in combination, they create gigantic potentials.

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<sup>112</sup> Dynamics; Relativity

## Nearby items

Items that are located nearby have a different effect. In general their influence will not have its strength equal to the average strength. Further these items are not uniformly distributed. Still their influence depends on inter-distance as  $f = -k/r$ . As a consequence their influences form a landscape of which the effects will become sensible in the action of the fields that surround the considered item. This landscape will form a curved action space. The considered item will try to follow a geodesic through that curved space.

## Rotational inertia

Besides linear inertia there exists rotational inertia. In a non-rotating universe hold near the origin  $\mathbf{A} = 0$  and  $\Phi = -c^2/G$ . We choose units such that  $c=G=1$ . In a universe rotating slowly with angular speed  $\omega$  hold

$$A_x = \omega \cdot y \quad (1)$$

$$A_y = -\omega \cdot x \quad (2)$$

$$A_z = 0 \quad (3)$$

$$\Phi = -\sqrt{1 + (\omega \cdot r)^2} \quad (4)$$

A constant angular movement meets the fields that correspond to a centripetal force.

The field  $E$  has the form

$$\mathbf{E} = \frac{\omega^2 \mathbf{r}}{\sqrt{1 + \omega^2 r^2}} \quad (5)$$

An added uniform speed  $v$  meets the fields corresponding to a Coriolis force.

$$\mathbf{H} = \nabla \times \mathbf{A} = 2 \cdot \boldsymbol{\omega} \quad (6)$$

$$\mathbf{v} \times \mathbf{H} = 2 \cdot \mathbf{v} \times \boldsymbol{\omega} \quad (7)$$

The forces are usually considered as *fictitious* but they are actually caused by inertia. Sciama treats them in section 5 of his paper. Like fields of linear inertia these rotation related fields correspond to actions of the manipulator.

### Storage, sign selections and virtual items.

The static fields act as storage media for the location and the speed of the charges of the physical items.

When the values of the fields are stored in hyper complex numbers, then the sign choices for these numbers will also be reflected in these fields. For example the handedness will show in the transverse fields. A right handed and a left handed version of the field will exist. The sign selection of the real part of the hyper complex numbers may also cause extra versions of fields.

The fields can be interpreted as reflections of the presence of virtual items that are ready to exchange roles with actual items.

### The proposition

This finding indicates that when our interpretation of Sciama's analysis is correct, the original proposition

*All items in universe influence each other's position.*

is not generally true. The universe of items does not influence position. It counteracts acceleration of individual items. Position is only influenced in an indirect way and presupposes an observation. If the item moves in a geodesic with uniform speed, then the position changes while the influences of all other items compensate each other. In such cases the summed influence is zero.

We may alter the original proposition (♠). If our analysis is correct, then the proposition

*All items in universe influence each other's acceleration.*  
is true.

## The origin of dynamics

If we want to discover the origin of dynamics, we must first determine what the static structure of nature is. We already found an important ingredient of this skeleton: the lattice structure of quantum logic and the corresponding lattice structure of the closed subspaces of a Hilbert space. Both structures are only defined in a static way. Nothing is said about their dynamics. Besides of these static relations the concept of wave functions and density operators offer insight in the probability and information content of these relations. These subjects correspond to private fields, which are the constituents of a covering field. This covering field can be seen as the superposition of all private fields. For a selected coordinate system the static covering field can be decomposed into a rotation free and a divergence free part. Depending on the configuration of the anchors of the private fields the decomposition does not run along straight coordinate lines. This defines a local curvature that depends on the selected coordinate system. That curvature can be used to define a curvature field. This together defines the ingredients of the static status quo. It all fits in a model that we call *extended quantum logic* or equivalently *extended Hilbert space*.

In the previous part of the paper the added component of the static structure of nature is investigated: the static structure of the influences. It appears that this structure is identical with the structure of static Hilbert fields. Both the analysis of inertia and the study of Hilbert fields showed the static relation between divergence free fields and rotation free fields. These analyses also showed the influence of dynamics on the coupling of these static fields. The analysis of Hilbert fields explained how these fields change as a function of the progression parameter  $q_0$ . Inertia showed how these fields get coupled when the uniform movement of a physical item is disturbed. We also explained that uniform movement may cause a reconfiguration of the field. On its turn, this change may initiate movement.

Next we try to find a more precise formulation for these origins of dynamics.

### **Extended quantum logic**

Wave functions represent the probability of finding properties of actual items. For example if a GPS type coordinate system is selected as its parameter space, then this property can be the position of the item. If it is a GMS type system, then the property can be momentum.

In quantum theory the values of fields are treated in equations of motion in a similar way as the wave functions of actual items are treated in such equations. In fact the wave function itself can be interpreted as part of the field.

When fields in general can be considered as representations of the probability of finding properties of actual as well as virtual items, then the fields get an interesting interpretation. In quantum logic the realistic physical item is represented by a proposition in the form of a statement that says everything about that item. For virtual items the new interpretation would mean that in extended quantum logic the virtual items are represented by potential propositions that are ready to become actual propositions or that were actual propositions in the past.

This would mean that traditional quantum logic is embedded in extended quantum logic such that it apart from propositions about actual physical items also contains potential proposition that represent virtual particles.

In this way, the set of propositions of extended quantum logic is much larger than the set of propositions of quantum logic.

### **Interpretation in logical terms**

The results of the analysis of inertia mean that when the redefinition of the set of vectors that belongs to the representation of the item occurs such that this corresponds to a uniform movement of the physical item,

then the influences of the universe of items tend to compensate each other. Otherwise, the universe of items reacts with a corresponding field. Besides of the universe wide response, a local variance in the distribution of items causes a variation in the influences. This local variance can partly be the consequence of a uniform displacement of particles.

It seems that quantum logic and Helmholtz decomposition together define an important part of the static relations that exist in physics. The fields appear to resist the disturbance of the interrelations in the lattice of quantum propositions. In dynamical sense this lattice might step from one static status quo to the next. After a step new conditions are established that again must fulfill the laws that govern the static situation. If this is a proper interpretation, then it is likely that the progression step is taken universe wide. After each step the positions of the physical items relative to the fields have changed, thus when the fields are not uniformly distributed, the items meet a different field configuration. The next step is taken with and due to these new conditions.

Quantum logic only defines a static skeleton in which the dynamics of quantum physics takes place. To make it a dynamic logic, the set of axioms must be extended. The new axioms must state that all propositions influence each other. The influence depends on their mutual (coordinate) distance. In stationary conditions, which include uniform motion, these influences compensate each other. When an atomic predicate that concerns an element of an ordered set is replaced in a non-ordered fashion, meaning that the distance between the replaced elements does not stay the same, then the universe of all propositions will react such that the influences of the other propositions no longer compensate each other. The disordered influences counteract the disordered replacement.

Besides of that the local variance in the distribution of the propositions, which corresponds to a variance of the distribution of the corresponding

physical items, also cause a variation in the influences that propositions have with respect to each other.

In Hilbert space these influences are implemented in the actions of Hilbert fields. In quantum physics the influence appears as a set of physical fields.

### Minkowski signature

One important step must still be taken. In physics observed spacetime has a Minkowski signature. Further we observe that space corresponds with the imaginary part of a position quaternion for which the real part seems to have no direct physical meaning. We must find an explanation for these facts. The Minkowski signature defines the following time-like relation between the space time step  $\Delta s$ , the space step  $\Delta q$  and the coordinate time step  $\Delta t$

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta q)^2/c^2 \quad (1)$$

During inertial motion this corresponds for the [proper time](#)<sup>113</sup>  $\tau$  to

$$(\Delta \tau)^2 = (\Delta t)^2 - (\Delta q)^2/c^2 \quad (2)$$

$$\Delta t = \Delta \tau + \Delta q/c \quad (3)$$

This is a triangle relation where  $\Delta t$  is at the hypotenuse.

If we substitute the Planck length for  $\Delta q$  and the Planck time for  $\Delta \tau$  then  $\Delta t$  equals the Planck time times the square root of 2. If we instead substitute the Planck time for  $\Delta t$  then  $\Delta \tau$  equals zero.

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<sup>113</sup> [http://en.wikipedia.org/wiki/Proper\\_time](http://en.wikipedia.org/wiki/Proper_time)

## Dynamics

### Schrödinger or Heisenberg picture

For global rotations around its origin the Hilbert unit sphere acts as an affine space. It does not matter whether the eigenvectors of operators or the subspace that represents the item is moved. We can take the picture in which the subspace stays fixed, while the vectors move and the operators change with them. This is the **Heisenberg picture**.

We can also take the picture in which the vectors and operators stay fixed and the subspace moves. This is the **Schrödinger picture**.

We are only interested in the consequences. These are determined by the relative movement, not by the absolute movement. For a given physical item, in both pictures the expectation values of the operators vary in the same way.

### Unitary transform

A unitary transform is a bounded normal operator. Unitary transforms keep the value of inner products untouched. If a unitary transform is applied to two vectors, then their inner product stays the same.

Unitary transforms need not have eigenvectors. For example Fourier transforms do not possess eigenvectors. In the rigged Hilbert space  $\mathbf{H}$  functions exist that apart from a scaling factor are invariant under Fourier transformation. These functions are not vectors. They define linear combinations of members of an orthonormal base that consists of eigenvectors of an operator. The corresponding eigenvalues form the parameters of the function.

If a unitary transform has eigenvectors then it has unit sized eigenvalues and to each of these eigenvalues correspond one or more eigenvectors that are mutually orthogonal. Unitary transforms are completely determined by their vector replacement characteristics, their eigenvectors

and the corresponding eigenvalues. An extra characteristic is for example the smoothness of their eigenspace.

When a unitary operator  $U$  is applied to the eigenvector  $|q\rangle$  of an operator  $Q$  with eigenvalue  $q$ , then the eigenvector is transferred into another vector  $|Uq\rangle$ . In general  $|Uq\rangle$  is not another eigenvector of  $Q$ . The expectation value for  $|QUq\rangle$  is no longer  $q$ , but

$$\langle q|U|QUq\rangle = \langle q|U^\dagger QUq\rangle \quad (1)$$

Or, with other words the operator  $Q$  is redefined to  $U^\dagger QU$ .

The norm of the expectation value  $\langle f|U|QUf\rangle$  for an arbitrary vector  $|f\rangle$  does not depend on  $U$ . It only depends on  $Q$  and  $|f\rangle$ .

### Trail of infinitesimal transforms

The effect of a single unitary transform  $U$  can also be achieved by a trail of infinitesimal unitary transforms  $\{U_t\}_t$ . This also holds for a set of unitary operators that operate in parallel.

The situation sketched above can be refined for any instant  $t$  occurring after  $t=0$ . We can treat it more generally by chopping the path from  $\{|f_s\rangle\}_s$  to  $\{|g_{st}\rangle\}_s$  into a **trail** of infinitesimal steps of size  $\Delta t$  that is achieved by a set of infinitesimal transforms  $\{U_{st}\}_{st'}$  where

$$|g_{st}\rangle = |\prod_s U_{st} f_s\rangle \quad (1)$$

and

$$U_{st} \approx 1 + \Delta S_{st} \quad (2)$$

The parameter  $t$  acts as the trail progression parameter. It is not identical with our common notion of time. If it has anything to do with time it will be confronted with a lower limit, which is set by the Planck time. The

infinitesimal transforms  $U_{st}$  work in parallel as well as in sequence.  $\Delta S_{st}$  represents the current local infinitesimal action step. It is an imaginary operator. Like time the action step also has a lower limit that is determined by a corresponding Planck unit.

The Heisenberg picture conforms to the description with unitary transforms where operators are redefined. When this is done in small steps, then the redefined operator becomes a function of progression parameter  $t$ .

### Unitary transform with full set of eigenvectors

When a unitary transformation  $U$  is applied to an arbitrary vector  $|f\rangle$ , which is not an eigenvector, then that vector is transferred into another vector  $|g\rangle = |Uf\rangle$ , which has the same norm. If  $|f\rangle$  is an eigenvector of  $U$  then  $|f\rangle$  is **not** transferred to a different vector, but it is multiplied with the corresponding eigenvalue. Also in this case the norm stays the same.

If a unitary transform contains a full set of eigenvectors, then multidimensional subspaces usually contain one or more eigenvectors of that unitary transform. In that case the transfer of a multidimensional closed subspace requires a set of parallel unitary transforms.

If we take a set of vectors  $\{|f_s\rangle\}_s$  that together span a closed subspace, then a set of suitable unitary transforms  $\{U_s\}_s$ , can in parallel transfer all vectors of this set such that after the transform  $|g_s\rangle = |U_s f_s\rangle$  the set  $\{|g_s\rangle\}_s$  spans the new subspace. Each of the members  $U_s$  of the set  $\{U_s\}_s$  can be split in a trail.  $\{U_{st}\}_t$

### Fourier transform as unitary transform

Unitary transforms exist that have no eigenvectors. For example in Hilbert space a Fourier transform has no eigenvectors and no eigenvalues. It does not leave a single Hilbert vector untouched.

The Fourier transform converts an orthonormal base into another orthonormal base, which is the canonical conjugate of the first.

Hilbert fields exist that apart from a scaling factor are invariant under Fourier transformation. They keep their form through Fourier transformation. For that reason they are often called eigenfunctions, but they do not correspond to eigenvectors. The [Hermite functions<sup>114</sup>](#) are notorious examples of Fourier “eigenfunctions”. [Even and odd functions](#) have an indirect relation to functions that are invariant under Fourier transformation.

*An invariant function is not an eigenfunction.* In extended separable Hilbert space, every Fourier transform causes a resampling of the analyzed field or function.

Each Fourier transform means a complete replacement of the current orthonormal base. For that reason, a Fourier transform that resides in separable Hilbert space can never be an infinitesimal unitary transform. Stated in different words this means: The transform  $U_{st} = 1 + \Delta S_{st}$  is not a Fourier transform. However, Fourier transforms  $U_{\mathcal{F}}$  and reverse Fourier transforms  $U_{\mathcal{F}}^{\dagger}$  can be member of a trail of unitary transforms, where each trail step contains a move up and down to Fourier space, while in Fourier space only an infinitesimal action is taken.

$$|g_{st}\rangle = |\prod_s U_{\mathcal{F}} U_{st} U_{\mathcal{F}}^{\dagger} f_s\rangle \tag{1}$$

Stepping through the Fourier space has the advantage that there derivation turns into multiplication and multiplication with a factor close to unity reduces to addition.

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<sup>114</sup> Functions and fields; Functions that are invariant under Fourier transform.

### Single infinitesimal step

The success of the [Feynman path integral formalism](#)<sup>115</sup> gives us guidance in the analysis of what happens during a single infinitesimal step. We analyze an arbitrary trail consisting of infinitesimal trajectory steps:

$$\langle q_{t_1} | U_{s_{t_1, t_0}} | q_{t_0} \rangle = \left\{ \prod_{t=t_0}^{t=t_1} (\langle q_t' | p_t \rangle \langle p_t | U_{s_t} | q_t \rangle) \right\} \langle q_{t_1} | p_{t_1} \rangle \quad (1)$$

During a single step the system moves from position  $q$  to  $q' = q + \Delta q$ ? Let us evaluate  $\langle q' | p \rangle \langle p | U_{s_t} | q \rangle$  for a single trajectory. Here  $U_{s_t}$  is an infinitesimal unitary transform. It is a member of the set of parallel unitary transforms that act on a target subspace. In the following text we leave the parallel trajectory index  $s$  in  $U_{s_t}$  unspecified. We concentrate on the sequence index  $t$ , which represents the progression parameter. The infinitesimal sequence step comprises three sub-steps:

1. Goto Fourier space. This is achieved by part  $\langle q' | p \rangle$ .
2. Perform the action. This is done by  $\langle p | U_t^\dagger |$ .
3. Go back to configuration space. This is achieved by  $\langle p | q \rangle$ .

The sense behind the first and the last part is a travel to and back from Fourier space. Step two means that in Fourier space the action of the operator is just a multiplication with factor  $\exp(\Delta s_t)$ .

$$\langle p | U_t^\dagger | = \exp(\Delta s_t) \langle p | \quad (2)$$

First we split  $\langle q' | p \rangle$ .

$$\langle q' | p \rangle = \langle q | p \rangle \exp\left(\frac{\Delta q p}{\hbar}\right) \approx \langle q | p \rangle \left(1 + \frac{\Delta q p}{\hbar}\right) \quad (3)$$

$$\langle p | q \rangle = \langle q | p \rangle^* \quad (4)$$

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<sup>115</sup> [http://en.wikipedia.org/wiki/Path\\_integral\\_formulation](http://en.wikipedia.org/wiki/Path_integral_formulation)

$$\begin{aligned}
\langle q'|p \rangle &\exp(\Delta s_t) \langle p|q \rangle & (5) \\
&\approx \langle q|p \rangle \left(1 + \frac{\Delta q p}{\hbar}\right) (1 + \Delta s_t) \langle p|q \rangle \\
&= \langle q|p \rangle C_t \langle q|p \rangle^*
\end{aligned}$$

This is a quaternionic rotation of the central term  $C_t$ , which is close to unity. The quaternionic rotation manipulators stands for the route to Fourier space and the route back from Fourier space. The central term  $C_t$  stands for what is done during a single step by the action in Fourier space.

$$C_t \approx \left(1 + \frac{\Delta q p}{\hbar}\right) (1 + \Delta s_t) \approx 1 + \frac{\Delta q p}{\hbar} + \Delta s_t = 1 + \Delta C \quad (6)$$

$$\Delta C = \frac{\Delta q p}{\hbar} + \Delta s_t \quad (7)$$

$$\hbar \Delta C - \hbar \Delta s_t = \Delta q p \quad (8)$$

$$= \Delta q_0 p_0 - \langle \Delta q, p \rangle + \Delta q_0 p + \Delta q p_0 + \Delta q \times p$$

We study the step  $\Delta C$  somewhat deeper. Since  $q$  and  $p$  are considered to be imaginary, we skip the parts containing  $\Delta q_0$  or  $p_0$ .

$$\hbar \Delta C - \hbar \Delta s_t = -\langle \Delta q, p \rangle + \Delta q \times p \quad (9)$$

We introduce  $K$ . It characterizes the infinitesimal step.

$$K \equiv -\hbar \frac{\Delta C}{\Delta t} \approx -\hbar \dot{C} \quad (10)$$

Both  $\Delta C$  and  $K$  are functions of progression parameter  $t$ .

$$\hbar \Delta C_0 = -K_0 \Delta t = -\langle \Delta q, p \rangle \quad (11)$$

$$K_0 = \left\langle \frac{\Delta \mathbf{q}}{\Delta t}, \mathbf{p} \right\rangle \approx \langle \dot{\mathbf{q}}, \mathbf{p} \rangle \quad (12)$$

$$\hbar \Delta \mathbf{C} - \hbar \Delta \mathbf{s}_t = -\mathbf{K} \Delta t - \hbar \Delta \mathbf{s}_t = \Delta \mathbf{q} \times \mathbf{p} \quad (13)$$

$$\mathbf{K} = -\hbar \frac{\Delta \mathbf{s}_t}{\Delta t} + \frac{\Delta \mathbf{q}}{\Delta t} \times \mathbf{p} \approx -\hbar \dot{\mathbf{s}}_t + \dot{\mathbf{q}} \times \mathbf{p} \quad (14)$$

The steps  $\Delta q_t$  and  $\Delta s_t$  depend on the step  $\Delta t$  of the trajectory parameter  $t$  that is used to chop the unitary transform  $U_{t_1, t_0}$ .

In the trail the imaginary part of  $K$  is rotated.

If  $\Delta \mathbf{q}$  is zero then

$$K = -\hbar \frac{\Delta \mathbf{s}_t}{\Delta t} \quad (15)$$

$\Delta t$  is never zero. If  $\Delta t$  equals the Planck time, then  $|\Delta \mathbf{q}|$  is either zero or it equals the Planck length. In that case

$$\left| \frac{\Delta \mathbf{q}}{\Delta t} \right| = c \quad (16)$$

### Ray tracing

Following a trail has much in common with ray tracing in [optics](#).

However in optics the use of characteristics that have their base in Fourier analysis seems to be more fruitful than ray tracing. Ray tracing follows the path of a sharp particle, while Fourier analysis is capable of following the life path of a blurred particle and include more of the influences of the environment in the analysis. It is sensible to expect that the advantages of Fourier analysis also hold for wave mechanics. As shown here, to a certain extent the path integral approach also makes use of Fourier analysis.

## Relativity

Einstein's own explanation of the origin of relativity was: "*There is no logical way to the discovery of these elementary laws. There is only the way of intuition.*" Read more in:

<http://www.time.com/time/magazine/article/0,9171,878733,00.html#ixzz15NlhWpWDu>

Thus, Einstein never gave a proper explanation for the existence of special relativity. He just provided a set of formulas that work properly. He left us the choice of finding the origin of special relativity or otherwise to follow his intuition. Let us give it a try:

The position operator  $\mathbf{Q}$  is modified by the unitary operators of the trail into another operator  $\mathbf{Q}_t$  that has different eigenvectors and different eigenvalues.

$$\mathbf{Q}_{t+\Delta t} = U_t \mathbf{Q}_t U_t^\dagger \quad (1)$$

$$U_t \approx 1 + \Delta \mathbf{S}_t \quad (2)$$

$$U_t^\dagger \approx 1 - \Delta \mathbf{S}_t \quad (3)$$

$$\mathbf{Q}_{t+\Delta t} \approx \mathbf{Q}_t + [\Delta \mathbf{S}_t, \mathbf{Q}_t] \quad (4)$$

$$\Delta \langle \mathbf{q} \rangle = \langle \mathbf{Q}_{t+\Delta t} \rangle - \langle \mathbf{Q}_t \rangle \approx [\Delta \mathbf{S}_t, \mathbf{Q}_t] = 2 \mathbf{Q}_t \times \Delta \mathbf{S}_t \quad (5)$$

This indicates that the step  $\Delta \langle \mathbf{q} \rangle$  in the expectation value  $\langle \mathbf{Q}_t \rangle$  of  $\mathbf{Q}_t$  is perpendicular to both  $\mathbf{Q}_t$  and  $\Delta \mathbf{S}_t$ . The steps  $\Delta \langle \mathbf{q} \rangle$  and  $\Delta \mathbf{S}_t$  form a right angular triangle with a hypotenuse:  $c \Delta \mathbf{t}_c$ , such that:

$$c \Delta \mathbf{t}_c = \Delta \langle \mathbf{q} \rangle + \Delta \mathbf{S}_t \quad (6)$$

Here we introduced a new imaginary variable  $\mathbf{t}_c$ .

With  $\Delta \sigma = \Delta \mathbf{S}_t / c$  the Minkowski signature of a new "observable" [spacetime](#) becomes visible.

$$\Delta t_c = \frac{\Delta \langle q \rangle}{c} + \Delta \sigma \quad (7)$$

$$\Delta \sigma = \Delta t_c - \Delta \langle q \rangle / c \quad (8)$$

$$|\Delta \sigma|^2 = |\Delta t_c|^2 - |\Delta \langle q \rangle|^2 / c^2 \quad (9)$$

Thus, the analysis of what occurs during a single infinitesimal step gives us an indication how relativity enters physics. However, it asks for the introduction of a local notion of time  $t_c = |\mathbf{t}_c|$  that differs considerably from the (global) progression parameter  $t$ . This new parameter is the [coordinate time](#)<sup>116</sup>  $t_c$ .

### Proper time

In relativity, [proper time](#)<sup>117</sup>  $t_p$  is time measured by a single clock between events that occur at the same place as the clock. It depends not only on the events but also on the motion of the clock between the events. An accelerated clock will measure a proper time between two events that is shorter than the coordinate time measured by a non-accelerated (inertial) clock between the same events.

$$|\Delta t_p|^2 = |\Delta t_c|^2 - |\Delta \langle q \rangle|^2 / c^2 \quad (1)$$

$$\Delta \sigma = \frac{\Delta \mathbf{S}_t}{c} = \mathbf{e}_\sigma \Delta t_p \quad (2)$$

Thus, proper time  $t_p$  is, via the action step  $\Delta \mathbf{S}_t$  related to our notion of progression parameter  $t$ .

For a photon the proper time step is always zero. This also holds in the realm of general relativity. In the vicinity of a black hole this leads to the

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<sup>116</sup> [http://en.wikipedia.org/wiki/Coordinate\\_time](http://en.wikipedia.org/wiki/Coordinate_time)

<sup>117</sup> [http://en.wikipedia.org/wiki/Proper\\_time](http://en.wikipedia.org/wiki/Proper_time)

fact that the radial velocity of a photon approaches zero when the photon approaches the border of the black hole. The border is located at the Schwarzschild radius  $r_{sh}$

$$r_{sh} = 2MG/c^2 \quad (3)$$

We use polar coordinates and the expression for the metric near the black hole

$$|\Delta t_p|^2 = (1 - \frac{r_{sh}}{r})|\Delta t_c|^2 - \left( \frac{|\Delta r|^2}{1 - r_{sh}/r} + r^2 |\Delta \Omega|^2 \right) / c^2 \quad (4)$$

Take  $\Delta \Omega = 0$ . Then with  $\Delta t_p = 0$

$$\frac{dr}{dt_c} = c \left( 1 - \frac{r_{sh}}{r} \right) \quad (5)$$

### Discussion

We have successfully introduced special relativity into our model.

By introducing relativity the way we did we played a few tricks.

- We neglect the real part of the position observable. In our model it plays no essential part in dynamics.
- We shift from Hilbert space via quaternionic 1+3D coordinate space to observed space, thereby losing one dimension.
- Clocks do not count progression steps. Instead they tend to measure coordinate time.
- We may shift from the global progression parameter  $t$  to the local coordinate time  $t_c$ .
- We may combine the resulting observed space with coordinate time into a Minkowski/Lorentzian space.

As a consequence

- We then shift from  $2^n$ -on/Riemannian space to Minkowski/Lorentzian space.

- Most physicists will use Clifford, Jordan and Grassmann algebras rather than  $2^n$ -on algebras.
- With these algebras they can use complex analysis instead of the more complicated  $2n$ -on analysis.
- But if they do so, they are confronted with unintuitive selection features.
- In the new space the quaternion waltz becomes an odd operation.
- Spinors can help in order to cope with these changes.

### Can we do without relativity?

Yes.

- Skip coordinate time.
- Use clocks that measure the progression parameter.

However, you would have to fight existing conventions. Specifying causality might become a problem.

### Inertia and progression step

The covering field represents the influence of the universe of all particles. According to the findings about [inertia<sup>118</sup>](#), the change  $\Delta E$  since the last progression step of the corresponding curvature field  $E$  determines the acceleration that a local particle senses during the current progression step.

This results in the acceleration  $\frac{\partial \mathbf{v}}{\partial t}$  of the particle.

$$\Delta E(\mathbf{r}, t) = G \cdot \frac{\partial \mathbf{v}}{\partial t} \tag{1}$$

### Redefinition

If we want to use the Schrödinger picture, rather than the Heisenberg picture, then it is better not to use unitary transforms, because they change the operators by acting on the eigenvectors of the operators. Instead the subspace should be redefined without touching eigenvectors.

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<sup>118</sup> Influence; Inertia

Let us suppose that there exists a dynamical equivalent of the traditional quantum logic. The equivalent of a move of a physical item in the lattice of propositions is a redefinition of a subset of the propositions. The redefinition occurs in terms of atomic predicates that describe the properties of the physical items. In the Hilbert space this corresponds with a redefinition of a relevant part of the Hilbert subspace in terms of the eigenvectors that belong to the new eigenvalues.

The redefinition concerns the Hilbert space which represents the current static status quo. The step transforms the current version of the Hilbert space into a past version of the Hilbert space and it transforms a future version of the Hilbert space into the new current version. This is interesting in the light that a Hilbert field exists that controls the relation between the past, the current and the future versions of the Hilbert fields. For that reason we will call this special Hilbert field the **adventure field**. A transform that controls dynamics converts a future Hilbert space into the new current Hilbert spaces and it converts the current Hilbert spaces into a past Hilbert space. This transform will be called **progression transform**. The local blurs that characterize the adventure field form boundary conditions for the local transfer characteristics of the progression transform. Each item type is surrounded by a characteristic blur.

A progression transform that moves Hilbert subspaces without touching the eigenvectors of normal operators will be called a **redefiner**. The effect of the action of the redefiner on expectation values of operators must be similar to the effect of the trail of parallel unitary transforms treated in the previous paragraphs. While the set of parallel trails of unitary transforms act in the Heisenberg picture, the redefiner acts in the Schrödinger picture. As indicated earlier, the redefiner has an equivalent in the dynamic version of quantum logic.

In order to achieve the same effect as the Heisenberg picture, the Hilbert subspace redefiner must to a large degree have similar properties as the

trails of parallel infinitesimal unitary transformations that are used to move the subspace in the Heisenberg picture. The redefinition keeps the inner products of vectors intact. Where unitary transforms rotate vectors around the origin of a Hilbert space, the redefiner takes subspaces of a potential future Hilbert space in order to redefine them into subspaces of the new current Hilbert space. In contrast to a unitary transform the redefiner does not change the eigenvectors of normal operators. Thus, it leaves the operators untouched. Like the trails of unitary transforms the redefinition works in infinitesimal steps. These infinitesimal actions also form trails. In this way the manipulated subspace can move close to continuously through Hilbert space. Where the redefiners act on subspaces, the trails of unitary transforms redefine operators.

During this process the subspace may change its configuration. This may include a change that corresponds to the change of type definitions of atomic predicates. The redefiner steps from one stationary situation to the next. The Schrödinger picture conforms to the description with a redefiner. The result for the position of the locator must be the same as it was under the influence of the set of parallel infinitesimal unitary operators in the Heisenberg picture. The redefiner moves the subspace such that the new locator position is similar to the value as was established by the redefined position operator. It means that during the redefiner step the position of the locator undergoes an infinitesimal number transform that is equivalent to the infinitesimal transform that is established by the redefined position operator. That redefinition was caused by the parallel infinitesimal unitary transforms.

### Trails

In fact the  $\Delta t$  step characterizes the redefinition step. The subsequent replacement of vectors and the replacement of the corresponding eigenvalues can be interpreted as a rather continuous movement of the corresponding characteristic subjects. Here we encountered ten different trails.

1. The trail of subsequent manipulators (infinitesimal unitary transforms or infinitesimal redefiners) that each perform an infinitesimal action.
2. The trail of subspaces, which with respect to the manipulators are characteristic for the considered item.
3. The trail of corresponding "action values" of the redefiner.
4. The trails of corresponding "action values" of the unitary transforms.
5. The trail of eigenvectors  $|q_t\rangle$
6. The trail of corresponding observables  $Q_t$ .
7. The trail of corresponding observed expectation values  $q_t$ .
8. The trail of values  $\psi(q_t)$  of a wave function.
9. This, on its turn corresponds to a trail of a state in coordinate space
10. And a trail of that state in Hilbert space.

## Cycles

It is quite possible that locally subsequent steps are done in cycles of two or more steps. It is obvious that movements inside an item are cyclic. In ideal circumstances these movements are harmonic.

## Redefiner

The concept of dynamic manipulator gives us reason to introduce a **new type of actuator**: the redefiner  $\mathcal{R}$ . This actuator moves subspaces, but leaves vectors untouched. It works in infinitesimal steps. In the Hilbert book model its activity fits in the conversion from an actual Hilbert space to the next Hilbert space. It is easily interpreted as a function  $\mathcal{R}_t$  of the progression parameter  $t$ . Its scope spans the subsequent Hilbert spaces. The effect of each step on an item is similar to the effect of a set of parallel infinitesimal unitary transforms  $\{U_{ts}\}_s$ . The current "action value" of the redefiner is a number, which is close to unity. It is an "average" of the "actions values" of the parallel infinitesimal uniforms that are active in the same step. The redefiner accepts 2<sup>n</sup>-ons as "action values".

The redefiner has an equivalent in a dynamic quantum logic, where it redefines propositions that concern the same objects as are represented by

the closed subspaces of the Hilbert space that are moved by  $\mathcal{R}_t$ . There seems to be no objection against the assumption that  $\mathcal{R}_t$  has a global scope. If we take that point of view, then the progression parameter  $t$  also has a global scope.

With this interpretation, the redefiner is a universe-wide stepper. It transforms the universe from one static situation to the next static situation. These static situations are governed by extended quantum logic, which combines traditional quantum logic, the blur of representations of physical items and the Helmholtz/Hodge decomposition theorems. After each step a new static status quo of subspaces and fields is established. After the step the conditions have been changed. After each step the position of the physical item relative to the fields has changed, thus when the fields are not uniformly distributed, the item meets a different field configuration. On the other hand the fields represent the blurs of the individual items. Thus, when the position or the type of the item has changed, then the local configuration of the field has changed. This is the way that macroscopic dynamics takes place in quantum physics.

## Equations of motion

### Private continuity equation

Existence, transport, generation and annihilation of information carrying quanta is governed by a continuity equation. In short this equation runs:

$$\text{Total change within } V = \text{flow into } V + \text{production inside } V \quad (1)$$

This integral equation corresponds to differential equations in which an information quantum density  $\rho_0$ , an information quantum current  $\rho$  and an information quantum source  $s$  will play a role.

Particles act as sources and drains. Private fields represent the currents and the static density distributions. Wave functions are private fields that represent the situation in the direct environment of particles.

The rotation free part of the private field corresponds to the divergence of the information probability amplitude distribution, whose squared modulus corresponds to the probability density. The transverse part of the private field corresponds to curl of the information current probability amplitude distribution, whose squared modulus corresponds to the information current. Together the private fields form the covering field. The covering field is the superposition of all private fields. The curvature field is derived from the decomposition properties of the covering field.

### Particles

Fields are superpositions of probability amplitude distributions. These probability amplitude distributions are typical for corresponding particles and are attached to one or a small set of Hilbert vectors.

Four boson types exist: photons, gluons, W type and Z type bosons. Bosons are attached to a single Hilbert vector. They all have spin  $\pm 1$ . However, apart from the W type, bosons do not carry a charge. They are the messengers that transfer interactions. Photons and gluons are massless bosons. Z and W type bosons have mass. They mediate weak field forces. The photons mediate EM field forces. The gluons mediate color (strong) field forces.

Six quark types exist: up, down, charm, strange, top and bottom. They all have spin  $\pm 1/2$  and fractional charge (color charge). They are massive. In the strand model the quarks are attached to two Hilbert vectors.

Six lepton types exist. All leptons have spin  $\pm\frac{1}{2}$ . The three neutrino types have no charge and relative little mass. The electron, the muon and the tau particle are all massive particles. According to the strand model the leptons are attached to three Hilbert vectors. Together these anchor points carry a centralized probability distribution that represents the influence of a single charge.

## Interactions

Probability amplitude distributions, which represent particles, move and rotate. That is interpreted as a movement / rotation of the corresponding item. Interactions may change the form of the probability amplitude distributions. Three types of change are discerned:

In strand theory the **first Reidemeister move**, or **type I move**, or **twist**, is the addition or removal of a twist in a corresponding strand. In Hilbert space it involves the approach of a single Hilbert vector into the realm of a particle. The twist, is related to the electromagnetic interaction. Two twist directions are possible. The twists form an SU(1) group.

In strand theory the **second Reidemeister move**, or **type II move**, or **poke**, is the addition or removal of a bend of one strand under (or over) a second strand. In Hilbert space it involves the interaction of two Hilbert vectors in the realm of a particle, where one Hilbert vector approaches the particle. The poke is related to electro-weak interaction. Three basic pokes exist. The pokes form an SU(2) group.

In strand theory the **third Reidemeister move**, or **type III move**, or **slide**, is the displacement of one strand segment under (or over) the crossing of two other strands. In Hilbert space it involves the simultaneous interaction of three Hilbert vectors in the realm of a particle, where one Hilbert vector approaches the particle. The slide is related to electro-strong interaction. The slides form an SU(3) group.

Each Reidemeister move generates a single corresponding observable quant or annihilates a single potentially observable quant.

## Schrödinger equation

When the spin has a constant direction:

The first term on the left side signifies the quantum generation rate per time step.

The second term indicates the influence of the electric field on this rate.

The first term on the right signifies the generation rate per path length.

The second term indicates the influence of the vector potential on this rate.

The square dependence indicates the increasing alignment of spin with the movement.

$$(\hbar\omega - q\tilde{V})\tilde{\psi}(x, t) = (\hbar\mathbf{k} - q\tilde{\mathbf{A}})\tilde{\psi}(x, t) \quad (1)$$

$$(i\hbar\partial_t - qV)\psi = \frac{1}{2m} (-i\hbar\nabla - q\mathbf{A})^2\psi \quad (2)$$

## Pauli equation

When the spin has no constant direction:

The density  $\rho(x, t)$  and the Euler angles  $\alpha, \beta, \gamma$  define the Pauli equation:

$$\Psi(x, t) = \sqrt{\rho} e^{\frac{i\alpha}{2}} \begin{bmatrix} \cos\left(\frac{\beta}{2}\right) e^{\frac{i\gamma}{2}} \\ i \sin\left(\frac{\beta}{2}\right) e^{-\frac{i\gamma}{2}} \end{bmatrix} \quad (1)$$

Due to the half angles, the two-component matrix is not a vector, but a spinor.

$$\boldsymbol{\sigma} = \left( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \quad (2)$$

$$\mathbf{1} \mapsto I, \quad \mathbf{i} \mapsto i\sigma_1, \quad \mathbf{j} \mapsto i\sigma_2, \quad \mathbf{k} \mapsto i\sigma_3 \quad (3)$$

Pauli's equation for the evolution of a free quantum particle with spin  $\frac{1}{2}$  is:

$$i\hbar\partial_t\Psi = -\frac{\hbar^2}{2m} (\boldsymbol{\sigma}\nabla)^2\Psi \quad (4)$$

$$(i\hbar\partial_t - qV)\Psi = \frac{1}{2m} (-i\hbar\nabla - q\mathbf{A})^2\Psi - \frac{q\hbar}{2m} \boldsymbol{\sigma}\mathbf{B}\Psi \quad (5)$$

The last term shows the influence of spin.

## Dirac equation

The final and most detailed description of elementary fermions, the Dirac equation, results from combining all three ingredients:

1. the relation between the quantum of action and the phase of the wave function,
2. the relativistic mass–energy relation,
3. spin 1/2.

$$i \frac{\partial \psi}{\partial t} = (\boldsymbol{\alpha} \nabla \pm i \gamma_5 m) \psi \quad (1)$$

$$\boldsymbol{\alpha} = \begin{bmatrix} 0 & \mathbf{i} \\ -\mathbf{i} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{j} \\ -\mathbf{j} & 0 \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{k} \\ -\mathbf{k} & 0 \end{bmatrix} \quad (2)$$

$$\gamma_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (3)$$

$$\psi = \sqrt{\rho} e^{i\delta} L(\mathbf{v}) R \left( \frac{\alpha}{2}, \frac{\beta}{2}, \frac{\gamma}{2} \right) \quad (4)$$

$\rho(\mathbf{r}, t)$  is the probability density.

$\delta$  is a phase which represents the relative importance of particle and antiparticle density.

$\alpha, \beta$  and  $\gamma$  are Euler angles. They describe the average local orientation and phase of the spin axis.

(this defines a rotating spin vector)

$\mathbf{v}(r, t)$  is the average local Lorentz boost.

$LR$  is an abbreviation for the boosted and rotated unit spinor.(quantum)

The probability amplitude  $\psi$  moves and rotates and individually the quanta carry position, momentum and angular momentum (including spin) information.

## Fields

It is clear that the physical fields play an important role in nature. They form an indispensable ingredient in the establishment of dynamics. Each physical item follows a path through a set of universe wide fields. The

static gravitational field, the electrostatic field and the electromagnetic field are all subjected to the Helmholtz decomposition theorem. The difference between the gravitational field and the electromagnetic field is that the masses are non-negative and the electric charges are, apart from a sign, always the same. All other fields also have charges that on the long range will compensate each other. The gravitation field can be seen as being derived from the curvature set by the decomposition of the covering field. The covering field is the superposition of all fields but the gravitation field.

When the path with respect to the gravitation field corresponds to a unit speed curve then that field executes no action onto that item. Only the gravitation field keeps its long range because its charges do not compensate each other's potentials. They only compensate each other's forces.

### More fields

There exists a list of fields with shorter ranges than the range of the gravitation field and the range of the electromagnetic fields. The electro-weak field and the electro-strong field are not treated here in detail.

The action represented by a complete Lagrangian indicates how fields appear in the argument of a manipulator. See [Lagrangian of the world<sup>119</sup>](#) for a complete survey of terms. [Mendel Sachs<sup>120</sup>](#) has also found a way to bring all terms under the same hood.

### Lagrangian

$$\mathcal{L} = \bar{\varphi}(i\hbar c\mathcal{D} - mc^2)\varphi - \frac{1}{4\mu_0}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\sum_{a=1}^3 W_{\mu\nu}^a W_a^{\mu\nu} - \frac{1}{4}\sum_{a=1}^8 G_{\mu\nu}^a G_a^{\mu\nu} \quad (1)$$

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<sup>119</sup> Appendix; Thoughts; The world's action

<sup>120</sup> Appendix; Thoughts; Representing multiple fields

Where  $\mu, \nu = 1, 2, 3$

The first term concerns the affected particle.

The second term concerns electromagnetic interactions. Reidemeister twists. SU(1).

The third term concerns unbroken weak interactions. Reidemeister moves. SU(2).

The fourth term concerns unbroken strong interactions. Reidemeister slides. SU(3).

$$\mathcal{D} = \gamma^\sigma D_\sigma = \gamma^\sigma (\partial_\sigma - iqA_\sigma) \quad (2)$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \quad (3)$$

$$W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a - g f^{abc} W_\mu^b W_\nu^c \quad (4)$$

$$G_{\mu\nu}^a = \partial_\mu G_\nu^a - \partial_\nu G_\mu^a - g f^{abc} G_\mu^b G_\nu^c \quad (5)$$

$\varphi$  is the private field of the affected particle.

$A_\nu$ ,  $W_\nu^a$  and  $W_\nu^a$  are vector potentials of the corresponding subfields

$g$  is the gauge [coupling constant](#)<sup>121</sup>. The quantity  $f^{abc}$  is the [structure constant](#)<sup>122</sup> of the gauge group.

## Path through field

The text in this section is borrowed from Wikipedia.

In a [Riemannian manifold](#)<sup>123</sup>  $M$  with [metric tensor](#)<sup>124</sup>  $g$ , the length of a continuously differentiable curve  $\gamma: [a, b] \rightarrow M$  is defined by

$$L(\gamma) = \int_a^b \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt \quad (1)$$

The distance  $d(p, q)$  between two points  $p$  and  $q$  of  $M$  is defined as the [infimum](#)<sup>125</sup> of the length taken over all continuous, piecewise continuously differentiable curves  $\gamma: [a, b] \rightarrow M$  such that  $\gamma(a) = p$  and

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<sup>121</sup> [http://en.wikipedia.org/wiki/Coupling\\_constant](http://en.wikipedia.org/wiki/Coupling_constant)

<sup>122</sup> [http://en.wikipedia.org/wiki/Algebra\\_over\\_a\\_field#Structure\\_coefficients](http://en.wikipedia.org/wiki/Algebra_over_a_field#Structure_coefficients)

<sup>123</sup> [http://en.wikipedia.org/wiki/Riemannian\\_manifold](http://en.wikipedia.org/wiki/Riemannian_manifold)

<sup>124</sup> [http://en.wikipedia.org/wiki/Metric\\_tensor](http://en.wikipedia.org/wiki/Metric_tensor)

<sup>125</sup> <http://en.wikipedia.org/wiki/Infimum>

$\gamma(b) = q$ . With this definition of distance, geodesics in a Riemannian manifold are then the locally distance-minimizing paths, in the above sense.

The minimizing curves of  $L$  in a small enough [open set](#)<sup>126</sup> of  $M$  can be obtained by techniques of [calculus of variations](#)<sup>127</sup>. Typically, one introduces the following [action](#)<sup>128</sup> or [energy functional](#)<sup>129</sup>

$$E(\gamma) = \frac{1}{2} \int_a^b g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) dt \quad (2)$$

It is then enough to minimize the functional  $E$ , owing to the [Cauchy–Schwarz inequality](#)<sup>130</sup>

$$L(\gamma)^2 \leq 2(b - a) E(\gamma) \quad (3)$$

with equality if and only if  $|d\gamma/dt|$  is constant.

The [Euler–Lagrange](#)<sup>131</sup> equations of motion for the functional  $E$  are then given in local coordinates by

$$\frac{d^2 x^\lambda}{dt^2} + \Gamma_{\mu\nu}^\lambda \cdot \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} = 0 \quad (4)$$

where  $\Gamma_{\mu\nu}^\lambda$  are the [Christoffel symbols](#)<sup>132</sup> of the metric. This is the **geodesic equation**.

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<sup>126</sup> [http://en.wikipedia.org/wiki/Open\\_set](http://en.wikipedia.org/wiki/Open_set)

<sup>127</sup> [http://en.wikipedia.org/wiki/Calculus\\_of\\_variations](http://en.wikipedia.org/wiki/Calculus_of_variations)

<sup>128</sup> [http://en.wikipedia.org/wiki/Action\\_\(physics\)](http://en.wikipedia.org/wiki/Action_(physics))

<sup>129</sup> [http://en.wikipedia.org/wiki/Energy\\_functional](http://en.wikipedia.org/wiki/Energy_functional)

<sup>130</sup> [http://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz\\_inequality](http://en.wikipedia.org/wiki/Cauchy%E2%80%93Schwarz_inequality)

<sup>131</sup> Appendix; Derivation of the one dimensional Euler Langrange equation

<sup>132</sup> Equations of motion; Path through field; Christoffel symbols

## Calculus of variations

Techniques of the classical [calculus of variations](#)<sup>133</sup> can be applied to examine the energy functional  $E$ . The [first variation](#)<sup>134</sup> of energy is defined in local coordinates by

$$\delta E(\gamma)(\varphi) = \left. \frac{\partial}{\partial t} \right|_{t=0} E(\gamma + t \varphi) \quad (1)$$

The [critical points](#)<sup>135</sup> of the first variation are precisely the geodesics. The second variation is defined by

$$\delta^2 E(\gamma)(\varphi, \psi) = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} E(\gamma + t \varphi + s \psi) \quad (2)$$

In an appropriate sense, zeros of the second variation along a geodesic  $\gamma$  arise along [Jacobi fields](#)<sup>136</sup>. Jacobi fields are thus regarded as variations through geodesics.

By applying variational techniques from [classical mechanics](#)<sup>137</sup>, one can also regard [geodesics as Hamiltonian flows](#)<sup>138</sup>. They are solutions of the associated [Hamilton–Jacobi equations](#)<sup>139</sup>, with (pseudo-)Riemannian metric taken as [Hamiltonian](#)<sup>140</sup>.

## Affine geometry

A **geodesic** on a smooth manifold  $M$  with an [affine connection](#)<sup>141</sup>  $\nabla$  is defined as a curve  $\gamma(t)$  such that [parallel transport](#)<sup>142</sup> along the curve preserves the tangent vector to the curve, so

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<sup>133</sup> [http://en.wikipedia.org/wiki/Calculus\\_of\\_variations](http://en.wikipedia.org/wiki/Calculus_of_variations)

<sup>134</sup> [http://en.wikipedia.org/wiki/First\\_variation](http://en.wikipedia.org/wiki/First_variation)

<sup>135</sup> [http://en.wikipedia.org/wiki/Critical\\_point\\_\(mathematics\)](http://en.wikipedia.org/wiki/Critical_point_(mathematics))

<sup>136</sup> [http://en.wikipedia.org/wiki/Jacobi\\_field](http://en.wikipedia.org/wiki/Jacobi_field)

<sup>137</sup> [http://en.wikipedia.org/wiki/Classical\\_mechanics](http://en.wikipedia.org/wiki/Classical_mechanics)

<sup>138</sup> [http://en.wikipedia.org/wiki/Geodesics\\_as\\_Hamiltonian\\_flows](http://en.wikipedia.org/wiki/Geodesics_as_Hamiltonian_flows)

<sup>139</sup> [http://en.wikipedia.org/wiki/Hamilton%E2%80%93Jacobi\\_equation](http://en.wikipedia.org/wiki/Hamilton%E2%80%93Jacobi_equation)

<sup>140</sup> [http://en.wikipedia.org/wiki/Hamiltonian\\_mechanics](http://en.wikipedia.org/wiki/Hamiltonian_mechanics)

<sup>141</sup> [http://en.wikipedia.org/wiki/Affine\\_connection](http://en.wikipedia.org/wiki/Affine_connection)

$$\nabla_{\dot{\gamma}} \dot{\gamma}(t) = 0 \tag{1}$$

at each point along the curve, where  $\dot{\gamma}$  is the derivative with respect to  $t$ . More precisely, in order to define the covariant derivative of  $\dot{\gamma}$  it is necessary first to extend  $\dot{\gamma}$  to a continuously differentiable imaginary Hilbert field in an [open set](#)<sup>143</sup>. However, the resulting value of the equation is independent of the choice of extension.

Using [local coordinates](#)<sup>144</sup> on  $M$ , we can write the **geodesic equation** (using the [summation convention](#)<sup>145</sup>) as

$$\frac{d^2 x^\lambda}{dt^2} + \Gamma_{\mu\nu}^\lambda \cdot \frac{dx^\mu}{dt} \cdot \frac{dx^\nu}{dt} = 0 \tag{2}$$

where  $x^\mu(t)$  are the coordinates of the curve  $\gamma(t)$  and  $\Gamma_{\mu\nu}^\lambda$  are the [Christoffel symbols](#)<sup>146</sup> of the connection  $\nabla$ . This is just an ordinary differential equation for the coordinates. It has a unique solution, given an initial position and an initial velocity.

From the point of view of classical mechanics, geodesics can be thought of as trajectories of free particles in a manifold. Indeed, the equation  $\nabla_{\dot{\gamma}} \dot{\gamma}(t) = 0$  means that the acceleration of the curve has no components in the direction of the surface (and therefore it is perpendicular to the tangent plane of the surface at each point of the curve). So, the motion is completely determined by the bending of the surface. This is also the idea of the general relativity where particles move on geodesics and the bending is caused by the gravity.

<sup>142</sup> [http://en.wikipedia.org/wiki/Parallel\\_transport](http://en.wikipedia.org/wiki/Parallel_transport)

<sup>143</sup> [http://en.wikipedia.org/wiki/Open\\_set](http://en.wikipedia.org/wiki/Open_set)

<sup>144</sup> [http://en.wikipedia.org/wiki/Local\\_coordinates](http://en.wikipedia.org/wiki/Local_coordinates)

<sup>145</sup> [http://en.wikipedia.org/wiki/Summation\\_convention](http://en.wikipedia.org/wiki/Summation_convention)

<sup>146</sup> [http://en.wikipedia.org/wiki/Christoffel\\_symbol](http://en.wikipedia.org/wiki/Christoffel_symbol)

## Christoffel symbols

If  $x^i, i = 1, 2, \dots, n$ , is a local coordinate system on a manifold  $M$ , then the tangent vectors

$$e_\mu = \frac{\partial}{\partial x_\mu}, \quad \mu = 1, 2, \dots, n \quad (1)$$

define a basis of the tangent space of  $M$  at each point. The Christoffel symbols  $\Gamma_{\mu\nu}^\lambda$  are defined as the unique coefficients such that the equation

$$\nabla_\mu e_\nu = \Gamma_{\mu\nu}^\lambda \cdot e_\lambda \quad (2)$$

holds, where  $\nabla_\mu$  is the [Levi-Civita connection](#)<sup>147</sup> on  $M$  taken in the coordinate direction  $e_\mu$ .

The Christoffel symbols can be derived from the vanishing of the covariant derivative of the metric tensor  $g_{ik}$ :

$$0 = \nabla_\lambda g_{\mu\nu} = \frac{\partial g_{\mu\nu}}{\partial x_\lambda} - g_{\eta\mu} \cdot \Gamma_{\mu\lambda}^\eta - g_{\mu\eta} \cdot \Gamma_{\nu\lambda}^\eta \quad (3)$$

By permuting the indices, and re-summing, one can solve explicitly for the Christoffel symbols as a function of the metric tensor:

$$\Gamma_{\nu\lambda}^\mu = \frac{1}{2} \cdot g^{\mu\nu} \cdot \left( \frac{\partial g_{\eta\nu}}{\partial x^\lambda} + \frac{\partial g_{\eta\lambda}}{\partial x^\nu} - \frac{\partial g_{\nu\lambda}}{\partial x^\eta} \right) \quad (4)$$

where the matrix  $(g^{\mu\nu})$  is an inverse of the matrix  $(g_{\mu\nu})$ , defined as (using the Kronecker delta, and Einstein notation for summation)

$$g^{\lambda\mu} \cdot g_{\mu\nu} = \delta_\nu^\lambda \quad (5)$$

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<sup>147</sup> [http://en.wikipedia.org/wiki/Levi-Civita\\_connection](http://en.wikipedia.org/wiki/Levi-Civita_connection)

Although the Christoffel symbols are written in the same notation as tensors with index notation, they are **not** tensors, since they do not transform like tensors under a change of coordinates.

Under a change of variable from  $(x^1, \dots, x^n)$  to  $(y^1, \dots, y^n)$ , vectors transform as

$$\frac{\partial}{\partial y^i} = \frac{\partial x^k}{\partial y^i} \cdot \frac{\partial}{\partial x^k} \quad (6)$$

and so

$$\Gamma_{ij}^k = \frac{\partial x^p}{\partial y^i} \cdot \frac{\partial x^q}{\partial y^j} \cdot \Gamma_{pq}^r \cdot \frac{\partial y^k}{\partial x^r} + \frac{\partial y^k}{\partial x^m} \cdot \frac{\partial^2 x^m}{\partial y^i \partial y^j} \quad (7)$$

where the underline denotes the Christoffel symbols in the  $y$  coordinate frame. Note that the Christoffel symbol does **not** transform as a tensor, but rather as an object in the jet bundle.

At each point, there exist coordinate systems in which the Christoffel symbols vanish at the point. These are called (geodesic) normal coordinates, and are often used in Riemannian geometry.

The Christoffel symbols are most typically defined in a coordinate basis, which is the convention followed here. However, the Christoffel symbols can also be defined in an arbitrary basis of tangent vectors  $e_\mu$  by

$$\nabla_{e_\mu} e_\nu = \Gamma_{\mu\nu}^\lambda \cdot e_\lambda \quad (8)$$

## The action along the live path

The integrated action  $S_{ab}$  is performed over a distance along the action trail or equivalently over a period of coordination time

$$S_{ab} = - \int_a^b m \cdot c^2 \cdot ds + \text{matter terms} \quad (1)$$

$$\begin{aligned}
&= - \int_{\tau_a}^{\tau_b} m \cdot c^2 \cdot \sqrt{1 - \left(\frac{v}{c}\right)^2} \cdot d\tau + \text{matter terms} \\
&= \int_{\tau_a}^{\tau_b} \mathcal{L} \cdot d\tau
\end{aligned}$$

$m$  is the mass of the considered item.

$v$  is the speed in Q space.

$\mathcal{L}$  is the Lagrangian.

The first line of this formula can be considered as an integral along the trail in coordinate space or equivalently over the trail in Hilbert space. The next lines concern integrals over the corresponding path in observed space combined with coordinate time. It must be noticed that these spaces have different signature.

$$\mathcal{L} = - m \cdot c^2 \cdot \frac{ds}{d\tau} + \text{matter terms} \quad (2)$$

In general relativity, the first term generalizes (includes) both the classical kinetic energy and interaction with the Newtonian gravitational potential. It becomes:

$$m \cdot c^2 \cdot \frac{ds}{d\tau} = -m \cdot c \cdot \sqrt{g_{\alpha\beta} \cdot \dot{q}_\alpha \cdot \dot{q}_\beta} \quad (3)$$

$g_{\alpha\beta}$  is the rank 2 symmetric metric tensor which is also the gravitational potential. Notice that a factor of  $c$  has been absorbed into the square root. The matter terms in the Lagrangian  $\mathcal{L}$  differ from those in the integrated action  $S_{ab}$ .

$$S_{ab\_matter} = - \int_a^b e \cdot A_\gamma \cdot dq^\gamma + \text{other matter terms} \quad (4)$$

The matter term in the Lagrangian due to the presence of an electromagnetic field is given by:

$$\mathcal{L} = -m \cdot c^2 \cdot \frac{ds}{d\tau} + e \cdot q^\gamma \cdot A_\gamma + \text{other matter terms} \quad (5)$$

$A_\gamma$  is the electromagnetic 4-vector potential.

## Optics

The optical Fourier transform (OTF) is an objective imaging quality characteristic for imaging devices in a similar way as the frequency transfer function qualifies the signal transfer function of a linearly operating electronic device. The transfer quality of a chain of linear signal transforming devices is characterized by the product of the frequency transfer functions of the elements of the chain. In a similar way the OTF of a chain of imaging devices is given by the product of the OTF's of the elements of the chain. However, this is a profound simplification of reality. The product rule only holds when the transfer characteristics of the imaging devices are spatially uniform over the complete input field of the separate imaging components. Further, the conditions in which the OTF's of the components are determined must be similar to the conditions in the chain. More in detail, this means that the angular distribution, the chromatic distribution and the homogeneity of the radiation must be identical.

In optics, the image sided spread function equals the convolution of the object sided spread function and the point spread function (PSF, the image of a point). The Fourier transform of the image sided spread function is equal to the product of the Fourier transform of the object sided spread function and the optical Fourier transforms (OTF's) of the imaging devices. When several imaging devices work in sequence, then the total optical transfer function of the imaging system equals the product of the transfer functions of the components.

If we restrict to a static situation and include the "depth" of the image, the static PSF is a three parametric function. Thus the OTF must have the same number of parameters. Like the PSF the three dimensional OTF has a longitudinal component and a two dimensional transverse component. In most cases only the transverse component is used as an imaging quality characteristic. On-axis the transverse component is rotationally symmetric. Off-axis its modulus, the MTF, is symmetric but not rotationally symmetric. On-axis the MTF is presented as a one parametric

curve in which only the positive axis is given. Off-axis the two extremes of the MTF are given. They correspond to radial and tangential directions.

Due to the fact that Hilbert fields are blurred Hilbert distributions, wave mechanics has much in common with wave optics. For each compact normal operator the Hilbert subspace that represents a physical item corresponds to a spread in Hilbert space and a corresponding spread in the eigenspaces of that normal operator. The distribution of this spread is represented in a [wave function](#), or more correctly, in a probability distribution. For example the wave function that has the position as a variable corresponds to the triple consisting of a physical item, its Hilbert subspace representation and the position operator.

After a move of a physical item its position related wave function has much in common with the spread function that characterizes the blur of the image sided pictures in a linear operating imaging system. The physical fields that influence the physical item have an equivalent in the chain of imaging devices that transfers the image.

The product formula for the transfer functions relies on several preconditions. First of all it relies on the fulfillment of the requirement for sufficient spatial uniformity of the transfer. At all places where information is passed, the transfer characteristics must be sufficiently identical. The product formula has only validity in the spatial area where this requirement is fulfilled.

The transfer characteristics will be different for each Fourier component. Their quality will reduce with higher spatial frequencies.

The final result can be computed in longitudinal direction by multiplication. In lateral direction these regions are restricted to areas where the transfer is locally sufficiently uniform. In the summation that is used to compute a sensible average the angular and chromatic distribution of the transferred information play a role. These distributions

determine the summation coefficients. The extent of the region in which the considered transfer function is considered valid depends on the accuracy that is required for the result of the computation. Sign selections inside the radiation determine the polarization. Often in optics this feature and its influence is ignored. Coherence plays a role as well, but in practice optical imaging uses either nearly completely incoherent light or nearly completely coherent light.

In wave mechanics the wave function, which is taken just before the item moves, gets the role of the object. After a movement through a region of the fields the wave function has been changed. Its Fourier transform then equals the product of the Fourier transform of the original wave function and the wave transfer functions (WTF's) of the fields that influence the item. If several steps are taken in sequence, then the transfer functions of the passed field pieces must be multiplied in order to get the overall result. This transfer is affected in a similar way by spatial non-uniformity as the optical case.

In cylindrical imaging systems Seidel aberrations take their toll. When the system is folded or when lenses are not perfectly in line, also non-cylindrical influences will influence the imaging quality. The measurement and the specification of the OTF must cope with the spatial non-uniformity of the imaging characteristics of the imaging devices and with the angular and chromatic distribution of the radiation. The OTF also depends on the longitudinal location of the object and where the image is detected. This also occurs with the WTF of physical fields. Both in optics and in wave mechanics the precise locations of the "object" and the "image" are often not well determined. They are defined by spatial distributions in three dimensions. In both cases the angular and chromatic distributions of the contributing radiation influence the transfer. The final result is constituted by the weighted sum of all contributions.

With inhomogeneous (= incoherent) imaging the phases are ignored. These facts indicate the difference between the particle view and the wave

view. From optics it is known that the modulation transfer function (MTF) is a proper imaging qualifier for inhomogeneous light imaging. In inhomogeneous imaging the imaging process can be properly described by ray tracing. Ray tracing has much similarities with the application of the path integral. However, ray tracing normally does not use arbitrary paths. In inhomogeneous imaging phases are scrambled. For holographic imaging the phase transfer function (PTF) or the whole OTF is the better measure. With holographic imaging the phases carry the depth information. Feynman's path integral can cover arbitrary paths because, according to Feynman's claim, interference via the phases eliminates the contributions of non-realistic paths. That is why in the path integral the angular distribution of the radiation plays no role.

In optics the image space is often a surface. In optics the OTF depends on the position in the object space. Off axis the OTF is not rotationally symmetric. The OTF also depends on the angular distribution and the chromatic distribution of the radiation. These dependencies also hold for the WTF in wave mechanics.

A longitudinal displacement of the image spread function with respect to the object spread function corresponds to an extra phase term in the longitudinal component of the Fourier transform of the image spread function. A lateral displacement corresponds to an extra phase term in the transverse component of the Fourier transform. In wave mechanics this holds for the respective components of the Fourier transform of the wave function after the move.

The resemblance between optics and wave mechanics becomes striking when the discrete lens pack is replaced by a medium with a continuously varying refraction. In optics this happens with electron optical lenses that are used in imaging with charged particles.

When the point spread function is a function of three-dimensional position, then the OTF is also a three-parametric function of spatial

frequency. The MTF is a symmetric function. However, the MTF is not rotationally symmetric (in 2D) or spherical symmetric (in 3D). On its vertical axis the MTF indicates the part of the energy of the radiation that is transferred by a given spatial frequency.

### Veiling glare and halo

Due to reflections on refracting surfaces some of the energy of the radiation loses much of its spatial information content. As a consequence the MTF shows a sharp peak near zero spatial frequency. This phenomenon is called **veiling glare**. When the drop is not so fast the phenomenon is called **halo**.

Equivalents of veiling glare and halo can also occur in wave mechanics. In this way spurious radiation and a spurious halo can enter space. This can happen in the form of energy or in the form of matter. Spurious radiation contains no spatial information.

## Strands

A strand is a notion introduced by [Christoph Schiller<sup>148</sup>](#). In this e-paper I try to embed the notion of a strand into the context of a sequence of Hilbert spaces or similarly into the context of a sequence of traditional quantum logics. The strand interpretation used in this e-paper is a mixture of the interpretation of Christoph Schiller and my personal interpretation. Any difference with the original interpretation is for my account. The reader must take Schiller's interpretation as the most original. See also [braid theory<sup>149</sup>](#).

*In my interpretation:*

Strands are chains of granules in the eigenspace of a strand operator. One of the granules represents the current state of the chain. That granule is the anchor location of a probability amplitude distribution. All granules in the chain obtain a position from a background coordinate system that is defined in the Gelfand triple. The anchor point coincides with a Hilbert vector, which is also an eigenvector of the strand operator.

## Schiller's strands

The fundamental principle of the strand model is:

**Planck units** are defined through crossing switches of strands.

An alternative fundamental postulate is:

An **event** is the switch of a crossing between two strand segments.

The original interpretation of strands can be found in [Motion Mountain, volume VI<sup>150</sup>](#). In Schiller's words:

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<sup>148</sup> <http://www.motionmountain.net/research.html>

<sup>149</sup> [http://en.wikipedia.org/wiki/Braid\\_symmetry](http://en.wikipedia.org/wiki/Braid_symmetry)

<sup>150</sup> <http://www.motionmountain.net/research.html>

**Strands** are one-dimensional curves in three-dimensional space that are closed or reach the border of space (or a horizon). Strands exist in relation to a 3D background coordinate system. Strands are fluctuating.

The one-dimensional strand curve fluctuates and as a consequence the fields in its surround exist. Fluctuation of a strand becomes apparent when the strand is averaged over a short time range. In this way the strands cause a blur. That blur represents the field that goes together with the strand.

Simple elementary types are bosons. All tangles made of one strand are elementary particles of spin 1, thus, they are elementary vector bosons. Conversely, all elementary spin-1 particles are made of one strand, because other tangles do not reproduce the spin-1 behavior under rotations: only one-stranded tangles return to the original strand after a core rotation by  $2\pi$ .

According to the strand model no Higgs boson are required.

The strand model predicts that apart from the six quarks and the graviton, no other two-stranded elementary particles exist in nature.

Leptons correspond with triples of tangled strands. The strand model predicts that apart from the six leptons, no other elementary particles made of three strands exist in nature.

More complex types are composed of the above mentioned elementary types.

Interaction is caused by one of three processes.

- The first process involves a single strand. It corresponds with normal electromagnetic interaction. It is characterized by the first Reidemeister move.
- The second process involves two strands. It corresponds with the electro-weak force. It is characterized by the second Reidemeister move.

- The third process involves three strands. It corresponds with the electro-strong force. It is characterized by the third Reidemeister move.

These interactions play in the direct environment of strand cores. According to Schiller, gravitational forces have their origin in the tails, relative far away of the cores. That is also the region where masses get their influence.

Strands and their fluctuations are *unobservable*. The only things that become observable from a strand are its crossing switches with itself or with other strands. Drawings of strands are made in order to clarify strand behavior. In that case strands are pictured in 3D space and the rotations are represented by rotating cores or knots.

The **tangle function** – the (short) time average of strand crossings – corresponds with a complex **probability amplitude distribution**.

The strand theory does not say anything about the transfer of information to quanta.

## Planck values

*Schiller*: Up to a numerical factor, the limit for every physical observable corresponds to the Planck value. (The limit values are deduced from the commonly used Planck unit values simply by substituting  $4G$  for  $G$ .) According to Schiller these limit values are the true natural units of nature. In fact, the ideal case would be to redefine the usual Planck values for all observables to these extreme values, by absorbing the numerical factor 4 into the respective definitions. In the strand model, Schiller calls the limit values the corrected Planck units and assumes that the factors have been properly included.

## Strand basics

A **crossing** between two strands has a position and a direction. It is the position where the distance between the strands has a minimum. The distance is measured

in terms of a selected background coordinate system. The distance is measured in corrected Planck length units.

A **crossing switch** is a turn of the crossing over  $\pi$  radians. Via its infinitesimal geometry the crossing switch defines the action  $\hbar/2$ , the corrected Planck length  $l_{pl}$ , the corrected Planck time  $t_{pl}$  and the Boltzmann constant  $k$ .

**Events** are observable crossing switches of unobservable strands. Every event in nature is characterized by the corrected Planck time, the corrected Planck length, the Planck entropy, i.e., the Boltzmann constant  $k$ , and Planck's quantum of action  $\hbar$  (for a full turn)

The **distance** between two particles is the maximum number of crossing switches that could appear between them. Length measurement is thus defined as counting corrected Planck lengths.

The **time interval** between two events is the maximum number of crossing switches that could appear between them. Time measurement is thus defined as counting corrected Planck times.

The physical **action** of a physical system evolving from an initial to a final state is the number of crossing switches that take place. Action measurement is thus defined as counting crossing switches. Physical action is thus a measure for the change that a system undergoes.

The **entropy** of any physical system is related to the total number of crossing switches that are possible. Entropy measurement is thus defined through the counting of potential crossing switches. The strand model thus states that any large physical system – be it made of matter, radiation, empty space or horizons – has entropy.

## Strand table

Typical strand configurations:

Physical system	Strands	Tangle type
<b>Vacuum</b>	many infinite unknotted strands	unlinked
<b>Dark energy</b>	many fluctuating infinite strands	unlinked
<b>Elementary vector boson</b>	one infinite strand	knotted or unknotted curve
<b>Quark</b>	two infinite strands	rational tangle

<b>Lepton</b>	three infinite strands	braided tangle
<b>Meson,baryon</b>	three or more infinite strands	rational tangle
<b>Higher-order propagating fermion</b>	two or more infinite strands	locally knotted or prime tangle
<b>Virtual particles</b>	open, unlinked and closed strands	trivial tangles, knots, links
<b>Composed systems</b>	many strands	separable tangles
<b>Graviton</b>	two infinite twisted strands	specific rational tangle
<b>Gravity wave</b>	many infinite twisted strands	many graviton tangles
<b>Horizon</b>	many tightly woven infinite strands	web-like tangle
<b>Earliest form of the universe</b>	Single closed strand	No tangles

See the [Toolkit](#) for more details on strands.

## Unique aspects of the model

### Fundament

- The model takes as basis the axioms of traditional quantum logic.
- It exploits the isomorphism between the set of propositions in this logic and the set of closed subspaces of an infinite dimensional separable Hilbert space  $\mathbf{H}$  in which the inner product is defined over the division ring of the quaternions.
- A proposition that treats everything that can be said about a physical item represents that item. Thus, the model represents physical items.
- Traditional quantum logic and its partner the separable Hilbert space  $\mathbf{H}$  cannot represent physical fields and they cannot represent dynamics.
- However, this basic model can be extended such that fields are attached to it. However, this extended model only represents a static status quo.
- A sequence of such extended models can represent dynamics.
- The separable Hilbert space  $\mathbf{H}$  does not contain a useable GPS coordinate operator. Due to the granularity of its eigenspace, such a normal operator would introduce preferred directions in the imaginary part of that eigenspace.
- Instead the corresponding continuous GPS operator that resides in the corresponding rigged Hilbert space  $\mathbf{H}$  can act as a background coordinate operator. Its eigenspace can be used to indicate the location of the field values. However, this operator cannot directly be used in order to locate the Hilbert vectors that represent particles.

- Instead a special normal operator whose eigenspace contains a set of freely located chains of granules can deliver the position observables. This operator is a strand operator.
- In each chain one granule represents the current position. It divides the chain in a past part and a future part.
- A probability amplitude distribution takes care of the smoothness in the surround of the current granule. This attachment extends the separable Hilbert space.
- Particles are represented by a single Hilbert vector or by a small set of Hilbert vectors. These vectors are eigenvectors of the strand operator they correspond with the current granule of a corresponding chain and are blurred by a spread function that can be interpreted as a probability amplitude distribution. The blur of the set of Hilbert vectors represents the private field of the particle and describes the cloud of quanta that carry the observable information about the particle. The quanta represent positions where the particle can be detected.
- The particle acts as the source or as the drain of these quanta. The cloud moves and rotates around a rotation axis.
- The superposition of all private fields constitutes a covering field.
- For a given coordinate system the static decomposition of the covering field into a rotation free part and one or two divergence free parts runs along curved lines. The local curvature value can be used to define a derived partner field of the covering field. This curvature field has all the characteristics of the gravitation field.
- The private fields of bosons are attached to a single unit size Hilbert vector and touch all other unit size Hilbert vectors.
- The private fields of quarks are attached to a pair of unit size Hilbert vectors and touch all other unit size Hilbert vectors.

- The private fields of leptons are attached to a triple of unit size Hilbert vectors and touch all other unit size Hilbert vectors.
- In interactions bosons take care of the transfer.
- Each electromagnetic interaction involves only one extra Hilbert vector.
- Each weak interaction involves an internal and an extra Hilbert vector.
- Each strong interaction involves two internal and one extra Hilbert vector.
- There are no more elementary kinds of interactions.
- The progression parameter that counts the subsequent Hilbert spaces is not our common notion of time, but it has certainly some relation with it.

## Insights

- The Minkowski signature of spacetime must have its explanation in what occurs during a progression step.
- The Minkowski signature of spacetime forbids that coordinate time acts as the fourth dimension that goes together with 3D coordinate space.
- Momentum acts as a precondition of the next displacement step.
- The fourth dimension must be as granular as the 3D displacement.
- Fields act as a precondition for the next action step.
- The displacement, measured in Planck length units, the progression step measured in Planck time units, the action step measured in Planck constant sized units and the entropy step in Boltzmann constant sized units form the basic steps during an observable event.
- Action represents change. Entropy represents potential change.
- A five-fold coverage of the separable Hilbert space exists

- A double coverage is done by the two extra members of the Gelfand triple that forms the corresponding rigged Hilbert space. This coverage delivers the background coordinate system.
- Another double coverage is done by the primary fields. Together they form the covering field. However, the static covering field can be decomposed into a rotation free part and a divergence free part.
- For a given coordinate system a curvature field can be derived from this covering field. This delivers the fifth cover.

Together these six elements deliver a sandwich that can characterize a static status quo of the universe.

## Discussion

### Macro and micro

Up to so far, the treatise confines to macroscopic dynamics. Micro dynamics concerns movements that occur inside the representation of small physical items. For each small physical item, it concerns the movements that occur inside the subsequent subspaces that represent this item.

In order to stay inside the item, the internal movements must be quasi periodical. They can be combinations of oscillations and rotations. The harmonics oscillator and the spherical harmonics are well known examples.

The local manipulator can be seen as a complicated (Fourier?) transform. The functions, which describe quantum harmonic movements, seem to be invariant under the action of this manipulator. Thus it appears that micro dynamics occurs via a different process than macro dynamics.

### Dynamic logic

The current trend in quantum logic development is to add axioms that change the static character of quantum logic in a more dynamic and operational logic. Logic of quantum actions ([LOA<sup>151</sup>](#)) adds unitary transforms as the source of dynamics. As we see in this article these transforms are not the real fundamental causes of dynamics. The fields that accompany the physical items form the more fundamental reason for the existence of dynamics. They control the redefinition of the actual propositions. To my knowledge the influences of physical fields are not yet covered by any dynamic logic theory.

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<sup>151</sup> [http://www.vub.ac.be/CLWF/SS/BethPaper\\_Final.pdf](http://www.vub.ac.be/CLWF/SS/BethPaper_Final.pdf)

## Conclusion

The fact that the set of propositions in traditional quantum logic is lattice isomorphic with the set of closed subspaces of an infinite dimensional separable Hilbert space  $\mathbf{H}$ , offers the possibility to study with mathematical means what happens with the propositions.

Quantum logic is only a partial description of the fundamentals of quantum physics. It only describes the static skeleton in which the quantum dynamics takes place. It does not treat physical fields. However, traditional quantum logic can be extended into a wider logic, such that fields are also included. When this is done, the task of the fields appears to bring coherence between past, current and future versions of extended quantum logics and dynamics can simply be considered as the simultaneous step from a future version, to a current version and from the current version to a past version.

An important ruler of quantum dynamics is the influence that is exposed by the universe of items in the phenomenon inertia. It indicates the laws that govern the exchange of atomic predicates from enveloping propositions. It characterizes the fields as the sticky resistance of the universe of quantum logical propositions against unordered redefinition of their members. This shapes the dynamics of the logic that describes dynamic quantum physics.

The fields consist of basic constituents that can be interpreted as probability amplitude distributions. The covering field is the superposition of these basic constituents and the gravitation field is a curvature field that can be derived from the static decomposition characteristics of the covering field. This picture carries on the assumption that the configuration of the covering field causes the curvature of the coordinate system.

Blurred Hilbert distributions form Hilbert fields. Blurred elementary Hilbert distributions form the private fields of corresponding elementary

particles. The blurs can be interpreted as probability distributions and as such they describe the sources of quantum noise. This means that the blurs also represent the probability of the generation, presence and annihilation of actual and virtual elementary particles.

The dynamics of the life path of an item can be described by a geodesic equation. The live environment can be considered as sets of  $2^n$ -ons that locally resemble quaternion spaces or in a still smaller region resemble complex number spaces. These numbers constitute the values of the fields that influence the dynamics of the items. The analysis of the local infinitesimal dynamic step also reveals the origin of special relativity.

In our model a universe wide progression stepper exists. This is the progression parameter clock. Due to this fact the redefinitions are universe wide synchronized. It also means that in the model universe is controlled by a single dynamic redefiner. However, its actions are locally influenced by fields, which are directly connected to the items that are present in this environment.

Inertia influences macroscopic dynamics. Microscopic movements are governed by a different process. They are directly controlled by the local manipulator and relate to its invariant functions.

Trying to implement a complex quantum logical proposition in Hilbert space is indeed an elucidating experience.

In the Hilbert book model, fields have several functions and interpretations:

- From the analysis of inertia you can derive that they represent the sticky resistance of the community of propositions/physical-items against unordered change. A uniform movement is still considered as a well ordered change. Acceleration is considered as unordered change and goes together with field activity.

- Fields are constituted of blurred sets of Hilbert vectors. With other words Hilbert fields are blurred Hilbert distributions.
- The blur renders the field differentiable.
- The blur can be interpreted as a probability amplitude distribution.
- Wave functions are probability amplitudes. No difference with private fields exists.
- Blurs can be squeezed and can be looked at in another coordinate representation, such as the canonical conjugated coordinates.
- Fields can be interpreted as the storage place of the conditions of future, present and past Hilbert spaces or equivalently as the storage place of the conditions of future, present and past versions of quantum logic systems.
- Like the Hilbert spaces and the quantum logics, the static fields describe a static status quo.
- Fields can be interpreted as the housing of annihilation and creation operators that act on actual or virtual particles.
- The probalistic nature of the fields invites their interpretation as clouds of quanta. These quanta represent potential realizations of Hilbert vectors that on their turn represent the anchor points of actual or virtual particles in past, present or future versions of traditional quantum logic propositions.
- In the view that uses the canonical conjugated coordinates the quantum cloud can be interpreted as a wave package.

## Appendix

### History of quantum logic

Around 1930 John von Neumann and Garrett Birkhoff were searching for an acceptable explanation of the results of experiments that showed that the execution of an observation of a very small object can completely destroy the validity of an earlier observation of another observable of that object. The Schrödinger equation that agreed with the dynamic behaviour of the particles already existed. Not much later Heisenberg's matrix formulation became popular as well. Quite soon the conclusion was made that something was fundamentally wrong with the logic behind the behaviour of small particles. These small objects show particle behaviour as well as wave behaviour and they show quantization effects. It was found that the distributive axiom of classical logic had to be changed. Soon it became apparent that the lattice structure of classical logic must be weakened from an orthocomplementary modular form to an orthocomplementary weakly modular lattice. The quantum logic was born. The next step was to find a useful mathematical presentation of this new logic. A historic review of what happened can be found in: "Quantum Theory: von Neumann" vs. Dirac; <http://www.illc.uva.nl/~seop/entries/qt-nvd/>. It includes extensions of the concept of Hilbert space and application of these concepts to quantum field theory. Another source is: [http://www.quantonics.com/Foulis\\_On\\_Quantum\\_Logic.html](http://www.quantonics.com/Foulis_On_Quantum_Logic.html).

### Quantum logic

Elementary particles behave non-classical. They can present themselves either as a particle or as a wave. A measurement of the particle properties of the object destroys the information that was obtained from an earlier measurement of the wave properties of that object.

With elementary particles it becomes clear that that nature obeys a different logic than our old trusted classical logic. The difference resides in the modularity axiom. That axiom is weakened. The classical logic is congruent to an orthocomplemented modular lattice. The quantum logic

is congruent to an orthocomplemented weakly modular lattice. Another name for that lattice is orthomodular lattice.

## Lattices

A subset of the axioms of the logic characterizes it as a half ordered set. A larger subset defines it as a lattice.

A lattice is a set of elements  $a, b, c, \dots$  that is closed for the connections  $\cap$  and  $\cup$ . These connections obey:

- The set is partially ordered. With each pair of elements  $a, b$  belongs an element  $c$ , such that  $a \subset c$  and  $b \subset c$ .
- The set is a  $\cap$  half lattice if with each pair of elements  $a, b$  an element  $c$  exists, such that  $c = a \cap b$ .
- The set is a  $\cup$  half lattice if with each pair of elements  $a, b$  an element  $c$  exists, such that  $c = a \cup b$ .
- The set is a lattice if it is both a  $\cap$  half lattice and a  $\cup$  half lattice.

The following relations hold in a lattice:

$$a \cap b = b \cap a \tag{1}$$

$$(a \cap b) \cap c = a \cap (b \cap c) \tag{2}$$

$$a \cap (a \cup b) = a \tag{3}$$

$$a \cup b = b \cup a \tag{4}$$

$$(a \cup b) \cup c = a \cup (b \cup c) \tag{5}$$

$$a \cup (a \cap b) = a \tag{6}$$

The lattice has a partial order inclusion  $\subset$ :

$$a \subset b \Leftrightarrow a \subset b = a \quad (7)$$

A complementary lattice contains two elements  $n$  and  $e$  with each element  $a$  an complementary element  $a'$  such that:

$$a \cap a' = n \quad (8)$$

$$a \cap n = n \quad (9)$$

$$a \cap e = a \quad (10)$$

$$a \cup a' = e \quad (11)$$

$$a \cup e = e \quad (12)$$

$$a \cup n = a \quad (13)$$

An orthocomplemented lattice contains two elements  $n$  and  $e$  and with each element  $a$  an element  $a''$  such that:

$$a \cup a'' = e \quad (14)$$

$$a \cap a'' = n$$

$$(a'')'' = a \quad (15)$$

$$a \subset b \Leftrightarrow b'' \subset a'' \quad (16)$$

$e$  is the unity element;  $n$  is the null element of the lattice

A distributive lattice supports the distributive laws:

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \quad (17)$$

$$a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \quad (18)$$

A modular lattice supports:

$$(a \cap b) \cup (a \cap c) = a \cap (b \cup (a \cap c)) \quad (19)$$

A weak modular lattice supports instead:

There exists an element  $d$  such that

$$a \subset c \Leftrightarrow (a \cup b) \cap c = a \cup (b \cap c) \cup (d \cap c) \quad (20)$$

where  $d$  obeys:

$$(a \cup b) \cap d = d \quad (21)$$

$$a \cap d = n \quad (22)$$

$$b \cap d = n \quad (23)$$

$$[(a \subset g) \text{ and } (b \subset g)] \Leftrightarrow d \subset g \quad (24)$$

In an atomic lattice holds

$$\exists p \in L \forall x \in L \{x \subset p \Rightarrow x = n\} \quad (25)$$

$$\forall a \in L \forall x \in L \{(a < x < a \cap p) \Rightarrow (x = a \text{ or } x = a \cap p)\} \quad (26)$$

$p$  is an atom

Both the set of propositions of quantum logic and the set of subspaces of a separable Hilbert space  $\mathbf{H}$  have the structure of an orthomodular lattice.

In this respect these sets are congruent.

In Hilbert space, an atom is a pure state (a ray spanned by a single vector).

Classical logic has the structure of an orthocomplemented distributive modular and atomic lattice.

Quantum logic has the structure of an orthomodular lattice. That is an orthocomplemented weakly modular and atomic lattice. The set of closed subspaces of a Hilbert space also has that structure.

### Proposition

In Aristotelian logic a proposition is a particular kind of sentence, one which affirms or denies a predicate of a subject. Propositions have binary values. They are either true or they are false.

Propositions take forms like "*This is a particle or a wave*". In quantum logic "*This is a particle.*" is not a proposition.

In mathematical logic, propositions, also called "propositional formulas" or "statement forms", are statements that do not contain quantifiers. They are composed of well-formed formulas consisting entirely of atomic formulas, the five [logical connectives](#)<sup>152</sup>, and symbols of grouping (parentheses etc.). Propositional logic is one of the few areas of mathematics that is totally solved, in the sense that it has been proven internally consistent, every theorem is true, and every true statement can be proved. Predicate logic is an extension of propositional logic, which adds variables and quantifiers.

In Hilbert space a vector is either inside or not inside a closed subspace. A proper quantum logical proposition is "*Vector  $|f\rangle$  is inside state  $s$* ".

In Hilbert space, an atomic predicate corresponds with a subspace that is spanned by a single vector.

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<sup>152</sup> [http://en.wikipedia.org/wiki/Logical\\_connective](http://en.wikipedia.org/wiki/Logical_connective)

Predicates may accept attributes and quantifiers. The predicate logic is also called first order logic. A dynamic logic can handle the fact that predicates may influence each other when atomic predicates are exchanged.

## Observation

In physics, particularly in quantum physics, a system **observable** is a property of the system state that can be determined by some sequence of physical operations. This paper distinguishes between measurements and observations.

- With an observation the state is considered as a linear combination of eigenvectors of the observable. An observation returns the statistical expectation value of the eigenvalue of the observable.
- A measurement transforms the observed state to one of the eigenvectors of the observable. What happens depends on the characteristics of the measuring equipment. The measurement can be seen as a combination of a transformation and an observation.

Depending on the characteristics of the measuring equipment a measurement and a clean observation can give the same result.

With this interpretation of the concept of observation it is possible to let states observe other states. A state might do a transformation before doing an observation but in general it fails the equipment to arrange that transformation. In nature observations are far more common than measurements.

## Quaternion coordinates

This part of the appendix describes candidates for the coordinates on the coordinate sphere.

### Polar coordinates

The equivalent to rectangular coordinates in quaternion space is  $(a_\tau, a_x, a_y, a_z)$

$$a = a_\tau + \mathbf{i} \cdot a_x + \mathbf{j} \cdot a_y + \mathbf{k} \cdot a_z \quad (1)$$

The equivalent to polar coordinates in quaternion space is

$$a_\tau = \|a\| \cos(\psi) \quad (2)$$

$$a_x = \|a\| \sin(\psi) \sin(\theta) \cos(\varphi) \quad (3)$$

$$a_y = \|a\| \sin(\psi) \sin(\theta) \sin(\varphi) \quad (4)$$

$$a_z = \|a\| \sin(\psi) \cos(\theta) \quad (5)$$

$\sin(\psi)$ , where  $\psi = (0, \pi)$ , is known as the (imaginary) amplitude of the quaternion. Angle  $\theta = (0, \pi)$  is the (co-)latitude and angle  $\varphi = (0, 2\pi)$  is the longitude.

For any fixed value of  $\psi, \theta$  and  $\varphi$  parameterize a 2-sphere of radius  $\sin(\psi)$ , except for the degenerate cases, when  $\psi$  equals 0 or  $\pi$ , in which case they describe a point.

This suggests the following structure of the argument  $\underline{a}$

$$a = \|a\| \exp(\mathbf{i} \cdot \psi) \quad (6)$$

$$= \|a\| (\cos(\psi) + \mathbf{i} \sin(\psi)) \quad (7)$$

$$= a_\tau + \|a\| \mathbf{i} \sin(\psi) = a_\tau + \mathbf{a} \quad (8)$$

The imaginary number  $\mathbf{i}$  may take any direction.

### 3 sphere

A 3-sphere is a compact, connected, 3-dimensional manifold without boundary. It is also simply-connected. What this means, loosely speaking, is that any loop, or circular path, on the 3-sphere can be continuously

shrunk to a point without leaving the 3-sphere. The [Poincaré conjecture](#)<sup>153</sup> proposes that the 3-sphere is the only three dimensional manifold with these properties (up to homeomorphism)<sup>154</sup>.

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\psi^2 + \sin^2(\psi) (d\theta^2 + \sin^2(\theta)d\varphi^2) \quad (1)$$

The volume form is given by

$$dV = \sin^2(\psi) \sin(\theta) d\psi \wedge d\theta \wedge d\varphi \quad (2)$$

The 3-dimensional volume (or **hyperarea**) of a 3-sphere of radius  $r$  is

$$2 \pi^2 r^3 \quad (3)$$

The 4-dimensional **hypervolume** (the volume of the 4-dimensional region bounded by the 3-sphere) is

$$\frac{1}{2} \pi^2 r^4 \quad (4)$$

The 3-sphere has constant positive sectional curvature equal to  $1/r^2$ .

The 3-sphere has a natural Lie group structure  $SU(2)$  given by quaternion multiplication.

The 3-sphere admits non-vanishing vector fields (sections of its tangent bundle). One can even find three linearly-independent and non-vanishing vector fields. These may be taken to be any left-invariant vector fields forming a basis for the Lie algebra of the 3-sphere. This implies that the 3-

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<sup>153</sup> [http://en.wikipedia.org/wiki/Poincar%C3%A9\\_conjecture](http://en.wikipedia.org/wiki/Poincar%C3%A9_conjecture)

<sup>154</sup> <http://en.wikipedia.org/wiki/3-sphere>

sphere is parallelizable. It follows that the tangent bundle of the 3-sphere is trivial.

There is an interesting action of the circle group  $\mathbb{T}$  on  $\mathbb{S}^3$  giving the 3-sphere the structure of a principal circle bundle known as the Hopf bundle. If one thinks of  $\mathbb{S}^3$  as a subset of  $\mathcal{C}^2$ , the action is given by

$$(z_1, z_2) \lambda = (z_1 \lambda, z_2 \lambda) \forall \lambda \in \mathbb{T}. \quad (5)$$

The orbit space of this action is homeomorphic to the two-sphere  $\mathbb{S}^2$ . Since  $\mathbb{S}^3$  is not homeomorphic to  $\mathbb{S}^2 \times \mathbb{S}^1$ , the Hopf bundle is nontrivial.

### Hopf coordinates

Another choice of hyperspherical coordinates,  $(\eta, \xi_1, \xi_2)$ , makes use of the embedding of  $\mathbb{S}^3$  in  $\mathcal{C}^2$ . In complex coordinates  $(z_1, z_2) \in \mathcal{C}^2$  we write

$$z_1 = \exp(i \xi_1) \sin(\eta) \quad (1)$$

$$z_2 = \exp(i \xi_2) \cos(\eta) \quad (2)$$

Here  $\eta$  runs over the range 0 to  $\pi/2$ , and  $\xi_1$  and  $\xi_2$  can take any values between 0 and  $2\pi$ . These coordinates are useful in the description of the 3-sphere as the Hopf bundle

$$\mathbb{S}^1 \rightarrow \mathbb{S}^3 \rightarrow \mathbb{S}^2 \quad (3)$$

For any fixed value of  $\eta$  between 0 and  $\pi/2$ , the coordinates  $(\xi_1, \xi_2)$  parameterize a 2-dimensional torus. In the degenerate cases, when  $\eta$  equals 0 or  $\pi/2$ , these coordinates describe a circle.

The round metric on the 3-sphere in these coordinates is given by

$$ds^2 = d\eta^2 + \sin^2(\eta) (d\xi_1^2 + \cos^2(\eta) d\xi_2^2) \quad (4)$$

and the volume form by

$$dV = \sin(\eta) \cos(\eta) d\eta \wedge d\zeta_1 \wedge d\zeta_2 \quad (5)$$

### Group structure

Because the set of unit quaternions is closed under multiplication,  $\mathbb{S}^3$  takes on the structure of a group. Moreover, since quaternionic multiplication is smooth,  $\mathbb{S}^3$  can be regarded as a real Lie group. It is a non-abelian, compact Lie group of dimension 3. When thought of as a Lie group  $\mathbb{S}^3$  is often denoted  $Sp(1)$  or  $U(1, \mathbb{H})$ .

It turns out that the only spheres which admit a Lie group structure are  $\mathbb{S}^1$ , thought of as the set of unit complex numbers, and  $\mathbb{S}^3$ , the set of unit quaternions. One might think that  $\mathbb{S}^7$ , the set of unit octonions, would form a Lie group, but this fails since octonion multiplication is non-associative. The octonionic structure does give  $\mathbb{S}^7$  one important property: [\*parallelizability\*<sup>155</sup>](#). It turns out that the only spheres which are parallelizable are  $\mathbb{S}^1$ ,  $\mathbb{S}^3$ , and  $\mathbb{S}^7$ .

By using a matrix representation of the quaternions,  $\mathbb{H}$ , one obtains a matrix representation of  $\mathbb{S}^3$ . One convenient choice is given by the Pauli matrices:

$$(a_\tau + a_x \cdot \mathbf{i} + a_y \cdot \mathbf{j} + a_z \cdot \mathbf{k}) = \begin{bmatrix} a_\tau + \tilde{\mathbf{i}} \cdot a_x & a_y + \tilde{\mathbf{i}} \cdot a_z \\ -a_y + \tilde{\mathbf{i}} \cdot a_z & a_\tau - \tilde{\mathbf{i}} \cdot a_x \end{bmatrix} \quad (1)$$

This map gives an injective algebra homomorphism from  $\mathbb{H}$  to the set of  $2 \times 2$  complex matrices. It has the property that the absolute value of a quaternion  $q$  is equal to the square root of the determinant of the matrix image of  $q$ .

The set of unit quaternions is then given by matrices of the above form with unit determinant. This matrix subgroup is precisely the special unitary group  $SU(2)$ . Thus,  $\mathbb{S}^3$  as a Lie group is isomorphic to  $SU(2)$ .

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<sup>155</sup> <http://en.wikipedia.org/wiki/Parallelizability>

Using our hyperspherical coordinates  $(\eta, \xi_1, \xi_2)$  we can then write any element of  $SU(2)$  in the form

$$\begin{bmatrix} \exp(\tilde{\mathbf{i}} \cdot \xi_1) \cdot \sin(\eta) & \exp(\tilde{\mathbf{i}} \cdot \xi_2) \cdot \cos(\eta) \\ -\exp(\tilde{\mathbf{i}} \cdot \xi_2) \cdot \cos(\eta) & \exp(-\tilde{\mathbf{i}} \cdot \xi_1) \cdot \sin(\eta) \end{bmatrix} \quad (2)$$

Another way to state this result is if we express the matrix representation of an element of  $SU(2)$  as a linear combination of the Pauli matrices. It is seen that an arbitrary element  $U \in SU(2)$  can be written as

$$U = \alpha_\tau \cdot 1 + \sum_{n=x,y,z} \alpha_n I_n \quad (3)$$

The condition that the determinant of  $U$  is +1 implies that the coefficients  $\alpha_n$  are constrained to lie on a 3-sphere.

### Versor

Any **unit quaternion**  $q$  can be written as a **versor**:

$$u = \exp(\tilde{\mathbf{i}} \psi) = \cos(\psi) + \tilde{\mathbf{i}} \sin(\psi) \quad (1)$$

This is the quaternionic analogue of Euler's formula. Now the unit imaginary quaternions all lie on the unit 2-sphere in  $\text{Im } \mathbb{H}$  so any such  $\tilde{\mathbf{i}}$  can be written:

$$\tilde{\mathbf{i}} = \mathbf{i} \cos(\varphi) \sin(\theta) + \mathbf{j} \sin(\varphi) \sin(\theta) + \mathbf{k} \cos(\theta) \quad (2)$$

### Symplectic decomposition

Quaternions can be written as the combination of two complex numbers and an imaginary number  $k$  with unit length.

### $2^n$ -on construction

The  $2^n$ -ons use the following doubling formula

$$(a, b)(c, d) = (ac - (bd^*)^*, (b^*c^*)^* + (b^*(a^*((b^{-1})^*d^*)^*)^*))^*) \quad (1)$$

Up until the 16-ons the formula can be simplified to

$$(a, b)(c, d) = (ac - bd^*, cb + (a^* b^{-1})(bd)) \quad (2)$$

Up to the octonions the Cayley Dickson construction delivers the same as the  $2^n$ -on construction. From  $n > 3$  the  $2^n$ -ons are 'nicer'.

### 2<sup>n</sup>-ons

Table of properties of the 2<sup>n</sup>-ons. See

[www.math.temple.edu/~wds/homepage/nce2.ps](http://www.math.temple.edu/~wds/homepage/nce2.ps).

Type	name	Lose
1-ons	<i>Reals.</i>	
2-ons	<i>Complex numbers</i>	$z^* = \bar{z}$ (the * denotes conjugating); the ordering properties that both $\{z > 0, -z > 0, \text{ or } z = 0\}$ and $\{w > 0, z > 0 \text{ implies } w + z > 0, wz > 0\}$ .
4-ons	<i>Quaternions</i>	commutativity $ab = ba$ ; the algebraic closedness property that every univariate polynomial equation has a root.
8-ons	<i>Octonions</i>	associativity $ab \cdot c = a \cdot bc$ .
16-ons	( <i>not Sedenions!</i> )	right-alternativity $x \cdot yy = xy \cdot y$ ; right-cancellation $x = xy \cdot y^{-1}$ ; flexibility $x \cdot yx = xy \cdot x$ ; left-linearity $(b + c)a = ba + ca$ ; anti-automorphism $ab = ba, (ab)^{-1} = b^{-1} a^{-1}$ ; left-linearity $(b + c)a = ba + ca$ ; continuity of the map $x \rightarrow xy$ ; Moufang and Bol identities; diassociativity
32-ons		generalized-smoothness of the map $x \rightarrow xy$ ; right-division properties that $xa = b$ has (generically) a solution $x$ , and the uniqueness of such an $x$ ; the "fundamental theorem of algebra" that every polynomial having a unique "asymptotically dominant monomial" must have a root; Trotter's formula: $\lim_{n \rightarrow \infty} [e^{x/n} e^{y/n}]^n = \lim_{n \rightarrow \infty} \left(1 + \frac{x+y}{n}\right)^n = e^{x+y}$

Type	name	Retain
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2 <sup>n</sup> -ons	<p>Unique 2-sided multiplicative &amp; additive identity elements 1 &amp; 0;</p> <p>Norm-multiplicativity <math> xy ^2 =  x ^2 \cdot  y ^2</math> ;</p> <p>Norm-subadditivity <math> a + b  \leq  a  +  b </math>;</p> <p>2-sided inverse <math>a^{-1} = a^*/ a ^2</math> (<math>a \neq 0</math>);</p> <p><math>a^{**} = a</math>;</p> <p><math>(x \pm y)^* = x^* \pm y^*</math> ;</p> <p><math>(a^{-1})^{-1} = a</math>;</p> <p><math>(a^*)^{-1} = (a^{-1})^*</math> ;</p> <p><math> a ^2 =  a ^2 = a^* a</math>;</p> <p>Left-alternativity <math>yy \cdot x = y \cdot yx</math>;</p> <p>Left-cancellation <math>x = y^{-1} \cdot yx</math>;</p> <p>Right-linearity <math>a(b + c) = ab + ac</math>;</p> <p>r<sup>th</sup> power-associativity <math>a^n a^m = a^{n+m}</math> ;</p> <p>Scaling <math>s \cdot ab = sa \cdot b = as \cdot b = a \cdot sb = a \cdot bs = ab \cdot s</math> (<math>s</math> real);</p> <p>Power-distributivity <math>(ra^n + sa^m)b = ra^n b + sa^m b</math> (<math>r, s</math> real);</p> <p>Vector product properties of the imaginary part: <b>ab</b> - re(<b>ab</b>) of the product for pure-imaginary 2<sup>n</sup>-ons <b>a, b</b> regarded as (2<sup>n</sup> - 1)-vectors;</p> <p><math>\langle xa, b \rangle = \langle a, x^*b \rangle</math>, <math>\langle xa, xb \rangle =  x ^2 \cdot \langle a, b \rangle</math> and <math>\langle x, y \rangle = \langle x^*, y^* \rangle</math></p> <p>Numerous weakened associativity, commutativity, distributivity, antiautomorphism, and Moufang and Bol properties including 9-coordinate "niner" versions of most of those properties; contains 2<sup>n-1</sup>-ons as subalgebra.</p>
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### The most important properties of 2<sup>n</sup>-ons

If  $a, b, x, y$  are 2<sup>n</sup>-ons,  $n \geq 0$ , and  $s$  and  $t$  are scalars (i.e. all coordinates are 0 except the real coordinate) then

**unit:** A unique 2<sup>n</sup>-on 1 exists, with  $1 \cdot x = x \cdot 1 = x$ .

**zero:** A unique 2<sup>n</sup>-on 0 exists, with  $0 + x = x + 0 = x$  and  $0 \cdot x = x \cdot 0 = 0$ .

**additive properties:**  $x+y = y+x$ ,  $(x+y)+z = x+(y+z)$ ;

$-x$  exists with  $x + (-x) = x - x = 0$ .

**norm:**  $|x|^2 = xx^* = x^*x$ .

**norm-multiplicativity:**  $|x|^2 \cdot |y|^2 = |x \cdot y|^2$ .

**scaling:**  $s \cdot x \cdot y = s \cdot x \cdot y = x \cdot s \cdot y = x \cdot s \cdot y = x \cdot y \cdot s$ .

**weak-linearity:**  $(x + s) \cdot y = x \cdot y + s \cdot y$  and  $x \cdot (y + s) = x \cdot y + x \cdot s$ .

**right-linearity:**  $x \cdot (y + z) = x \cdot y + x \cdot z$ .

**inversion:** If  $x \neq 0$  then a unique  $x^{-1}$  exists, obeying  $x^{-1} \cdot x = x \cdot x^{-1} = 1$ . It is  $x^{-1} = x \cdot |x|^{-2}$ .

**left-alternativity:**  $x \cdot xy = x^2 \cdot y$ .

**left-cancellation:**  $x \cdot x^{-1} \cdot y = y$ .

**effect on inner products:**  $\langle x \cdot a, b \rangle = \langle a, x^* \cdot b \rangle$ ,  $\langle x, y \rangle = \langle x^*, y^* \rangle$ ,  $\langle x^* \cdot a, x^{-1} \cdot b \rangle = \langle a, b \rangle$ ,

and  $\langle x \cdot a, x \cdot b \rangle = |x|^2 \cdot \langle a, b \rangle$ .

**Conjugate of inverse:**  $(x^{-1})^* = (x^*)^{-1}$ .

**Near-anticommutativity of unequal basis elements:**  $e_k^2 = -1$  and  $e_k \cdot e_l^* = -e_l \cdot e_k^*$  if  $k \neq l$ .

(Note: the case  $k; l > 0$  shows that unequal pure-imaginary basis elements anticommute.)

**Alternative basis elements:**  $e_k \cdot e_l \cdot e_k = e_k \cdot e_l \cdot e_k$ ,  $e_l \cdot e_k \cdot e_k = e_l \cdot e_k \cdot e_k$ , and  $e_k \cdot e_k \cdot e_l = e_k \cdot e_k \cdot e_l$ . (However, when  $n \geq 4$  the  $2^n$ -ons are not flexible i.e. it is not generally true that  $x \cdot y \cdot x = x \cdot y \cdot x$  if  $x$  and  $y$  are  $16$ -ons that are not basis elements. They also are not right-alternative.)

**Quadratic identity:** If  $x$  is a  $2^n$ -on (over any field  $F$  with  $\text{char} F \neq 2$ ), then  $x^2 + |x|^2 = 2 \cdot x \text{ re } x$

**Squares of imaginaries:** If  $x$  is a  $2^n$ -on with  $\text{re } x = 0$  ("pure imaginary") then  $x^2 = -|x|^2$  is nonpositive pure-real.

**Powering preserves  $\text{im} x$  direction**

## Niners

Niners are  $2^n$ -ons whose coordinates with index  $> 8$  are zero. The index starts with 0.

**9-flexibility**  $x^p \cdot x = x \cdot p x$ ,  $p x \cdot p = p \cdot x p$ .

**9-similitude unambiguity**  $x^p \cdot x^{-1} = x \cdot p x^{-1}$ ,  $p x \cdot p^{-1} = p \cdot x p^{-1}$ .

**9-right-alternativity**  $x^p \cdot p = x \cdot p^2$ ,  $p x \cdot x = p \cdot x^2$ .

**9-right-cancellation**  $x^{p^{-1}} \cdot p = x$ ,  $p x^{-1} \cdot x = p$ .

**9-effect on inner products**  $\langle x, y^p \rangle = \langle x p, y \rangle$ ,  $\langle x p, y^p \rangle = |p|^2 \langle x, y \rangle$ .

**9-left-linearity**  $(x + y)^p = x^p + y^p$ ,  $(p + q)x = p x + q x$ .

**9-Jordan-identity**  $x^p \cdot x x = x(p \cdot x x)$ ,  $p y \cdot p p = p(y \cdot p p)$ .

**9-coordinate-distributivity**  $([x + y]z)_{0;\dots;8} = (x z + y z)_{0;\dots;8}$ .

**9-coordinate-Jordan-identity**  $[xy \cdot xx]_{0;\dots;8} = [x(y \cdot xx)]_{0;\dots;8}$ .

**9-anticommutativity for orthogonal imaginary  $2^n$ -ons**

If  $\langle p, x \rangle = \text{re } p = \text{re } x = 0$  then  $px = -xp$ .

**9-reflection** If  $|a| = 1$  and the geometric reflection operator is defined below then  $-\{\text{refl}[a](y)\}_{0;\dots;8} = (a \cdot y^* a)_{0;\dots;8}$ , and  $-\{\text{refl}[a](y)\}^*_{0;\dots;8} = (a^* y \cdot a^*)_{0;\dots;8}$ , and

if either  $a$  or  $y$  is a niner then  $-\text{refl}[a](y) = a \cdot y^* a$  and  $-\text{refl}[a](y) = a^* y \cdot a^*$ .

$$\text{refl}[\vec{x}](\vec{t}) \stackrel{\text{def}}{=} \vec{t} - \frac{2\langle \vec{x}, \vec{t} \rangle}{|\vec{x}|^2} \vec{x} \quad (3)$$

What holds for the niners, also holds for the octonions.

## Regular quaternionic functions

See: <http://www.zipcon.net/~swhite/docs/math/quaternions/analysis.html>

and <http://world.std.com/~sweetser/quaternions/ps/Quaternionic-analysis-memo.pdf>).

The differential  $df$  is given by

$$df = \frac{\partial f}{\partial q_0} dq_0 + \frac{\partial f}{\partial q_i} dq_i + \frac{\partial f}{\partial q_j} dq_j + \frac{\partial f}{\partial q_k} dq_k \quad (1)$$

A regular function  $f$  obeys:

$$\frac{\partial f}{\partial q_0} + \mathbf{i} \frac{\partial f}{\partial q_i} + \mathbf{j} \frac{\partial f}{\partial q_j} + \mathbf{k} \frac{\partial f}{\partial q_k} = 0 \quad (2)$$

In addition the regular function  $f$  obeys:

$$\int_C D_q f = 0 \quad (3)$$

where  $C$  is any smooth closed 3-manifold in  $\mathbb{H}$ .  $D_q$  is the quaternion representing an element  $\delta C$  of the 3-manifold, its magnitude being equal to the volume of  $\delta C$  and its direction being normal to  $\delta C$ .

$$f(p) = \frac{1}{2\pi^2} \int_{\mathfrak{D}} \left\{ \frac{(q-p)^{-1}}{|q-p|^2} D_q f(q) \right\} \quad (4)$$

where  $\mathfrak{D}$  is a domain in  $\mathbb{H}$  in which  $f$  is regular and  $p$  is a point inside  $\mathfrak{D}$ .

$$D_q = dq_i \wedge dq_j \wedge dq_k - \mathbf{i} dq_0 \wedge dq_j \wedge dq_k - \mathbf{j} dq_0 \wedge dq_k \wedge dq_i - \mathbf{k} dq_0 \wedge dq_i \wedge dq_j \quad (5)$$

$$d(dq \wedge dq f) = Dq f'(q) \quad (6)$$

$$dq \wedge dq = \mathbf{i} dq_j \wedge dq_k + \mathbf{j} dq_k \wedge dq_i + \mathbf{k} dq_i \wedge dq_j \quad (7)$$

Here  $a \wedge b$  is the external vector product between vectors  $a$  and  $b$ . It is not the quaternionic external product.

$$d(Dq f) = 0 \quad (8)$$

$$d \left[ \frac{(q-p)^{-1}}{|q-p|^2} \right] Dq f(q) = \Delta \left( \frac{1}{|q-p|^2} \right) f(q) dq_0 \wedge dq_i \wedge dq_j \wedge dq_k \quad (9)$$

where  $\Delta$  is the Laplacian on  $\mathbb{R}^4$  and  $dq_0 \wedge dq_i \wedge dq_j \wedge dq_k$  is the standard volume 4-form. Since  $\frac{1}{|q-p|^2}$  is the Green's function for the Laplacian in  $\mathbb{R}^4$ , (4) follows from (9).

$$\Delta f = \frac{\partial f^2}{\partial q_0^2} + \frac{\partial f^2}{\partial q_i^2} + \frac{\partial f^2}{\partial q_j^2} + \frac{\partial f^2}{\partial q_k^2} \quad (10)$$

If  $f$  is regular in an open set  $U$ , then it has a power series expansion about each point of  $U$ . Thus, point-wise differentiability, together with the four real conditions (2) on the sixteen partial derivatives of  $f$ , is sufficient to ensure analyticity.

The set of homogeneous regular functions of degree  $n$  forms a quaternionic vector space of dimension  $(n+1)(n+2)/2$ ;

This is true for any integer  $n$  if for negative  $n$  it is understood that the functions are defined and regular everywhere except at 0. The functions with negative degree of homogeneity correspond to negative powers of a complex variable, and occur in the quaternionic Laurent series which exists for any regular function which is regular in an open set except at one point.

On the unit sphere in  $\mathbb{H}$  the homogeneous regular functions form a group isomorphic to  $SU(2)$ . The harmonic analysis of these functions bears the same relation to quaternionic analysis as the theory of complex Fourier series does to complex analysis.

Because the quaternions are four-dimensional, there is no counterpart to the geometrical description of complex analytic functions as conformal mappings. The zeros of a quaternionic regular function are not necessarily isolated, and its range is not necessarily open; neither of these sets needs even be a sub-manifold of  $\mathbb{H}$ .

**Definition:** A function  $f : \mathbb{H} \rightarrow \mathbb{H}$  is quaternion-differentiable on the left at  $q$  if the limit

$$\frac{df}{dq} = \lim_{h \rightarrow 0} \frac{f(q + h) - f(q)}{h} \quad (11)$$

exists.

**Theorem:** Suppose the function  $f$  is defined and quaternion-differentiable on the left throughout a connected open set  $U$ . Then on  $U$ ,  $f$  has the form:

$$f(q) = a + q b \quad (12)$$

for some  $a, b \in \mathbb{H}$ .

Even if  $f$  is quaternion-differentiable, it will not in general satisfy Cauchy's theorem in the form

$$\int d_q f = 0 \tag{13}$$

where the integral is round a closed curve; in fact the only functions satisfying this equation for all closed curves are the *constant* functions.

**Definition 2:** A function  $f : \mathbb{H} \rightarrow \mathbb{H}$  is left-regular at  $q \in \mathbb{H}$  if it is **real-differentiable** at  $q$  and there exists a quaternion  $f'_l(q)$  such that

$$d(dq \wedge dq f) = -2D_q f'_l(q) \tag{14}$$

It is right-regular if there exists a quaternion  $f'_r(q)$  such that

$$d(f dq \wedge dq) = -2f'_r(q)D_q:$$

Clearly, the theory of left-regular functions will be entirely equivalent to the theory of right-regular functions. For definiteness, we will only consider left-regular functions, which we will call simply **regular**. We will write

$$f'(q) = f'_l(q) \tag{15}$$

and call it the **derivative** of  $f$  at  $q$ .

**Theorem2:** (about the Cauchy-Riemann-Fueter equations)  
A real-differentiable function  $f$  is regular at  $q$  if and only if

$$\frac{\partial f}{\partial q_0} + \mathbf{i} \frac{\partial f}{\partial q_i} + \mathbf{j} \frac{\partial f}{\partial q_j} + \mathbf{k} \frac{\partial f}{\partial q_k} = 0 \tag{16}$$

This is formula (2)

**Theorem 3:** A differentiable function  $f$  is regular at  $q$  if and only if

$$D_q \wedge df = 0 \tag{17}$$

**Theorem 4:** If  $f$  is regular at every point of the 4-parallelepiped  $C$ ,

$$\int_c D_q f = 0 \tag{18}$$

This is equivalent to formula (3).

$$G(q) = \frac{q^{-1}}{|q|^2} \tag{19}$$

Note that

$$G(q) = -\partial_t \frac{1}{|q|^2} = -\partial_r \frac{1}{|q|^2} \tag{20}$$

It follows that  $\bar{\partial}_t G = 0$ , i.e.  $G$  is regular except at 0.

**Theorem 5:** A function which is regular in an open set  $U$  is real-analytic in  $U$

This follows from (4).

**Theorem 6:** (Cauchy's theorem for a differentiable contour)

Suppose  $f$  is regular in an open set  $U$ , and let  $C$  be a differentiable 3-chain in  $U$ , which is homologous to 0 in the differentiable singular homology of  $U$ , i.e.  $C = \partial C'$  for some differentiable 4-chain  $C'$  in  $U$ . Then

$$\int_c D_q f = 0 \tag{21}$$

In order to state the general form of the integral formula, we need an analogue of the notion of the winding number of a curve round a point in the plane.

**Definition 3:** Let  $q$  be any quaternion, and let  $C$  be a closed 3-chain in  $\mathbb{H} \setminus \{q\}$ . Then  $C$  is homologous to a 3-chain  $C' : \partial I^4 \rightarrow S$ , where  $S$  is the unit sphere with center  $q$ .

The wrapping number of  $C$  about  $q$  is the degree of the map  $C'$ .

**Theorem 7:** (The integral formula for a differentiable contour)

Suppose  $f$  is regular in an open set  $U$ . Let  $p \in U$ , and let  $C$  be a differentiable 3-chain

in  $U \setminus \{p\}$  which is homologous, in the differentiable singular homology of  $U \setminus \{p\}$ , to

a 3-chain whose image is  $\partial B$  for some ball  $B \subset U$ . Then

$$\frac{1}{2\pi^2} \int_C \left\{ \frac{(q-p)^{-1}}{|q-p|^2} D_q f(q) \right\} = n f(p) \quad (22)$$

where  $n$  is the wrapping number of  $C$  about  $p$ .

Formulas (21) and (22) also hold for a rectifiable 3-chain  $C$ .

Since regular functions are harmonic, they satisfy a maximum-modulus principle and a Liouville theorem. As with functions of a complex variable, Liouville's theorem follows immediately from the Cauchy-Fueter integral formula.

**Theorem 8:** (Morera's theorem) Suppose that the function  $f$  is continuous in an open set  $U$  and that

$$\int_c D_q f = 0 \quad (21)$$

for every 4-parallelepiped  $C$  contained in  $U$ . Then  $f$  is regular in  $U$ .

**Theorem 9:** Let  $u$  be a real-valued function defined on a star-shaped open set  $U \in \mathbb{H}$ .

If  $u$  is harmonic and has continuous second derivatives, there is a regular function  $f$  defined on  $U$  such that  $Re f = u$ .

This shows that there are as many regular functions of a quaternion variable as there are harmonic functions of four real variables. However,

these functions do not include the simple algebraic functions, such as powers of the variable, which occur as analytic functions of a complex variable.

## The separable Hilbert space $\mathcal{H}$

### Notations and naming conventions

$\{f_x\}_x$  means ordered set of  $f_x$ . It is a way to define functions.

*The use of bras and kets differs slightly from the way Dirac uses them.*

$|f\rangle$  is a ket vector,  $f\rangle$  is the same ket

$\langle f|$  is a bra vector,  $\langle f$  is the same bra

$A$  is an operator.  $|A$  is the same operator

$A^\dagger$  is the adjoint operator of operator  $A$ .  $A|$  is the same operator

$|$  on its own, is a nil operator

$|A|$  is a self-adjoint (Hermitian) operator

We will use capitals for operators and lower case for quaternions, eigenvalues, ket vectors, bra vectors and eigenvectors. Quaternions and eigenvalues will be indicated with *italic* characters. Imaginary and anti-Hermitian objects are often underlined and/or indicated in **bold** text.

$\sum_k$  means: sum over all items with index  $k$ .

$\int_x$  means: integral over all items with parameter  $x$ .

### Quaternionic Hilbert space

The Hilbert space is a **linear space**. That means for the elements  $|f\rangle$ ,  $|g\rangle$  and  $|h\rangle$  and numbers  $a$  and  $b$ :

#### **Ket vectors**

For **ket** vectors hold

$$|f\rangle + |g\rangle = |g\rangle + |f\rangle = |g + f\rangle \quad (1)$$

$$(|f\rangle + |g\rangle) + |h\rangle = |f\rangle + (|g\rangle + |h\rangle) \quad (2)$$

$$|(a + b)f\rangle = |f\rangle \cdot a + |f\rangle \cdot b \quad (3)$$

$$(|f\rangle + |g\rangle) \cdot a = |f\rangle \cdot a + |g\rangle \cdot a \quad (4)$$

$$|f\rangle \cdot 0 = |0\rangle \quad (5)$$

$$|f\rangle \cdot 1 = |f\rangle \quad (6)$$

Depending on the number field that the Hilbert space supports,  $a$  and  $b$  can be real numbers, complex numbers or (real) quaternions.

### Bra vectors

The **bra** vectors form the dual Hilbert space  $\mathbf{H}^\dagger$  of  $\mathbf{H}$ .

$$\langle f| + \langle g| = \langle g| + \langle f| = \langle g + f| \quad (1)$$

$$(\langle f| + \langle g|) + \langle h| = \langle f| + (\langle g| + \langle h|) \quad (2)$$

$$\langle f(a + b)| = \langle f| \cdot a + \langle f| \cdot b = a^* \cdot \langle f| + b^* \cdot \langle f| \quad (3)$$

$$(\langle f| + \langle g|) \cdot a = \langle f| \cdot a + \langle g| \cdot a = a^* \cdot \langle f| + a^* \cdot \langle g| \quad (4)$$

$$0 \cdot \langle f| = \langle 0| \quad (5)$$

$$1 \cdot \langle f| = \langle f| \quad (6)$$

### Scalar product

The Hilbert space contains a **scalar product**, also called **inner product**,  $\langle f|g\rangle$  that combines  $\mathbf{H}$  and  $\mathbf{H}^\dagger$  in a direct product that we also indicate with  $\mathbf{H}$ .

The scalar product  $\langle f|g\rangle$  satisfies:

$$\langle f | g + h \rangle = \langle f | g \rangle + \langle f | h \rangle \quad (1)$$

$$\langle f | \{ |g\rangle \cdot a \}_g = \{ \langle f | g \rangle \}_g \cdot a \quad (2)$$

With each ket vector  $|g\rangle$  in  $\mathbf{H}$  belongs a bra vector  $\langle g |$  in  $\mathbf{H}^\dagger$  such that for all bra vectors  $\langle f |$  in  $\mathbf{H}^\dagger$

$$\langle f | g \rangle = \langle g | f \rangle^* \quad (3)$$

$$\langle f | f \rangle = 0 \text{ when } |f\rangle = |0\rangle \quad (4)$$

$$\langle f | a g \rangle = \langle f | g \rangle \cdot a = \langle g | f \rangle^* \cdot a = \langle g | a f \rangle^* = (a^* \cdot \langle g | f \rangle)^* = \langle f | g \rangle \cdot a \quad (5)$$

In general is  $\langle f | a g \rangle \neq \langle f | g \rangle \cdot a$ . However for real numbers  $r$  holds  $\langle f | r g \rangle = \langle f | g \rangle \cdot r$

Remember that when the number field consists of quaternions, then also  $\langle f | g \rangle$  is a quaternion and a quaternion  $q$  and  $\langle f | g \rangle$  do in general not commute.

The scalar product defines a **norm**:

$$\| |f| \| = \sqrt{\langle f | f \rangle} \quad (6)$$

And a **distance**:

$$D(f,g) = \| |f - g| \| \quad (7)$$

The Hilbert space  $\mathbf{H}$  is closed under its norm. Each converging row of elements of converges to an element of this space.

### **Separable**

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence  $\{x_n\}_{n=1}^\infty$  of elements

of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Every continuous function on the separable space  $\mathbf{H}$  is determined by its values on this countable dense subset.

### Base vectors

The Hilbert space  $\mathbf{H}$  is **separable**. That means that there exist a countable row of elements  $\{|f_n\rangle\}$  that **spans** the whole space.

If  $\langle f_n | f_m \rangle = \delta(m,n) = [1 \text{ when } n = m; 0 \text{ otherwise}]$   
then  $\{|f_n\rangle\}$  forms an **orthonormal base** of the Hilbert space.

A ket base  $\{|k\rangle\}$  of  $\mathbf{H}$  is a minimal set of ket vectors  $|k\rangle$  that together span the Hilbert space  $\mathbf{H}$ .

Any ket vector  $|f\rangle$  in  $\mathbf{H}$  can be written as a linear combination of elements of  $\{|k\rangle\}$ .

$$|f\rangle = \sum_k (|k\rangle \cdot \langle k | f \rangle) \quad (1)$$

A bra base  $\{\langle b | \}$  of  $\mathbf{H}^\dagger$  is a minimal set of bra vectors  $\langle b |$  that together span the Hilbert space  $\mathbf{H}^\dagger$ .

Any bra vector  $\langle f |$  in  $\mathbf{H}^\dagger$  can be written as a linear combination of elements of  $\{\langle b | \}$ .

$$\langle f | = \sum_b (\langle f | b \rangle \cdot \langle b |) \quad (2)$$

Usually base vectors are taken such that their norm equals 1. Such a base is called an orthonormal base.

### Operators

Operators act on a subset of the elements of the Hilbert space.

#### Linear operators

An operator  $Q$  is linear when for all vectors  $|f\rangle$  and  $|g\rangle$  for which  $Q$  is defined and for all quaternionic numbers  $a$  and  $b$ :

$$|Q \cdot a \rangle + |Q \cdot b \rangle = |a \cdot Q \rangle + |b \cdot Q \rangle = |Q \rangle \cdot a + |Q \rangle \cdot b = \quad (1)$$

$$Q(|\rangle \cdot a + |\rangle \cdot b) = Q(|a \rangle + |b \rangle) \quad (2)$$

B is **colinear** when for all vectors  $|\rangle$  for which B is defined and for all quaternionic numbers  $a$  there exists a quaternionic number  $c$  such that:

$$|B \cdot a \rangle = |a \cdot B \rangle = |B \rangle \cdot c \cdot a \cdot c^{-1} \quad (3)$$

If  $|\rangle$  is an eigenvector of operator A with quaternionic eigenvalue  $a$ , then is  $|b \rangle$  an eigenvector of A with quaternionic eigenvalue  $b \cdot a \cdot b^{-1}$ .

$A^\dagger = A^\dagger$  is the **adjoint** of the **normal** operator A.  $|A$  is the same as A.

$$\langle f | A | g \rangle = \langle f | A^\dagger | g \rangle^* \quad (4)$$

$$A^{\dagger\dagger} = A \quad (5)$$

$$(A \cdot B)^\dagger = B^\dagger \cdot A^\dagger \quad (6)$$

$|B|$  is a **self adjoint** operator.

$|$  is a nil operator.

The construct  $|\rangle \langle g|$  acts as a linear operator.  $|g\rangle \langle f|$  is its adjoint operator.

$$\sum_n \{ |f_n\rangle \cdot a_n \cdot \langle f_n| \}, \quad (7)$$

where  $a_n$  is real and acts as a density function.

The set of eigenvectors of a normal operator form an orthonormal base of the Hilbert space.

A self adjoint operator has real numbers as eigenvalues.

$\langle q | f \rangle_q$  is a function  $f(q)$  of parameter  $q$ .

$\langle g | q \rangle_q$  is a function  $g(q)$  of parameter  $q$ .

When possible, we use the same letter for identifying eigenvalues, eigenvalues and the corresponding operator.

So, usually  $|q\rangle$  is an eigenvector of a normal operator  $Q$  with eigenvalues  $q$ .

$\{q\}$  is the set of eigenvalues of  $Q$ .

$\{q\}_q$  is the ordered field of eigenvalues of  $q$ .

$\{|q\rangle\}_q$  is the ordered set of eigenvectors of  $Q$ .

$\langle q | f \rangle_q$  is the **Q view** of  $|f\rangle$ .

Normal operators

The most common definition of continuous operators is:

A **continuous** operator is an operator that creates images such that the inverse images of open sets are open.

Similarly, a **continuous** operator creates images such that the inverse images of closed sets are closed.

A normal operator is a continuous linear operator.

A normal operator in  $\mathbf{H}$  creates an image of  $\mathbf{H}$  onto  $\mathbf{H}$ . It transfers closed subspaces of  $\mathbf{H}$  into closed subspaces of  $\mathbf{H}$ .

Normal operators represent continuous quantum logical observables.

The normal operators  $N$  have the following property.

$$N: \mathbf{H} \Rightarrow \mathbf{H} \tag{1}$$

$N$  commutes with its **(Hermitian) adjoint**  $N^\dagger$

$$N \cdot N^\dagger = N^\dagger \cdot N \tag{2}$$

Normal operators are important because the spectral theorem holds for them.

Examples of normal operators are

- **unitary** operators:  $U^\dagger = U^{-1}$ , unitary operators are bounded;
- **Hermitian** operators (i.e., self-adjoint operators):  $N^\dagger = N$ ;
- **Anti-Hermitian** or anti-self-adjoint operators:  $N^\dagger = -N$ ;
- **Anti-unitary** operators:  $I^\dagger = -I = I^{-1}$ , anti-unitary operators are bounded;
- **positive operators**:  $N = MM^\dagger$
- **orthogonal projection** operators:  $N = N^\dagger = N^2$

### Spectral theorem

For every compact self-adjoint operator  $T$  on a real, complex or quaternionic Hilbert space  $\mathbf{H}$ , there exists an orthonormal basis of  $\mathbf{H}$  consisting of eigenvectors of  $T$ . More specifically, the orthogonal complement of the kernel (null space) of  $T$  admits, either a finite orthonormal basis of eigenvectors of  $T$ , or a countable infinite orthonormal basis  $\{e_n\}$  of eigenvectors of  $T$ , with corresponding eigenvalues  $\{\lambda_n\} \subset \mathbb{R}$ , such that  $\lambda_n \rightarrow 0$ . Due to the fact that  $\mathbf{H}$  is separable the set of eigenvectors of  $T$  can be extended with a base of the kernel in order to form a complete orthonormal base of  $\mathbf{H}$ .

If  $T$  is compact on an infinite dimensional Hilbert space  $\mathbf{H}$ , then  $T$  is not invertible, hence  $\sigma(T)$ , the spectrum of  $T$ , always contains 0. The spectral theorem shows that  $\sigma(T)$  consists of the eigenvalues  $\{\lambda_n\}$  of  $T$ , and of 0 (if 0 is not already an eigenvalue). The set  $\sigma(T)$  is a compact subset of the real line, and the eigenvalues are dense in  $\sigma(T)$ .

A normal operator has a set of eigenvectors that spans the whole Hilbert space  $\mathbf{H}$ .

In quaternionic Hilbert space a normal operator has quaternions as eigenvalues.

The set of eigenvalues of a normal operator is NOT compact. This is due to the fact that  $\mathbf{H}$  is separable. Therefore the set of eigenvectors is countable. As a consequence the set of eigenvalues is countable. Further, the eigenspace of normal operators has no finite diameter.

A continuous bounded linear operator on  $\mathbf{H}$  has a compact eigenspace. The set of eigenvalues has a closure and it has a finite diameter.

### Eigenspace

The set of eigenvalues  $\{q\}$  of the operator  $Q$  form the eigenspace of  $Q$

### Eigenvectors and eigenvalues

For the eigenvector  $|q\rangle$  of normal operator  $Q$  holds

$$|Qq\rangle = |q\rangle = |q\rangle \cdot q \tag{1}$$

$$\langle q|Q^\dagger| = \langle q|q^*| = q^* \cdot \langle q| \tag{2}$$

$$\forall |f\rangle \in \mathbf{H} [\{\langle f|Qq\rangle\}_q = \{\langle f|q\rangle q\}_q = \{\langle q|Q^\dagger|f\rangle^*\}_q = \{q^* \langle q|f\rangle^*\}_q] \tag{3}$$

The eigenvalues of  $2^n$ -on normal operator are  $2^n$ -ons

$$Q = \sum_{j=0}^{n-1} I_j Q_j \tag{4}$$

The  $Q_j$  are self-adjoint operators.

Generalized Trotter formula

For bounded operators  $\{A_j\}$  hold:

$$\lim_{n \rightarrow \infty} \left( \prod_{j=1}^p e^{A_j/n} \right)^n = \exp \left( \sum_{j=1}^p A_j \right) = \lim_{n \rightarrow \infty} \left( 1 + \frac{\sum_{j=1}^p A_j}{n} \right)^n \quad (1)$$

In general

$$\exp \left( \sum_{j=1}^p A_j \right) \neq \prod_{j=1}^p e^{A_j} \quad (2)$$

Unitary operators

For unitary operators holds:

$$U^\dagger = U^{-1} \quad (1)$$

Thus

$$U \cdot U^\dagger = U^\dagger \cdot U = 1 \quad (2)$$

Suppose  $U = I + C$  where  $U$  is unitary and  $C$  is compact. The equations  $U U^\dagger = U^\dagger U = I$  and  $C = U - I$  show that  $C$  is normal. The spectrum of  $C$  contains 0, and possibly, a finite set or a sequence tending to 0. Since  $U = I + C$ , the spectrum of  $U$  is obtained by shifting the spectrum of  $C$  by 1.

The unitary transform can be expressed as:

$$U = \exp(\tilde{I} \cdot \Phi / \hbar) \quad (3)$$

$$\hbar = h / (2 \cdot \pi) \quad (4)$$

$\Phi$  is Hermitian. The constant  $\hbar$  refers to the granularity of the eigenspace. Unitary operators have eigenvalues that are located in the unity sphere of the  $2^n$ -ons field.

The eigenvalues have the form:

$$u = \exp(\mathbf{i} \cdot \phi / \hbar) \quad (5)$$

$\phi$  is real.  $\mathbf{i}$  is a unit length imaginary number in  $2^n$ -on space. It represents a direction.

$u$  spans a sphere in  $2^n$ -on space. For constant  $\mathbf{i}$ ,  $u$  spans a circle in a complex subspace.

### *Polar decomposition*

Normal operators  $N$  can be split into a real operator  $A$  and a unitary operator  $U$ .  $U$  and  $A$  have the same set of eigenvectors as  $N$ .

$$N = ||N|| \cdot U = A \cdot U \quad (1)$$

$$N = A \cdot U = U \cdot A \quad (2)$$

$$= A \cdot \exp(\tilde{\mathbf{I}} \cdot \Phi / \hbar)$$

$$= \exp(\Phi_r + \tilde{\mathbf{I}} \cdot \Phi) / \hbar$$

$\Phi_r$  is a positive normal operator.

Ladder operator

### *General formulation*

Suppose that two operators  $X$  and  $N$  have the commutation relation:

$$[N, X] = c \cdot X \quad (1)$$

for some scalar  $c$ . If  $|n\rangle$  is an eigenstate of  $N$  with eigenvalue equation,

$$|N n\rangle = |n\rangle \cdot n \quad (2)$$

then the operator  $X$  acts on  $|n\rangle$  in such a way as to shift the eigenvalue by  $c$ :

$$\begin{aligned}
 |N \cdot X n\rangle &= |(X \cdot N + [N, X]) n\rangle \\
 &= |(X \cdot N + c \cdot X) n\rangle \\
 &= |X \cdot N n\rangle + |X n\rangle \cdot c \\
 &= |X n\rangle \cdot n + |X n\rangle \cdot c \\
 &= |X n\rangle \cdot (n+c)
 \end{aligned}
 \tag{3}$$

In other words, if  $|n\rangle$  is an eigenstate of  $N$  with eigenvalue  $n$  then  $|X n\rangle$  is an eigenstate of  $N$  with eigenvalue  $n + c$ .

The operator  $X$  is a *raising operator* for  $N$  if  $c$  is real and positive, and a *lowering operator* for  $N$  if  $c$  is real and negative.

If  $N$  is a Hermitian operator then  $c$  must be real and the Hermitian adjoint of  $X$  obeys the commutation relation:

$$[N, X^\dagger] = -c \cdot X^\dagger \tag{4}$$

In particular, if  $X$  is a lowering operator for  $N$  then  $X^\dagger$  is a raising operator for  $N$  and vice-versa.

### Unit sphere of $H$

The ket vectors in  $H$  that have their norm equal to one form together the **unit sphere**  $\Theta$  of  $H$ .

Base vectors are all member of the unit sphere. The eigenvectors of a normal operator are all member of the unit sphere.

The end points of the eigenvectors of a normal operator form a **grid** on the unit sphere  $\Theta$  of  $H$ .

### Closure

The closure of  $H$  means that converging rows of vectors converge to a vector of  $H$ .

In general converging rows of eigenvalues of  $Q$  do not converge to an eigenvalue of  $Q$ .

Thus, the set of eigenvalues of  $Q$  is open.

At best the density of the coverage of the set of eigenvalues is comparable with the set of  $2^n$ -ons that have rational numbers as coordinate values.

With other words, compared to the set of real numbers the eigenvalue spectrum of  $Q$  has holes.

The set of eigenvalues of operator  $Q$  includes 0. This means that  $Q$  does not have an inverse.

The rigged Hilbert space  $\mathbf{H}$  can offer a solution, but then the direct relation with quantum logic is lost.

### **Canonical conjugate operator $P$**

The existence of a canonical conjugate represents a stronger requirement on the continuity of the eigenvalues of canonical eigenvalues.

$Q$  has eigenvectors  $\{|q\rangle\}_q$  and eigenvalues  $q$ .

$P$  has eigenvectors  $\{|p\rangle\}_p$  and eigenvalues  $p$ .

For each eigenvector  $|q\rangle$  of  $Q$  we define an eigenvector  $|p\rangle$  and eigenvalues  $p$  of  $P$  such that:

$$\langle q|p \rangle = \langle p|q \rangle^* = \exp(i \cdot p \cdot q/\hbar) \quad (1)$$

$\hbar = h/(2\pi)$  is a scaling factor.  $\langle q|p \rangle$  is a quaternion.  $i$  is a unit length imaginary quaternion.

### **Displacement generators**

Variance of the scalar product gives:

$$i \hbar \delta \langle q|p \rangle = -p \langle q|p \rangle \delta q \quad (1)$$

$$i \hbar \delta \langle p|q \rangle = -q \langle p|q \rangle \delta p \quad (2)$$

In the rigged Hilbert space  $\mathbf{H}$  the variance can be replaced by differentiation.

Partial differentiation of the function  $\langle q|p\rangle$  gives:

$$i \hbar \partial / \partial q_s \langle q|p\rangle = -p_s \langle q|p\rangle \quad (3)$$

$$i \hbar \partial / \partial p_s \langle p|q\rangle = -q_s \langle p|q\rangle \quad (4)$$

## Derivation of the one dimensional Euler Lagrange equation

This is taken from [Wikipedia](#)<sup>156</sup>.

### Equation

The Euler–Lagrange equation is an equation satisfied by a function,  $q$ , of a [real](#)<sup>157</sup> argument,  $t$ , which is a stationary point of the [functional](#)<sup>158</sup>

$$S(q) = \int_a^b L(t, q(t), \dot{q}(t)) dt \quad (1)$$

where:

$q$  is the function to be found:

$$\begin{aligned} q: [a, b] \subset \mathbb{R} &\rightarrow X \\ t &\mapsto v = \dot{q}(t) \end{aligned} \quad (2)$$

such that  $q$  is differentiable,  $q(a) = x_a$ , and  $q(b) = x_b$ ;  
 $\dot{q}$  is the derivative of  $q$ :

$$\begin{aligned} \dot{q}: [a, b] &\rightarrow T_{q(t)}X \\ t &\mapsto v = \dot{q}(t) \end{aligned} \quad (3)$$

$TX$  being the [tangent bundle](#) of  $X$  (the space of possible values of derivatives of functions with values in  $X$ );

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<sup>156</sup> <http://en.wikipedia.org/wiki/Euler%E2%80%93Lagrange>

<sup>157</sup> [http://en.wikipedia.org/wiki/Real\\_number](http://en.wikipedia.org/wiki/Real_number)

<sup>158</sup> [http://en.wikipedia.org/wiki/Functional\\_\(mathematics\)](http://en.wikipedia.org/wiki/Functional_(mathematics))

$L$  is a real-valued function with [continuous](#)<sup>159</sup> first [partial derivatives](#)<sup>160</sup>:

$$\begin{aligned} L: [a, b] \times X \times TX &\rightarrow \mathbb{R} \\ t, x, v &\mapsto L(t, x, v) \end{aligned} \quad (4)$$

The Euler–Lagrange equation, then, is given by

$$L_x(t, q(t), \dot{q}(t)) - \frac{d}{dt} L_v(t, q(t), \dot{q}(t)) = \frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} = 0 \quad (5)$$

where  $L_x$  and  $L_v$  denote the partial derivatives of  $L$  with respect to the second and third arguments, respectively.

If the dimension of the space  $X$  is greater than 1, this is a system of differential equations, one for each component:

$$\frac{\partial L}{\partial x_i} - \frac{d}{dt} \frac{\partial L}{\partial v_i} = 0; i = 1, \dots, n \quad (6)$$

## Derivation

Given a functional

$$J = \int_a^b F(t, y(t), y'(t)) dt \quad (1)$$

on  $C^1([a, b])$  with the boundary conditions  $y(a) = A$  and  $y(b) = B$ , we proceed by approximating the extremal curve by a polygonal line with  $n$  segments and passing to the limit as the number of segments grows arbitrarily large.

Divide the interval  $[a, b]$  into  $n + 1$  equal segments with endpoints  $t_0 = a, t_1, t_2, \dots, t_n, t_{n+1} = b$  and let  $\Delta t = t_k - t_{k-1}$ . Rather than a smooth function  $y(t)$  we consider the polygonal line with vertices  $(t_0, y_0), \dots, (t_{n+1}, y_{n+1})$ , where  $y_0 = A$  and  $y_{n+1} = B$ . Accordingly, our functional becomes a real function of  $n$  variables given by

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<sup>159</sup> [http://en.wikipedia.org/wiki/Continuous\\_function](http://en.wikipedia.org/wiki/Continuous_function)

<sup>160</sup> [http://en.wikipedia.org/wiki/Partial\\_derivatives](http://en.wikipedia.org/wiki/Partial_derivatives)

$$J(y_1, \dots, y_n) \approx \sum_{k=0}^n F\left(t_k, y_k, \frac{y_{k+1}-y_k}{\Delta t}\right) \Delta t \quad (2)$$

Extremals of this new functional defined on the discrete points  $t_0, \dots, t_{n+1}$  correspond to points where

$$\frac{\partial J(y_1, \dots, y_n)}{\partial y_m} = 0 \quad (3)$$

Evaluating this partial derivative gives

$$\begin{aligned} \frac{\partial J}{\partial y_m} = & F_y\left(t_m, y_m, \frac{y_{m+1}-y_m}{\Delta t}\right) \Delta t + F_{y'}\left(t_{m-1}, y_{m-1}, \frac{y_m-y_{m-1}}{\Delta t}\right) \\ & - F_y\left(t_m, y_m, \frac{y_{m+1}-y_m}{\Delta t}\right) \end{aligned} \quad (4)$$

Dividing the above equation by  $\Delta t$  gives

$$\begin{aligned} \frac{\partial J}{\partial y_m \Delta t} = & F_y\left(t_m, y_m, \frac{y_{m+1}-y_m}{\Delta t}\right) \\ & + \frac{F_{y'}\left(t_{m-1}, y_{m-1}, \frac{y_m-y_{m-1}}{\Delta t}\right) - F_y\left(t_m, y_m, \frac{y_{m+1}-y_m}{\Delta t}\right)}{\Delta t} \end{aligned} \quad (5)$$

and taking the limit as  $\Delta t \rightarrow 0$  of the right-hand side of this expression yields

$$F_y - \frac{dF_{y'}}{dt} = 0 \quad (6)$$

The left hand side of the previous equation is the [functional derivative](#)<sup>161</sup>  $\frac{\delta J}{\delta y}$  of the functional  $J$ . A necessary condition for a differentiable functional

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<sup>161</sup> [http://en.wikipedia.org/wiki/Functional\\_derivative](http://en.wikipedia.org/wiki/Functional_derivative)

to have an extremum on some function is that its functional derivative at that function vanishes, which is granted by the last equation.

## Zak transform

(See also <http://eom.springer.de/Z/z130030.htm>)

### Definition

The Weil-Brezin-Zak transform  $Z_\alpha(f)$  of a function  $f$  is defined by

$$Z_\alpha[f](t, \omega) = Z_\alpha f(t, \omega) = \sqrt{\alpha} \sum_{k=-\infty}^{\infty} f(\alpha t + \alpha k) \exp(-2\pi i k \omega) \quad (1)$$

Where  $\alpha > 0$  and  $t$  and  $\omega$  are real. When  $\alpha = 1$ , one denotes  $Z_\alpha f$  by  $Zf$ .

If  $f$  represents a signal, then its Zak transform can be considered as a mixed time-frequency representation of  $f$ , and it can also be considered as a generalization of the discrete Fourier transform of  $f$  in which an infinite sequence of samples in the form  $f(\alpha t + \alpha k)$ ,  $k = 0, \pm 1, \pm 2, \dots$ , is used

### Elementary properties.

- 1) (linearity): for any complex numbers  $\alpha$  and  $\beta$ ,

$$Z[\alpha f(t) + \beta g(t)](t, \omega) = \alpha Z[f(t)](t, \omega) + \beta Z[g(t)](t, \omega) \quad (2)$$

- 2) (translation): for any integer  $m$ ,

$$Z[f(t + m)](t, \omega) = \exp(2\pi i k m \omega) Z[f](t, \omega) \quad (3,4)$$

in particular,

$$(Zf)(t + 1, \omega) = \exp(2\pi i k \omega) Zf(t, \omega)$$

- 3) (modulation):

$$Z[\exp(2\pi i k m t)f](t, \omega) = \exp(2\pi i k m t)(Zf)(t, \omega) \quad (5)$$

- 4) (periodicity): The Zak transform is periodic in  $\omega$  with period one, that is,

(6)

$$(Z f)(t, \omega + 1) = (Z f)(t, \omega)$$

5) (translation and modulation): By combining 2) and 3) one obtains (7)

$$Z [\exp(2 \pi \mathbf{k} m t) f(t + n)](t, \omega) = \exp(2 \pi \mathbf{k} m t) \exp(2 \pi \mathbf{k} n \omega) (Z f)(t, \omega)$$

6) (conjugation): (8)

$$(Z \bar{f})(t, \omega) = \overline{(Z f)(t, -\omega)}$$

7) (symmetry): If  $f$  is even, then

$$(Z f)(t, \omega) = (Z f)(-t, -\omega) \quad (9)$$

and if  $f$  is odd, then

$$(Z f)(t, \omega) = -(Z f)(-t, -\omega) \quad (10)$$

From 6) and 7) it follows that if  $f$  is real-valued and even, then

$$(Z f)(t, \omega) = \overline{(Z f)(t, -\omega)} = (Z f)(-t, -\omega) \quad (11)$$

Because of 2) and 4), the Zak transform is completely determined by its values on the unit square  $Q = [0,1] \times [0,1]$ .

8) (convolution): Let

$$h(t) = \int_{-\infty}^{\infty} R(t-s) f(s) ds \quad (12)$$

then

$$(Z h)(t, \omega) = \int_0^1 (Z R)(t - s, \omega) (Z f)(s, \omega) ds \quad (13)$$

### Analytic properties.

If  $f$  is a continuous function such that

$$f(t) = O((1 + |t|)^{-1-\epsilon}) \text{ as } t \rightarrow \infty \text{ for some } \epsilon > 0 \quad (1)$$

Then  $Z f$  is continuous on  $Q$ . A rather peculiar property of the Zak transform is that if  $Z f$  is continuous, it must have a zero in  $Q$ . The Zak transform is a unitary transformation from  $L^2(\mathbb{R})$  onto  $L^2(Q)$ .

### Inversion formulas.

The following inversion formulas for the Zak transform follow easily from the definition, provided that the series defining the Zak transform converges uniformly:

$$f(t) = \int_0^1 (Z f)(t, \omega) d\omega; -\infty < t < \infty \quad (1)$$

$$\tilde{f}(-2\pi\omega) = \frac{1}{\sqrt{2\pi}} \int_0^1 \exp(-2\pi\mathbf{k}\omega t) (Z f)(t, \omega) dt \quad (2)$$

and

$$f(2\pi\omega) = \frac{1}{\sqrt{2\pi}} \int_0^1 \exp(-2\pi\mathbf{k}x t) (Z \tilde{f})(x, t) dx \quad (3)$$

where  $\tilde{f}$  is the Fourier transform of  $f$ , given by

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(\mathbf{k}\omega x) dx \quad (4)$$

### Applications.

The Gabor representation problem can be stated as follows: Given  $g \in L^2(\mathbb{R})$  and two real numbers,  $\alpha, \beta$ , different from zero, is it possible to represent any function  $f \in L^2(\mathbb{R})$  by a series of the form

$$f = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} c_{mn} g_{m\beta, n\alpha}, \quad (1)$$

where  $g_{m\beta, n\alpha}$  are the Gabor functions, defined by:

$$g_{m\beta, n\alpha}(x) = \exp(2\pi\beta x) g(x - n\alpha) \quad (2)$$

and  $c_{mn}$  are constants? And under what conditions is the representation unique?

Fix a coordinate  $x$  in a line  $\mathbb{R}$ ; the family of functions in  $\mathbb{R}$

$$e_{\lambda}(x) = \sqrt[4]{2} \exp(-\pi(x-p)^2 + 2\pi k\theta x) \quad (3)$$

are called Gabor functions. Here  $\lambda = (p, \theta)$  is a point in the phase space  $\Phi = \mathbb{R} \otimes \mathbb{R}$ .

The operators

$$\mathcal{A} = \frac{1}{2\pi} \frac{d}{dx} + x \quad (4)$$

$$\mathcal{A}^{\dagger} = -\frac{1}{2\pi} \frac{d}{dx} + x \quad (5)$$

in  $L^2$  are adjoint one to another. They are called the *annihilation* and the *creation* operators.

Any Gabor function is an eigenvector of the annihilation operator:

$$\mathcal{A}e_{\lambda} = \lambda e_{\lambda} \quad (6)$$

where

$$\lambda = (p, \theta) \quad (7)$$

and

$$\lambda = p + k\theta \quad (8)$$

For any  $\varphi$  in the domain of the operator  $a$  we have

$$Z (\mathcal{A}\varphi) = AZ\varphi \quad (9)$$

$$A = \frac{1}{2\pi k} \left( \frac{\partial}{\partial \xi} + \frac{\partial}{\partial y} \right) + y \quad (10)$$

## Thoughts

The following texts represent collections of thoughts that still have to be brought in proper order and in mutual consistency.

### Spin and dyadic product

As factors of the dyadic product we consider imaginary quaternionic numbers or vectors in  $\mathbb{R}_3$ . The product corresponds to a matrix. This matrix acts as an operator.

$$u \otimes v \rightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} [v_1 \quad v_2 \quad v_3] = \begin{bmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{bmatrix} \quad (1)$$

The product of quaternions contains sign selections. For the imaginary parts this selection has to do with the handedness of the external product. Dyadic products are well suited to store the product such that the sign selections are stored as well. The sign selection plays its role in the dyad  $\mathbf{ij}$ , which consists of two imaginary base numbers. The dyad  $\mathbf{ij} = -\mathbf{ji}$ , and  $\mathbf{k}$  can be  $\pm \mathbf{ij}$ . Let us apply this to the definition of  $S_z$ .

(2)

$$\begin{aligned}
S_z &= -i\hbar \begin{bmatrix} 0 & \mathbf{e}_x \mathbf{e}_y - \mathbf{e}_y \mathbf{e}_x & 0 \\ \mathbf{e}_y \mathbf{e}_x - \mathbf{e}_x \mathbf{e}_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
&= i\hbar \begin{bmatrix} 0 & -2\mathbf{e}_x \mathbf{e}_y & 0 \\ 2\mathbf{e}_x \mathbf{e}_y & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\end{aligned}$$

This shows that the definition of  $S_z$  via the dyadic product reflects the choice in handedness of the external product of  $\mathbf{e}_x$  and  $\mathbf{e}_y$ .

### Wave package

The linear momentum is interpretable as a displacement operator. This operator is better treated in Fourier space than in configuration space. In Fourier space a particle becomes a wave package. The Fourier transforms of the fields describe the wave package.

Operator  $P$  has eigenfunctions  $\tilde{f}(p)$  with eigenvalues  $p$ :

$$\langle q|p \rangle = \tilde{f}(p) = \langle p|q \rangle^* = f^*(q) = \exp\left(\mathbf{k} \cdot \frac{p q}{\hbar}\right) \quad (1)$$

$$\langle p|P p \rangle = p \quad (2)$$

A pure particle can be represented by a single Hilbert vector  $|f\rangle$ . Its wave function is given by:

$$\psi(q) = \langle \psi|q \rangle \quad (3)$$

Or by:

$$\tilde{\psi}(p) = \langle p|\psi \rangle \quad (4)$$

A mixed particle takes a [Hilbert distribution](#)<sup>162</sup> in order to define its presence.

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<sup>162</sup> Functions and fields; Distributions in quaternionic Hilbert space

$$\rho(q) = \langle \rho | q \rangle \quad (5)$$

A blurred Hilbert distribution is a Hilbert field.

$$\phi(q) = \rho(q) \circ \varphi(q) \quad (6)$$

A different type of blur gives a different type of Hilbert field.

The [wave functions](#)<sup>163</sup> and private Hilbert fields represent particles. Their Fourier transforms represent wave packages. A very particular Hilbert field is a probability density that is based on a [probability density operator](#)<sup>164</sup>.

A single wave mode represents a plane wave. Look at the linear momentum of the field contained in a volume  $V$  surrounded by surface  $S$ :

$$\mathbf{P}_{field} = \int_V \mathbf{g}_{field} dV = \int_V \rho_0 \boldsymbol{\phi} dV + \int_V \langle \nabla \boldsymbol{\phi}, \mathbf{E} \rangle dV + \oint_S \langle \hat{\mathbf{n}}, \mathbf{E} \boldsymbol{\phi} \rangle dS \quad (7)$$

For each temporal Fourier mode of the field in free space (vanishing charge density  $\rho_0$ , no variance of scalar potential  $\phi_0$ ), where  $\mathbf{E}\boldsymbol{\phi}$  falls off rapidly, we can neglect the first and the third term.

$$\mathbf{P}_{field} \approx \int_V \langle \nabla \boldsymbol{\phi}, \mathbf{E} \rangle dV \quad (8)$$

Further:

$$\mathbf{E} = \nabla \phi_0 - \nabla_0 \boldsymbol{\phi} \approx -\nabla_0 \boldsymbol{\phi} \quad (9)$$

$$(10)$$

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<sup>163</sup> States

<sup>164</sup>States; State definition; Probability density

$$\begin{aligned}
\mathbf{P}_{field} &\approx - \int_V \langle \nabla \phi, \nabla_0 \phi \rangle dV \\
&= \int_{V_p} \langle \mathbf{p} \tilde{\phi}, \nabla_0 \tilde{\phi} \rangle dV_p \\
&= \int_{V_p} \mathbf{p} \langle \tilde{\phi}, \nabla_0 \tilde{\phi} \rangle dV_p \\
&= \int_{V_p} \omega(p) \mathbf{p} \langle \tilde{\phi}, \tilde{\phi} \rangle dV_p
\end{aligned}$$

If the function  $\langle \tilde{\phi}(p), \tilde{\phi}(p) \rangle$  gives the probability density for eigenvalue  $p$ . Then, this gives reason to interpret  $\langle \phi(q), \phi(q) \rangle$  as probability density for the position  $q$  of the particle.

### Fourier mode

A Fourier mode is a single frequency wave. It can be interpreted as a “particle” or as a train of particles whose charge is blurred by a very wide spread function. The corresponding current is blurred by that same spread function. It means that the divergence along the wave reduces to zero.

Often waves of the same frequency that belong to different mutually perpendicular fields combine to form polarized waves. The waves may differ in their phase shifts. The combination then forms a polarized wave. Depending on the phase difference it may be an elliptical polarized wave, a circular polarized wave or a linearly polarized wave.

### Systems

A system is a local assembly of physical items that act as a single physical item. Its [state<sup>165</sup>](#) is mixed. When a redefinition of physical items in terms of

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<sup>165</sup> States

atomic predicates goes together with influences between items in the form of fields, then a redefinition of a system in terms of its components will certainly also have such effects. The redefinition may take different forms. It may be represented by an emission or absorption of a component or it may be a reshuffling of the components. The simplest case of reshuffling is a permutation of items that belong to the same category. A more complex situation is a periodic movement of one or more components within the realm of a system. In addition each sequence of creation and annihilation is a form of redefinition.

The system has its own characteristic vectors. The wave function may depend on the permutation state of the system. For example for fermions an odd permutation changes the sign of the (position related) wave function. For bosons a permutation does not affect the wave function. Permutations of different categories of components go together with their own type of influence. Thus, there are fermionic fields and there are bosonic fields. Each of these fields has its own type of creation and annihilation. Being fermion or boson relates to the spin type of the component. The annihilation and creation operators are closely related to the type of components involved and are also closely related to the type of fields involved. The annihilation/creation operators of fermions anti-commute and the annihilation/creation operators of bosons commute.

### Entropy

A [system](#) is a local assembly of [physical items](#)<sup>166</sup> that act as a single physical item. The Density operator  $\rho$  relates to the currently considered observable  $Q$ . A pure state is a ray spanned by an eigenvector of the operator  $Q$ .

[The von Neumann entropy](#)<sup>167</sup>  $S(\rho)$  of a physical system that is characterized by a [state](#)<sup>168</sup>  $|\psi\rangle$  is given by

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<sup>166</sup> Logic; Items

<sup>167</sup> [http://en.wikipedia.org/wiki/Von\\_Neumann\\_entropy](http://en.wikipedia.org/wiki/Von_Neumann_entropy)

<sup>168</sup> States

$$\rho = \sum_q \{|q\rangle\langle q|\} = \sum_q \{\lambda_q \cdot \rho_q\} \quad (1)$$

$$\rho_q = |q\rangle\langle q| \quad (2)$$

$$\lambda_q = |\langle \psi | q \rangle|^2$$

$$S(\rho) = -k_B \cdot \sum_q \{\lambda_q \cdot \ln(\lambda_q)\} \quad (3)$$

The entropy  $S(\rho)$  describes the departure of the system from a pure state. In other words, it measures the degree of mixture ([entanglement<sup>169</sup>](#)) of the state  $|\psi\rangle$ .

Some properties of the von Neumann entropy:

- $S(\rho)$  is only zero for pure states.
- $S(\rho)$  is maximal and equal to  $\log_2 N$  for a maximally mixed state,  $N$  being the dimension of the Hilbert space.
- $S(\rho)$  is invariant under changes in the basis of  $\rho$ , that is,  $S(\rho) = S(U\rho U^\dagger)$ , with  $U$  a unitary transformation.
- $S(\rho)$  is concave, that is, given a collection of positive numbers  $\lambda_q$  which sum to unity ( $\sum_q \lambda_q = 1$ ) and density operators  $\rho_q$ , we have

$$S\left(\sum_q \lambda_q \rho_q\right) \geq \sum_q \lambda_q S(\rho_q) \quad (4)$$

- $S(\rho)$  is additive. Given two density matrices  $\rho_A, \rho_B$  describing independent systems  $A$  and  $B$ , then

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<sup>169</sup> [http://en.wikipedia.org/wiki/Quantum\\_entanglement](http://en.wikipedia.org/wiki/Quantum_entanglement)

$$S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) \quad (5)$$

Instead, if  $\rho_A, \rho_B$  are the reduced density operators of the general state  $\rho_{AB}$ , then

$$|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B) \quad (6)$$

While in Shannon's theory the entropy of a composite system can never be lower than the entropy of any of its parts, in quantum theory this is not the case, i.e., it is possible that  $S(\rho_{AB}) = 0$  while  $S(\rho_A) > 0$  and  $S(\rho_B) > 0$ .

Intuitively, this can be understood as follows: In quantum mechanics, the entropy of the joint system can be less than the sum of the entropy of its components because the components may be [entangled](#)<sup>170</sup>. The left-hand inequality can be roughly interpreted as saying that entropy can only be canceled by an equal amount of entropy. If system  $A$  and system  $B$  have different amounts of entropy, the lesser can only partially cancel the greater, and some entropy must be left over. Likewise, the right-hand inequality can be interpreted as saying that the entropy of a composite system is maximized when its components are uncorrelated, in which case the total entropy is just a sum of the sub-entropies.

- The von Neumann entropy is also *strongly sub-additive*. Given three Hilbert spaces,  $A, B, C$ ,

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}) \quad (7)$$

### Isolated systems

With isolated systems we mean systems in a geometrically compound environment where influences from the environment compensate each other, possibly including the influences on the environment that are caused by the system under consideration. This includes e.g. the gravitation field. The gravitation potential cannot be zero, but the

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<sup>170</sup> [http://en.wikipedia.org/wiki/Quantum\\_entanglement](http://en.wikipedia.org/wiki/Quantum_entanglement)

influence of other items can be negligible. Internal influences are internally compensated such that they are not felt by other systems. For example the sum of the charges, which are related to electromagnetic fields is zero. It means that the Fourier transforms of the local fields consist of linear combinations of discrete terms. This holds for the electrostatic fields and the magneto-static fields. It holds for rectangular components as well as for polar components. These components are the germs of quanta and are the source of creations and annihilations. For example consider the vector potential  $A$ . Its Fourier transform can be written as:

$$A(\mathbf{r}, t) = \sum_k \sum_{\mu=-1,1} \{ \mathbf{e}_\mu \cdot a_{\mu k}(t) \cdot \exp(i(\mathbf{k}, \mathbf{r})) + \bar{\mathbf{e}}_\mu \cdot \bar{a}_{\mu k}(t) \cdot \exp(-i(\mathbf{k}, \mathbf{r})) \} \quad (1)$$

Where  $\mathbf{e}_\mu$  are unit sized polarization vectors. They depend on the orthonormal vectors  $\mathbf{e}_x$  and  $\mathbf{e}_y$  that represent quaternionic imaginary base numbers. The index  $\mu$  labels the photon spin. The product  $\mathbf{e}_\mu \cdot a_{\mu}$  represents a quaternionic imaginary number. The number  $i$  can be interpreted as a base imaginary number in the direction of  $\mathbf{k}$ .

$$\mathbf{e}_1 \stackrel{\text{def}}{=} \frac{-1}{\sqrt{2}} (\mathbf{e}_x + i \cdot \mathbf{e}_y) \quad (2)$$

$$\mathbf{e}_{-1} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} (\mathbf{e}_x - i \cdot \mathbf{e}_y) \quad (3)$$

$$(\mathbf{e}_x, \mathbf{k}) = 0 \quad (4)$$

$$(\mathbf{e}_y, \mathbf{k}) = 0 \quad (5)$$

$$[a_\mu(\mathbf{k}), a_{\mu'}(\mathbf{k}')] = 0 \quad (6)$$

$$[a_{\mu}^{\dagger}(\mathbf{k}), a_{\mu'}^{\dagger}(\mathbf{k}')] = 0 \quad (7)$$

$$[a_{\mu}(\mathbf{k}), a_{\mu'}^{\dagger}(\mathbf{k}')] = \delta_{\mu\mu'} \cdot \delta_{\mathbf{k}\mathbf{k}'} \quad (8)$$

Here the  $\sqrt{\frac{\hbar}{2\omega V \epsilon_0}} a_{\mu}(\mathbf{k})$  are the operator equivalents of the coefficients  $a_{\mu\mathbf{k}}$  and  $\omega = c |\mathbf{k}| = ck$ .

This results in:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\mathbf{k}, \mu} \sqrt{\frac{\hbar}{2\omega V \epsilon_0}} \{ \mathbf{e}_{\mu(\mathbf{k})} \cdot a_{\mu}(\mathbf{k}, t) \cdot \exp(i(\mathbf{k}, \mathbf{r})) + \bar{\mathbf{e}}_{\mu(\mathbf{k})} \cdot a_{\mu}^{\dagger}(\mathbf{k}, t) \cdot \exp(-i(\mathbf{k}, \mathbf{r})) \} \quad (9)$$

$$\mathbf{E}(\mathbf{r}, t) = i \cdot \sum_{\mathbf{k}, \mu} \sqrt{\frac{\hbar}{2\omega V \epsilon_0}} \{ \mathbf{e}_{\mu(\mathbf{k})} \cdot a_{\mu}(\mathbf{k}, t) \cdot \exp(i(\mathbf{k}, \mathbf{r})) - \bar{\mathbf{e}}_{\mu(\mathbf{k})} \cdot a_{\mu}^{\dagger}(\mathbf{k}, t) \cdot \exp(-i(\mathbf{k}, \mathbf{r})) \} \quad (10)$$

$a_{\mu}(\mathbf{k}, t)$  is an annihilation operator and  $a_{\mu}^{\dagger}(\mathbf{k}, t)$  is a creation operator.

$$a_{\mu}^{\dagger}(\mathbf{k}, t) |n\rangle = |n+1\rangle \sqrt{n+1} \quad (11)$$

$$a_{\mu}^{\dagger}(\mathbf{k}, t) |0\rangle = |1\rangle \quad (12)$$

$$a_{\mu}(\mathbf{k}, t) |n\rangle = |n-1\rangle \sqrt{n} \quad (13)$$

$$a_{\mu}(\mathbf{k}, t) |0\rangle = 0 \quad (14)$$

$$[a_{\mu}(\mathbf{k}), (a_{\mu}^{\dagger}(\mathbf{k}))^n] = (a_{\mu}(\mathbf{k}))^n \quad (15)$$

The Hamiltonian is:

$$\mathbf{H}(t) = \hbar\omega \sum_{\mathbf{k}, \mu} \{ a_{\mu}^{\dagger}(\mathbf{k}, t) \cdot a_{\mu}(\mathbf{k}, t) + 1/2 \} \quad (16)$$

The number operator  $N_\mu$  gives the number of quanta:

$$N_\mu(\mathbf{k}, t) = a_\mu^\dagger(\mathbf{k}, t) \cdot a_\mu(\mathbf{k}, t) \quad (17)$$

The quanta discussed here are bosons. With the electromagnetic field they are photons. Photons have integer spin 1. With the dyadic product  $\otimes$  follows:

$$S_z \stackrel{\text{def}}{=} -i\hbar(\mathbf{e}_x \otimes \mathbf{e}_y - \mathbf{e}_y \otimes \mathbf{e}_x) \text{ and cyclically for } x \rightarrow y \rightarrow z \rightarrow x \quad (18)$$

$$[S_x, S_y] = i\hbar S_z \quad (19)$$

$$S_z \cdot \mathbf{e}_\mu = \mu \cdot \mathbf{e}_\mu \quad (20)$$

Fermions have half integer spin. With fermions the creation and annihilation operators  $a$  and  $a^\dagger$  have different commutation relations. Instead of commuting, these operators anti-commute.

### Measurement

We differentiate between a measurement using a piece of equipment and an observation as is done between items in universe. In the particle view the measuring equipment scrambles the phases. After that scrambling an observation is done. In the wave view the measuring equipment takes care that the phases stay intact, while the amplitudes are ignored during the next observation.

In measurement terms the scramble of the phases is called **de-coherence**. In the same sense the care to keep phases pure and the neglecting of the amplitudes could be called re-coherence. Both actions can be related with the Fourier transforms that convert the wave view into the particle view or vice versa.

### Measurement preparation

In a measurement the observation follows after a preparation phase by the measuring equipment. Such a preparation may squeeze the shape of

the private field that represents the item. For example, a preparation for precise position measurement may squeeze the private field and change the item's subspace such that its range of covered position eigenvectors becomes very short and that its range of covered momentum eigenvectors extends very far. Similarly, when a preparation is made for precise momentum measurement then the item's private field is squeezed and its subspace is changed in the other direction, such that it covers a huge range of position eigenvectors and a very short range of momentum eigenvectors. A Fourier transform does not change the item's subspace. It changes the private field of the item from position based coordinates to momentum based coordinates or vice versa.

Changing the item's subspace such that its range of covered position eigenvectors becomes very short and that its range of covered momentum eigenvectors extends very far is called **decoherence**. In case of a system it reduces the entanglement of that system.

### Hamilton-Jacobi

The Hamilton-Jacobi equation shows how the Hamiltonian relates to the action  $S$  of the current manipulator. In this section we consider  $t$  to be the manipulator time!

$$H \cdot U_t = \tilde{I}_t \cdot \hbar \cdot \frac{\partial U_t}{\partial t} \quad (1)$$

For the eigenvalues holds

$$\Delta u_t \approx \Delta S_{lt} \cdot u_t \quad (2)$$

Thus, we can put

$$H \cdot U_t = - \left( \frac{\partial S_t}{\partial t} \right) \cdot U_t \quad (3)$$

$$(4)$$

$$H = - \left( \frac{\partial S_t}{\partial t} \right)$$

For the expectation values  $s_t$  of the action operator  $S_t$  holds

$$\begin{aligned} \tilde{\mathbf{i}}_t \cdot \hbar \cdot \frac{\Delta S_t}{\Delta t} = & \mathbf{e}_{t0} + \mathbf{e}_{t1} \cdot \Delta t \cdot \frac{\chi_{t1}}{2} + \mathbf{e}_{t2} \cdot \Delta t^2 \cdot \frac{\chi_{t1} \cdot \chi_{t2}}{6} \\ & - \mathbf{e}_{t3} \cdot \Delta t^3 \cdot \frac{\chi_{t1} \cdot \chi_{t2} \cdot \chi_{t3}}{24} + \mathcal{O}(\Delta t^3) \end{aligned} \quad (5)$$

This derivation is completely independent from the observation of Q. Thus  $S_t$  has nothing to do with the Minkowski metric that appears during observations of position.

### The Lagrangian

The Lagrangian is equivalent to the local geodesic equation.

The Lagrangian  $\mathcal{L}_\tau$  is related with the action  $s_t$ .

$$s_t = \int_a^b \mathcal{L}_\tau d\tau \quad (1)$$

The integral is taken over the trail with the observed path. The index  $t$  of the action  $S_t$  is the trail progression parameter. The integration parameter stands for the coordinate time. The right side of the equation plays in Lorentzian space.

The Euler Lagrange equations explicitly use observations. For that reason the Lagrangian is considered to be a function of the observed  $q$ , the velocity  $\dot{q}$  and the coordinate time  $\tau$ . The velocity is measured with the coordinate time.

$$\mathcal{L}_\tau = \mathcal{L}_\tau(\tau, q, \dot{q}) \quad (2)$$

$$\dot{q} = \frac{dq}{d\tau} \quad (3)$$

The Euler-Lagrange equations are:

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial q_i} - \frac{d}{d\tau} \frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial \dot{q}_i} = 0 \quad (4)$$

for  $i = x, y, z$

When the Lagrangian does not vary with one or more of its parameters, then this corresponds with a corresponding symmetry of the system. By [Noether's theorem](#)<sup>171</sup>, such symmetries of the system correspond to [conservation laws](#)<sup>172</sup>. In particular, the invariance of the Lagrangian with respect to time  $\tau$  implies the conservation of energy.

By partial differentiation of the above Lagrangian, we find:

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial q_i} = \frac{\partial U}{\partial q_i} = F_i \quad (5)$$

$$\frac{\partial \mathcal{L}_\tau(\tau, q, \dot{q})}{\partial \dot{q}_i} = m \cdot \dot{q}_i = p_i \quad (6)$$

where the force is  $F = -\nabla U$  (the negative gradient of the potential, by definition of conservative force), and  $p$  is the momentum. By substituting these into the Euler–Lagrange equation, we obtain a system of second-order differential equations for the coordinates on the particle's trajectory,

$$F_i = \frac{d(m\dot{q}_i)}{dt} = m \cdot \ddot{q}_i = \dot{p}_i \quad (7)$$

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<sup>171</sup> [http://en.wikipedia.org/wiki/Noether%27s\\_theorem](http://en.wikipedia.org/wiki/Noether%27s_theorem)

<sup>172</sup> [http://en.wikipedia.org/wiki/Conservation\\_law](http://en.wikipedia.org/wiki/Conservation_law)

which is Newton's second law.

### The world's action

The action  $S_t$  represents the influences that the rest of the world via unitary operator  $U_t$  release onto the state  $\{|f\rangle_s\}$ .

In his book about quantum gravity Rovelli writes:

"In the general relativistic parlance 'matter' is anything which is not the gravitational field. As far as current physics knows, the world is made up of the gravitational field, Yang Mills fields, fermion fields and, presumably, scalar fields."<sup>173</sup>

All these fields give a contribution to the action S.

$$\begin{aligned} S(e, \omega, A, \psi, \phi) & \tag{1} \\ & = S_{GR}[e, \omega] + S_{matter}[e, \omega, A, \psi, \phi] = S_{GR}[e, \omega] + S_{YM}[e, A] + S_f(e, \omega, A, \psi) + S_{sc}[e, A, \phi] \end{aligned}$$

$e$  is the gravitational field.

$A(q)$  is the electromagnetic field.

$\omega(q)$  is the spin connection. It is a one form in the Lie algebra of the Lorentz group  $so(3,1)$

$\psi(q)$  is a scalar field, possibly with values in the representation of the Yang Mills group.

$\phi(q)$  is a field in the spinor representation of the Lorentz group.

$A(q)$  has a non Abelian connection to the Yang Mills group.

The local characteristics of these fields must be represented in the eigenvalue of the current manipulator.

### Representing multiple fields

Professor Mendel Sachs recently wrote a few books in which he promotes the inclusion of more terms in the metric than Einstein did. Sachs uses a

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<sup>173</sup> Carlo Rovelli, book: Quantum gravity, 2004, chapter 2, paragraph 2.1.2

four vector with quaternionic coefficients in order to specify the metric. Sachs uses all sixteen terms, while Einstein skipped six due to symmetry considerations. The argument of Sachs is that the symmetry is broken due to the characteristics of the quaternion number space. See:

<http://www.compukol.com/mendel/publications/publications.html>.

16-ons contain the required 16 real numbers that can be arranged as a four vector with quaternion coefficients. Sachs still uses the Minkowski metric. So, his view concerns observed spacetime.

## Planck limits for all physical observables

This is taken from: <http://www.motionmountain.net/research.html> .

### Basic measures

The basic measures of physics are:

$$\text{The Planck length, } l_{Pl} = \sqrt{\hbar G/c^3} = 1.6 \cdot 10^{-35} \text{ m} \quad (1)$$

$$\text{The Planck time, } t_{Pl} = \sqrt{\hbar G/c^5} = 5.4 \cdot 10^{-44} \text{ s} \quad (2)$$

$$\text{The Planck energy is given by: } E_{Pl} = \sqrt{\hbar c^5/G} = 2.0GJ \quad (3)$$

### Fundamentals

A large part of modern physics can be summarized in four simple and fundamental statements on motion:

quantum theory on action:	$W \geq \hbar$
thermodynamics on entropy:	$S \geq k$
special relativity on speed:	$v \leq c$
general relativity on force:	$F \leq \frac{c^4}{4G}$

These limits are valid for all physical systems, whether composite or elementary, and for all observers. Note that the limit quantities of special relativity, thermodynamics, quantum theory and general relativity can also be seen as the right-hand sides of the respective indeterminacy relations:

length $l$ and acceleration $a$	$\Delta l \Delta a \leq c^2$
the displacement $d$ and momentum $p$	$\Delta d \Delta p \geq \frac{\hbar}{2}$
temperature $T$ and energy $U$	$\Delta \frac{1}{T} \Delta U \geq \frac{k}{2}$
Energy flow $E$ and size $l$	$\Delta E \Delta l \leq \frac{c^4}{4G}$

By combining the three fundamental limits, we can obtain limits on a number of physical observables. The following limits are valid generally, for both composite and elementary systems:

time interval:

$$t \geq \sqrt{\frac{4G\hbar}{c^5}} = 1.1 \cdot 10^{-43} \text{ s}$$

time-distance product:

$$td \geq \frac{4G\hbar}{c^4} = 3.5 \cdot 10^{-78} \text{ ms}$$

acceleration:

$$a \leq \sqrt{\frac{c^7}{4G\hbar}} = 2.8 \cdot 10^{51} \text{ m/s}^2$$

angular frequency:

$$\omega \leq 2\pi \sqrt{\frac{c^5}{2G\hbar}} = 5.8 \cdot 10^{43} / \text{s}$$

Adding the knowledge that space and time can mix, we get:

distance:

$$d \geq \left(\frac{4G\hbar}{c^3}\right)^{1/2} = 3.2 \cdot 10^{-35} \text{ m}$$

area:

$$A \geq \frac{4G\hbar}{c^3} = 1.0 \cdot 10^{-69} \text{ m}^2$$

volume:

$$V \geq \left(\frac{4G\hbar}{c^3}\right)^{3/2} = 3.4 \cdot 10^{-104} \text{ m}^3$$

curvature:

$$K \leq \frac{c^3}{4G\hbar} = 1.0 \cdot 10^{69} / \text{m}^2$$

mass density:

$$\rho \leq \frac{c^5}{16G^2\hbar} = 3.2 \cdot 10^{95} \text{ kg/m}^3$$

### Elementary particles

A particle is elementary if the system size  $l$  is smaller than any conceivable dimension, thus for elementary particles:

$$l \leq \frac{\hbar}{mc} \tag{1}$$

Using this limit, we find the well-known mass, energy and momentum limits, valid only for elementary particles:

$$m \leq \sqrt{\frac{\hbar c}{4G}} = 1.1 \cdot 10^{-8} \text{ kg} = 0.60 \cdot 10^{19} \text{ GeV}/c^2 \quad (2)$$

$$E \leq \sqrt{\frac{\hbar c^5}{4G}} = 9.8 \cdot 10^8 \text{ J} = 0.60 \cdot 10^{19} \text{ GeV} \quad (3)$$

$$p \leq \sqrt{\frac{\hbar c^3}{4G}} = 3.2 \text{ kgm/s} = 0.60 \cdot 10^{19} \text{ GeV}/c \quad (4)$$

### Virtual particles

Virtual particles do not obey the mentioned limits.

### EM limits

Our discussion of limits can be extended to include electromagnetism. Using the (lowenergy)

electromagnetic coupling constant  $\alpha$ , we get the following limits for physical systems interacting electromagnetically:

electric charge	$q \geq \sqrt{4\pi\epsilon_0\alpha\hbar c} = e = 0.16 \text{ aC}$
electric field	$E \leq \sqrt{\frac{c^7}{64\pi\epsilon_0\alpha\hbar G^2}} = \frac{c^4}{4Ge} = 1.9 \cdot 10^{62} \text{ V/m}$
magnetic field	$B \leq \sqrt{\frac{c^5}{64\pi\epsilon_0\alpha\hbar G^2}} = \frac{c^3}{4Ge} = 6.3 \cdot 10^{53} \text{ T}$
Voltage	$U \leq \sqrt{\frac{c^4}{16\pi\epsilon_0\alpha G}} = 1e \sqrt{\frac{\hbar c^5}{4G}} = 6.1 \cdot 10^{27} \text{ V}$
Inductance	$L \geq \frac{1}{4\pi\epsilon_0\alpha} \sqrt{\frac{4G\hbar}{c^7}} = \frac{1}{e^2} \sqrt{4G\hbar^3 c^5}$ $= 4.4 \cdot 10^{-40} \text{ H}$

With the additional assumption that in nature utmost one particle can occupy one Planck volume, we get

charge density

$$\rho_e \leq \sqrt{\frac{\pi \epsilon_0 \alpha c^5}{16 G^3 \hbar}} = e \sqrt{\frac{c^9}{64 G^3 \hbar^3}}$$

$$= 4.7 \cdot 10^{84} \text{ C/m}^3$$

Capacitance

$$C \geq 4 \pi \epsilon_0 \alpha \sqrt{\frac{4 G \hbar}{c^3}} = e^2 \sqrt{\frac{4 G}{c^5 \hbar}}$$

$$= 2.6 \cdot 10^{-47} \text{ F}$$

For the case of a single conduction channel, we get

electric resistance

$$R \geq \frac{1}{4 \pi \epsilon_0 \alpha c} = \hbar e^2 = 4.1 \text{ k}\Omega$$

electric conductivity

$$G \leq 4 \pi \epsilon_0 \alpha c = e^2 \hbar = 0.24 \text{ mS}$$

electric current

$$I \leq \sqrt{\frac{\pi \epsilon_0 \alpha c^6}{G}} = e \sqrt{\frac{c^5}{4 \hbar G}} = 1.5 \cdot 10^{24} \text{ A}$$

Indeterminacy relations:

$$C \text{ capacity and } U \text{ potential difference} \quad \Delta C \Delta U \geq e$$

$$\text{Electric current } I \text{ and time } t \quad \Delta I \Delta t \geq e$$

### Derived limits

The ratio of angular momentum  $D$  to energy  $E$  times length  $L$  has the dimensions of inverse speed. Since speeds are limited by the speed of light, we get

$$D_{system} \leq \frac{1}{c} LE \tag{1}$$

The action limit

$$W \leq LE/c \quad (2)$$

is not exceeded in any physical process.

Since action values in nature are limited from below by  $\hbar$ , we get a limit for the speed of a system:

$$v_{system} \geq \hbar c^2 \frac{1}{LE} \quad (3)$$

This is not a new result; it is just a form of the indeterminacy relation of quantum theory.

Thanks to the connection  $W = FLT$  between action  $W$ , force  $F$ , distance  $L$  and time  $T$ , we can deduce

$$F_{system} \geq \frac{\hbar}{2c} \frac{1}{T^2} \quad (4)$$

The power  $P$  emitted by a system of size  $L$  and mass  $M$  is limited by

$$c^3 \frac{M}{L} \geq P_{system} \geq 2\hbar G \frac{M}{L^3} \quad (5)$$

In 1973 Bekenstein discovered a famous limit that connects the entropy  $S$  of a physical system with its size and mass. No system has larger entropy than one bounded by a horizon. The larger the horizon surface, the larger the entropy.

$$\frac{S}{S_{limit}} \leq \frac{A}{A_{limit}} \quad (6)$$

which gives

$$S \leq k \frac{c^3}{4G\hbar} A \quad (7)$$

where  $A$  is the surface of the system. Equality is realized only for black holes.

We assume that the limits for vacuum are opposite to those for matter.

We can then write

$$c^2/4G \leq M/L \quad (8)$$

for the vacuum.

Using

$$\frac{S}{S_{c.Planck}} \leq \frac{M}{M_{c.Planck}} \frac{A}{A_{c.Planck}} \frac{L_{c.Planck}}{L} \quad (9)$$

we get

$$S \leq \frac{\pi kc}{\hbar} ML = \frac{2\pi kc}{\hbar} MR \quad (10)$$

This is called Bekenstein's entropy bound.

A lower limit for the temperature  $T$  of a thermal system can be found using the idea that the number of degrees of freedom of a system is limited by its surface, or more precisely, by the ratio between the surface and the Planck surface. We get the limit

$$T \geq \frac{4G\hbar}{\pi kc} \frac{M}{L^2} \quad (11)$$

Lower limit for the electric field  $E$ :

$$E \geq 4Ge \frac{M^2}{Q^2 L^2} \quad (12)$$

Lower limit for the magnetic field  $B$ :

$$B \geq \frac{4Ge}{c} \frac{M^2}{Q^2 L^2} \quad (13)$$

### Cosmological limits

Cosmology is characterized via the cosmological constant  $\Lambda$  by the inequality:

$$l \approx \frac{1}{\Lambda} \quad (1)$$

For single particles, the absolute lower speed limit, the cosmological speed limit, is given by:

$$v_{particle} \geq \frac{\sqrt{4G\hbar/c}}{L_{universe}} = L_{corr.Planck} \sqrt{\Lambda} c \approx 7 \cdot 10^{-53} m/s \quad (2)$$

The negative energy volume density  $-\Lambda c^4/4\pi G$  corresponds to a force value

$$F = \frac{\Lambda \hbar c}{2\pi} = 4.8 \cdot 10^{-79} N \quad (3)$$

This is also the gravitational force between two corrected Planck masses located at the cosmological distance  $\sqrt{\pi/4\Lambda}$ .

In nature there is a minimum time interval,  $l_{Pl}/c = t_{Pl}$ , the Planck time.

A recent prediction derived from the standard model of elementary particles give as an upper limit for the electron dipole moment  $d_e$  a value of

$$\frac{|d_e|}{e} < 3 \cdot 10^{-21} m \quad (4)$$

The mass  $m$  of any elementary particle is constrained by the Planck mass  $m_{Pl}$

$$m < \frac{\hbar}{c l_{Pl}} = \sqrt{\frac{\hbar c}{G}} = m_{Pl} = 2.2 \cdot 10^{-8} kg = 1.2 \cdot 10^{19} GeV/c^2. \quad (5)$$

The maximum possible value for mass density  $\rho_{Pl}$  is

$$\rho_{Pl} = \frac{c^5}{G^2 \hbar} = 5.2 \cdot 10^{96} kg/m^3 \quad (6)$$

Within a factor of order one, we find

$$K < \frac{c^3}{G\hbar} = 0.39 \cdot 10^{70} m^{-2} \quad (7)$$

as a limit for the surface curvature  $K$  in nature. In other words, the universe has never been a point, never had zero age, never had infinite density, and never had infinite curvature.

### Limit quality

Nature provides two limits for each observable: a Planck limit and a cosmological limit.

All measurements are limited in precision.

Because of the fundamental limits to measurement precision, the measured values of physical observables do not require the full set of real numbers. In fact, limited precision implies that observables cannot be described by the real numbers.

At Planck scales it is impossible to distinguish between matter and vacuum.

Vacuum and matter do not differ at Planck scales. Similarly, at the Planck length it is impossible to distinguish between positive and negative time values: so particles and antiparticles are not clearly distinguished at Planck scales.

The strictest upper limits are those with the smallest exponent for length, and the strictest lower limits are those with the largest exponent of length.

The accuracy of time measurements is limited by the Planck time  $t_{\text{Pl}}$ .

The accuracy of length measurements is limited by the Planck length  $l_{\text{Pl}}$ .

All measurements – be they measurements of position, speed, mass or any other observable – are electromagnetic. In other words, all measurements in nature are detection of photons. And in strand theory photon absorption and detection are intimately related to the crossing switch.

All electromagnetic information is communicated by directed information carrying quanta in the form of shot noise. However, secondary information can be derived from the shape of the quantum cloud.

### References:

More useful stuff is collected in the [toolkit](#)

Axiomatic Quantum Theory, W. Lücke, [http://arxiv.org/PS\\_cache/quant-ph/pdf/9510/9510024v2.pdf](http://arxiv.org/PS_cache/quant-ph/pdf/9510/9510024v2.pdf)

An overview of gravity theories:

[http://arxiv.org/PS\\_cache/arxiv/pdf/0909/0909.4672v2.pdf](http://arxiv.org/PS_cache/arxiv/pdf/0909/0909.4672v2.pdf).



# Part three

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How the brain works

# How the brain works

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## Abstract

This paper analyses the associative capabilities of the brain and takes the consequences of that capability.

## Pre-processing

A study on how the environment is observed and interpreted should start with an investigation of how the sense-organs and the brain cooperate. Between the sense-organs and the brain exists a series of pre-processors that encode and pre-interpret the incoming signals. This process also performs some noise filtering, such that later stages of the processing are not bothered by misinformation. For that reason the pre-processors act as decision centres where the signal transfer is blocked when the signal to noise ratio stays underneath a given level, e.g. 2.3 (Crozier's law. The level may differ in different persons.). In this way the visual trajectories run via a cross-over to the cortex. The cross-over encodes and adds depth information. After a series of additional pre-processing steps the signal arrives in the fourth cortex layer. Here about four square millimetres is devoted to the direct environment of each receptor of the fovea. In this area a complete geometric encoding of the local geometry and dynamics of the perceived picture is presented. This includes whether the detected detail is a line or an edge or another form, in which direction it is positioned and whether the detail moves. (See the papers of Hubel and Wiesel on the visual trajectory and the visual cortex for more detailed information).

## Processing

Thus, the brain does not work with a pictorial copy of the picture that is received on the fovea. In further steps the encoded map is interpreted.

That part of the brain tries to associate the details of the map with remembered and recognized items. When dynamics is considered then it must also be considered that the eyes are continuously scanning the input scene.

## **Image intensification**

I studied visual perception because I needed this to specify useful measuring standards for night vision and X-ray imaging equipment (~1975). Many of the known visual illusions are due to the pre-processing in the visual trajectory. The viewing chain includes lenses, image intensifier tubes and either a camera or the human visual system. This last component includes the eye ball. The object is noisy and can be considered as a Poisson process. With respect to the noise the optical components act as binomial processes. Their point spread functions act as integration area. Image intensification is usually a Poisson process, but channel plates are characterized by an exponential distribution rather than by a Poisson distribution. Chains that include Poisson processes and binomial processes can be considered as one generalized Poisson process. Thus, imaging chains that include channel plates are more difficult to characterize.

## **Imaging quality characteristics**

When the imaging chain can be characterized by a Poisson process, then its quantum detection efficiency can be characterized by the Detective Quantum Efficiency (DQE). Its optical imaging quality can be characterized by the Optical Transfer Function (OTF). With inhomogeneous light imaging it is sufficient to use the modulus, the Modulation Transfer Function (MTF). The MTF of the chain is the product of the MTF's of the components of the imaging chain.

## Vision of noisy images

The intensification of image intensifiers is such that at low radiation levels the output image is formed by large numbers of separate light dots that together give the impression of a snowy picture. The visual trajectory contains a sequence of pre-processors that each performs a part of the encoding of the object. At its input the visual cortex gets an encoded image rather than an optical image of the perceived scene. This encoded image is further encoded and interpreted in channels higher in the brain. This is done by associating the elements of the encoded image that is entering the visual cortex. The folded visual cortex offers about four square millimetres for the encoding of the environment of each separate receptor in the fovea. The pre-processors act as decision centres. When the offered signal to noise ratio is too low then nothing is passed. This is a general principle in the encoding process and also governs the association of encoded data in other parts of the brain.

The research resulted in a significant contribution of our laboratory to the world standards for the measurement of the OTF and the DQE.

## Information association

The associative nature of the process is common for all kinds of objects and parts of objects. That includes objects that did not enter through one of the sense-organs. For example a house is not stored in the brain as a complete concept. It is stored as a series of details that can be associated to the concept. If a sufficient number of these details are detected then a decision centre in the brain decides that the whole concept is present. In this way not only a particular house can be recognized, the process can recognize a series of objects that resemble the original house. It classifies houses. By adding details that can be associated with it, the concept of a house can be widened. The resulting information, i.e. the information that passed the decision centre, is used for further reasoning. Together with other details the same details can also be used to detect other concepts by a different association. When the association act still produces too much

noise, then the information is not produced and further reasoning is neither disturbed nor triggered by this fact. High enough in the hierarchy individuals can be discerned. The brain is not static. The network of communication paths and decision centres is dynamically adapted to the changing needs.

## Noise filter

The decision level for the signal to noise ratio may vary from person to person. If the level becomes too low the person may start hallucinating. Further, the level may be influenced by body owned messenger stuff, drugs, poisons and medicines.

## Brain waves

On the other hand one must not underestimate the positive value of noise. Noise may be the stimulus of new thoughts. These are built upon existing association networks to which via noise or via a new impression a new aspect is added. This may lead to several conclusions that did not exist before.

## Reasoning

The brain is capable to perform complex reasoning. However it must be trained to perform the reasoning in a logical way. For example, it must learn that the start from a false presumption can cause the deduction of any conclusion, just or false. When a path of reasoning is helpful, then it is stored in a similar way as an observation. Not the reasoning itself is stored, but the details that are part of the reasoning path. Also here association of the details and a suitable noise threshold plays its role. The reasoning can be identified as a theory and its concept can be widened. The brain can also generate new details that together with existing details can act as a reasonable theory. Even noise can generate such signals. These details can be perceived as a dream or as a newly invented theory. It depends whether the theory is accepted as realistic. That means that the

brain must be capable of testing the realism of a theory. This testing can be improved by training. The brain can forget stored details and stored concepts. This holds for objects as well as theories. Valuable concepts are regularly refreshed and become better remembered.

## **Other species**

Hubel and Wiesel did their experiments on several kinds of vertebrates, such as goldfishes, cats and humans. Their main target was visual perception. Where the handling of the signals of sense organs in the brains is quite similar for all vertebrates, the handling of paths of reasoning by humans is superior in comparison to other vertebrates.

## **Humans**

Humans have an advantage over other vertebrates. Apart from direct observation the theories and the concepts of things can also be retrieved by communication with other parties. This occurs by education, discussion, reading books, papers or journals, seeing films or videos or surfing the internet. These media can also act as a reference medium that extends the storage capacity of the brain.

## **Science**

Mathematics is a particularly helpful tool that extends the capability of the brain to perform reasoning in a logical and precise way. Physics extends this capability further with focus on observables. Philosophy adds self-reflection and focuses on the why and how of existence. Every branch of science adds to the capabilities of the individuals and to the effectiveness of the community.

## **Physical reality**

Our brain has a limited storage capability. We cannot comprehend things that have an enormous complexity. However we can detect regularities.

Our brain is optimized to detect regularities. The laws of physics appear regularly in our observations or can be deduced from regularly returning observations. More complex laws are derived using tools and in combination with other people. Nature is not only controlled by laws. It is also controlled by boundary conditions. These boundary conditions may be caused by the influence of items that lay beyond the reach of our direct observations. The number and complexity of boundary conditions far outgrows the number of recognized laws of nature. The laws of nature play a role in our theories. However, the boundary conditions play a much smaller role. This is because the laws of nature that we detect treat a simplified version of the environment. In this abstraction the boundary conditions play no realistic role. This is another reason why our theories differ from physical reality.

## Theories

These deliberations learn that theories are a product of our mind. They can be used as a looking glass that helps in the observation and interpretation of physical reality. However, it is false to interpret the theories as or as part of physical reality. When a theory fits, then it is congruent, to some extent, with physical reality. That does not say that we as human beings and the environment from which we take our observations are not part of reality. It says that what our brain produces is another thing than physical reality.

## Inventions of the human mind

Infinity is typically an invention by the human mind. There exist strong indications that nature does not support infinity. In the same sense unlimited precision real numbers are prohibited in the physical universe by the holographic principle and the Bekenstein bound. However, we can embed the results of our observations in a model that includes infinities and unlimited precision. For example classical mechanics and field theories use these concepts. Quantum mechanics shows us that as soon as

we introduce unlimited precision we are immediately confronted with Heisenberg's uncertainty principle. We need infinity and unlimited precision in order to resolve the paradoxes that otherwise creep into our theories. We use theories that are in direct conflict with each other. One forbids infinity, the other theory uses and requires it. This says at least one thing; none of the theories describes physical reality correctly. Thus none of the theories can replace the concept of physical reality. Still it appears useful to use both theories side by side. It means that great care must be taken with the interpretation of the theories.

## History

Mathematical theories and physical theories tend to build upon the results of other exact theories. After some generations a very complex building is obtained. After a while it becomes humanly impossible to check whether the building elements are correct and whether the binding is done correctly. So, complex exact theories should be questioned.

## Dreams

In this sense, only when we study our own dreams, fantasies or theories, then we observe these items and the dreams; fantasies and theories become part of "physical reality". If the theory is congruent with a part of physical reality, it will become useful as a view on physical reality.



# Part four

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**A law of nature**

# A law of nature

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## Abstract

Complexity plays an important role in all kinds of human activity as well as in nature. Complexity can be defined in terms of the number of relations that must be handled accounted relative to the number of potential relations. Modularization is an efficient methodology that helps reducing the number of relevant relations. It has the property that it becomes more efficient when the availability and the diversity of modules that can be coupled increase. Its efficiency can grow exponentially when modules can be generated out of simpler modules. In nature this effect leads to the generation of very complex creatures, such as intelligent species. In fact it is possible to interpret this tendency as a new law of nature.

## Complexity

Relations can be encountered in several areas of physics and in human interactions.

We will define complexity in terms of the number potential relations divided by the number of relevant relations.

There exists a tendency in nature to reduce complexity via modularization.

This tendency grows when more suitable modules become available. Finally this tendency enables nature to create very sophisticated and intelligent creatures.

## Potential complexity

Potential complexity of a set of objects is a measure that is defined by the number of potential relations that exist between the members of that set.

Actual complexity of a set of objects is a measure that is defined by the number of relevant relations that exist between the members of the set.

It takes time and other resources to determine whether a relation is relevant or not. Only an expert has the knowledge that a given relation is relevant. Thus it is advantageous to have as little irrelevant potential relations as is possible, such that mainly relevant and preferably usable relations result.

If there are  $n$  elements in the set, then there exist  $n*(n-1)$  potential relations.

## Physical relations

Physics is based on relations. Quantum logic is a set of axioms that restrict the relations that exist between quantum logical propositions. Via its isomorphism with Hilbert spaces quantum logic forms a fundament for quantum physics. Classical logic is a similar set of restrictions that define how we can communicate logically. Quantum logic only describes static relations. Traditional quantum logic does not treat physical fields and it does not touch dynamics. However, traditional logic can be extended such that physical fields are included as well and by assuming that dynamics is the travel along subsequent versions of extended quantum logics, also dynamics will be treated. The set of propositions of traditional logic is isomorphic with the set of closed subspaces of a Hilbert space. This is a mathematical construct in which quantum physicists do their investigations and calculations. In this way fundamental physics can be constructed. Here holds very strongly that only relevant relations have significance.

## Modularization

Actual complexity can be reduced by modularization, by applying standard module interfaces and standard inter-module communications, by increasing the diversity of modules and by increasing the availability of modules. Another important factor is the ease of modular system configuration.

Modularization can easily reduce the relational complexity of large complex systems by several orders of magnitude. The resources needed to devise and generate that system reduce by the same factor.

Nature secures its modules literally by brute force or when those binding forces are not available by enveloping the module in a skin such that only controlled access must be accepted by the module.

## Modular system design

The system configuration process profits most from modularization, but reuse of modules and interfaces also saves resources and building time. The capability to create modules out of simpler modules gives the improvement an enormous boost.

The complexity of a modular system can be many orders of magnitude less than the complexity of a monolith.

Systems can be atoms, molecules, (in-) organic stuff, organs, living objects, products, modules, modular subsystems, organizations, governments, stars, galaxies, et cetera.

Two kinds of modular system design exist. Modularization can be done randomly, as is done in nature by the evolution process, or it can be done in an intelligent way as is done by human system designers.

1. Stochastic module and modular system design. In this way modules and their coupling are created via stochastic processes. This way of system creation is used by nature.
2. Intelligent module and modular system design. In this way modules and their coupling are designed via well planned processes that are controlled by intelligent individuals.

The efficiency of modular system generation is influenced by the following factors:

- Resources needed to generate modules
- Availability of modules
- Diversity of modules
- Availability of provide interfaces
- Availability of require interfaces
- Diversity of interfaces
- Mutual static adaptation of provide and require interfaces
- Mutual dynamic adaptation of modules via their interfaces

In intelligent modular system design the following extra factors play an important and stimulating role:

- Public knowledge of available modules and their static and dynamic specification
- Public knowledge of available interfaces and their specification
- Availability of standardized module design tools
- Availability of standardized interface design tools
- Availability of standardized modular system configuration tools
- A market of standardized modules
- Publicly accessible repositories where automatically and humanly readable modules are published
- Publicly accessible repositories where automatically and humanly readable interfaces are published

A large availability, couple-ability and diversity of modules ease and stimulate the system configuration both in the random trial and error approach as well as in the case of intelligent system configuration.

Too much diversity works negative. So there exists an optimum diversity both for modules and for interfaces.

The same holds for availability. Too much availability exhausts precious resources, which are not used effectively and could be better used otherwise.

## Intelligent modularization

Generation, management, improvement and support of modular systems normally cost a fraction of the time and the resources that the equivalent for a monolith or a layered system takes.

In intelligent modularization publication of the capabilities of modules and interfaces will reduce the complexity and increase the efficiency of system configuration and will increase the effectiveness of module and interface design.

Intelligent designers of modular systems secure the integrity of their modules by power such as a patent mechanism or by proper encapsulation. The first measure secures the intelligence property that went into the design. The second measure guards against unwanted access that may hamper the integrity of the module. Still the module must have publicly accessible interfaces.

Providing tools for component and interface generation and providing system configuration tools will also reduce the complexity of system generation. Via modularization the complexity of system configuration can be reduced so strongly that it becomes possible to automate the configuration process.

In intelligent modularization interfaces make only sense when they are standardized, when they are well known and when their use is widely spread.

When they are well known, connected interfaces can be seen as a single relevant relation. In this way they replace a number of potential relations.

## Coupling

Provide interfaces couple the server module with the require interface(s) of one or more client modules. Communication occurs via relations and via relation paths. It must be done via a well-defined protocol. Otherwise the communication makes no sense. Interfaces can couple modules directly or they can couple via communication channels. Another way of coupling occurs via broadcasting controls or messages.

Intelligent communication is usually regulated via standardized protocols. The regulation may concern a handshake, a send with immediate return or a single send or receive. A broadcast is a special type of communication.

In high urgency communication and in message streaming measures must be taken against deadlocks and race conditions.

An important aspect of functioning of interfaces is the quality of coupling. It is important with which partner they couple. It might even be a matter of life and death. If the serving partner has no intention to save the integrity of the coupled module and the module has insufficient defense against such attacks, than that client module may get disturbed.

When the modules have the capability to generate siblings, as is the case in biologic systems, then the quality of mating is important for the success of the siblings. Thus in evolution not the survival of the fittest, but the quality of coupling is of crucial importance for the survival of species. In more abstract sense, the quality of coupling is important for the abundance of module types.

## Interfaces

Interfaces have a type definition. Not only the static aspects of the interface coupling play a role, also the dynamic properties may be important.

Interfaces have several general aspects. Interfaces have static characteristics and dynamical characteristics that mostly can be specified separately. An interface can be a require-interface or it is a provide-interface. Often a require-interface and a provide-interface are combined in a single physical realization. In intelligent system design the require-interface is often just a link with a type definition. It may fit at several types of provide-interfaces. The capabilities of this server may be larger than required by the client.

For example in software there exist SW-SW interfaces, SW-HW interfaces, streaming interfaces and notification interfaces. A given environment may ask for an adapted physical implementation of the interface.

Communication-channels may connect require interfaces to provide interfaces.

In nature in most cases physical fields play the role of interfaces. In biology and technology all kinds of interfaces are explored.

Thus, there is a lot to be said about abstract interfaces.

## **Principle of minimal action**

The principle of minimal action is based on the tendency of the fields to keep coherence between subsequent dynamical steps.

If you accept that action stands for change, thus for change of properties of particles, thus for change in the relations between particles and that entropy stands for potential change, then the ratio of action and entropy stands for the relational complexity of nature. Thus the principle of minimal action stands for minimizing the relational complexity. Thus, even in its fundamentals, nature has a built-in tendency to minimize relational complexity.

## **The power of modularization**

Modularity enables and stimulates nature to create sophisticated creatures via a random process called evolution.

The intelligent modular system creation process enables humans to create very sophisticated systems in much shorter time than nature took in order to create smart individuals.

The laws of entropy are directed towards the increase of chaos. The tendency to reduce complexity via modularization works in the reverse direction. When more capable modules come into being, then it becomes possible to generate more complex systems and it becomes also possible to construct more complex modules out of these modules.

Both nature and intelligent system designers make use of the advantages of modularization. It enabled nature to create intelligent and very sophisticated creatures, such as human beings.

Evolution is a stochastic process. Set against the fact that the system generation process acted by human individuals profits from the intelligence of the actor, nature has enormous resources like time, energy and matter. This gives nature an enormous potential. If one starts from the assumption that universe is its own creator, then it creates all of its parts. That also holds for such immensely complex creatures as human beings. The law of increasing complexity reduction via advancing modularization forces nature to create such hyper-complex things. Modularization is a very complex concept. Its aspects may easily fill a thick book.

It would be best to put the influence of modularization into a law of nature, but due to the complexity of the modularization process it is difficult to formulate such a law.

## Abusing modularization

The fact that the advantages of modularization can be denied is shown by several human system generation processes. The most prominent are the ways software is generated and the way that fusions of large organizations are promoted.

See:

[Story of the war against software complexity.pdf](#); part six of this book  
[Managing the generation process.pdf](#) ;part seven of this book

## Software

The fact that modularization is not or hardly applied in software is responsible for the exponential growth of the costs of the development and generation of complex software systems. The software industry is responsible for this deficiency. Their customers suffer.

A demo project showed that if modularization is fully implemented in embedded software generation, then:

- The whole world can contribute in the design of a large variety of modules.
- The operating systems can be generated by the system configuration tool from dedicated modules that are adapted to the needs of the collected application modules.
- The configuration tool can simulate and test a large part of the envisioned system that is generated from such a dedicated operating system and skeleton components that are generated on the basis of specifications that are taken from machine readable repositories. The skeletons can be replaced one by one by real binaries and the system can be tested at each phase.
- Such a generation process can be largely automated and can be controlled by a creative system designer rather than by a genial system architect that takes a high risk to get a burnout when the project fails.
- The overall system generation will take a few weeks of a few specialists compared to the hundreds of man years that are currently spent on comparable projects.
- Compared to the current way of system generation the process is very democratic. Everybody who knows a niche of expertise can contribute his modules.
- Compared to the open software solution, everybody who generates modules is allowed to earn money from the investment of his intelligent property.



# Part five

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**Story of a war against software complexity**

# Story of a war against software complexity

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## Abstract

This is the account of the course of a project that had the aim to improve the efficiency of embedded software generation with several orders of magnitude. All factors that determined the success of the project are treated honestly and in detail.

## Prelude

In 1995 the physicist Hans van Leunen was employee of an internal software house of a large electronics company. His speciality was the creation of scientific software. Then he got the invitation from a software strategist Henk de Vries to join the semiconductor department in order to resolve a quickly emerging problem. The costs of complex embedded software were growing exponentially and this would cause severe problems in the next future. The reasons why the costs of software generation grow exponentially are the growing size and the growing complexity of the embedded software that goes into high-tech appliances. One of the reasons of the non-linear growth of costs is the growth of complexity. But the exponential growth of costs is mainly caused by surpassing of the available resources, which on its turn required measures against expected internal and external damage claims. In many cases software projects were stopped when the costs were expected to explode, or when they did not seem to reach the expected result.

## Analysis

There exist several possible solutions to this dilemma. One is to move the software development to low wage countries. Another is to apply open source software. A third possibility is to increase the quality of the

software generation process. One way to do this is to improve the control of the flow of the generation process. Another way is to improve the way that software is generated. The management of the electronics company tried all these possibilities. Improving the control of the flow of the software generation process has little sense when the generation process itself has severe defects.

## Setting

The electronics company is successful in the generation of hardware. This is for a large part due to the fact that hardware is generated via a modular approach. This is one of the reasons that the research lab of the firm created a dedicated way to create modular embedded software. However, this is a rather closed system and it is directed to the direct need, the generation of software for consumer appliances. Still it is intuitively felt that a modular approach will improve the effectiveness of software generation. There are also many objective reasons for this point of view.

## History

The electronics firm was not very successful with its software projects. Many software projects were stopped after having burned hundreds of man years and millions of dollars, leaving the project leaders, system architects and the software designers back in despair. For that reason relief was sought in outsourcing of the software generation. One form of it is the use of open source software. Parallel to it the internal software generation was moved for a significant part to low wage countries like India. This was only a short time solution. The exponential growth of costs took its toll there as well.

Also the switch to open source software was no smart decision. The electronics firm had no control over the way that the open source software was evolving and the open source software generation suffered the same bad habits as the present-day commercial software generation

process does. Commercial software generation and open source software generation are both non-modular. There exists no healthy and lively software modules market that stimulates the diversity, availability, accessibility and favourable quality/price ratio that characterizes the hardware modules market. Thus, the high-tech hardware appliances industry is still confronted with the negative aspects of the current software generation technology. It drives their costs high and the fragility of the software is transferred to the hardware products that include the software. The ineffectiveness of the software generation affects the affordability and the time to market of the hardware products.

## Strategy

A small group of experts consisting of the software specialist Hans van Leunen, the software strategist Henk de Vries and a software marketing specialist William Vanderboon Ringer studied the resulting possibilities and concluded that a drastic change in the way that software is generated is a promising solution of the problem. The way that hardware is generated was taken as an example. Hardware is generated mostly in a modular way. Modularization reduces the relational complexity of the design and construction process. It also enables partition and delegation of the design and construction work. It even enables a flourishing modules market.

## Approach

The group tried to interest vendors of embedded software generation tools to join the enterprise. It was obvious that international standards would play a crucial role. So, the group stimulated the management of the electronics firm to involve other electronics firms and the OMG. All these measures lacked sufficient success. The tool vendors were interested, but used the opportunity to monitor whether their current way of operation was endangered. They did not really take part in the development. The other electronics companies took the role of an observer and asked for a

convincing demo of the concept. OMG lets standards create by the interested parties. It is not usage that the standard is introduced by a single company.

## What happened

The group encountered severe resistance against their intentions from internal software development groups, because it was expected that the generation of the modules would be outsourced to the suppliers of the software modules market. This fear is realistic. On the other hand it became more and more clear that the internal software generation capabilities were not measured up against the task to create large and complex embedded software systems. Several costly debacles proved this. Especially managers, including the managers of software groups, showed that they lacked a proper feeling for the factors that influence complex software generation.

## Attack

The group decided to create a demo version of the modular software generation system that included major parts of the envisioned system. This includes software module development tools, system configuration tools, web and local file based repositories that act as searchable exchange places for machine and humanly readable specifications of modules and interfaces and central services that act as a marketplace for software modules. The module development tool can generate skeleton modules and it can generate the interface definitions from specifications that are retrieved from web based or local repositories. The tool helps filling the skeletons with working code. The configuration tool retrieves specifications of modules and interfaces from the repositories. It can retrieve the binaries of modules from the market place or from a local store. It enables the mostly automatic assembly of modules into target systems. It adds a dedicated RTOS that consists of automatically generated modules. The RTOS provides automatic memory garbage

collection. The central service collects specifications from the module developers and distributes these to the web based repositories. The central service also collect the binaries of the modules and stores the specifications and the binaries in its banks. The central service acts as a modules market.

## Set-back

The project of the group was severely hampered by the dot com crisis in 2001. This stopped all long term research projects and brought the funding of the group to a minimum. The development of the demo continued at a low pace and stopped in 2004. At that time most planned parts of the demo worked at least for a large part.

## Remnants

The central service worked partly. The development tools are functioning. Modules can be generated and the configuration tool can assemble systems from these modules and add a service layer that consists of automatically generated dedicated modules. The service layer includes garbage collection. It uses connection schemes and scheduling schemes that dynamically control the switch between system modes. The created system does not contain a HAL and it does not contain interrupt services. Instead it relies on the services of a virtual machine or a POSIX OS. This is not the target to work on top of hardware but it is good enough for most demonstration purposes. The tools generate software in C++, but as a bonus it can deliver C# code. That code works on top of a dotNet virtual machine. The tools and central services are written in C#.

Apart from SW/SW interfaces the modules may contain HW/SW interfaces. Streaming interfaces and the notification interfaces that handle interrupts were planned. The skeleton of the modules are modelled after Microsoft's Component Object Model (COM), but the IUnknown interface is replaced by the IAccessor interface. That interface replaces the AddRef

and Release functions with a ResetInstance routine. Instead of the designer, the system is made responsible for the garbage collection. For that reason the new module skeleton is named Robust Component Object Model (RCOM).

## Goal

The demo was planned to demonstrate the generation of real-time embedded software. That goal is not reached. However, many aspects of the planned target are shown in the completed part of the demo. That part offers trust in the feasibility of the final goal.

## Lessons

The project also learned many valuable lessons.

- The current suppliers of software generation tools are not interested in a drastic change in the way that software is generated.
- Despite the fact that embedded software is causing major problems, the companies that produce high-tech appliances or high-tech systems are hesitating to cooperate in improving the software generation process. Software generation is not their strength.
- This world is not good in organizing actions that are rather complex. For that reason it is difficult to arrange standards on new subjects.
- It is difficult to motivate management to enter new inroads when the reasons are not very simple and require insight in the topic.
- Managers of these days are interested in short term low risk solutions. They are not interested in long term solutions even when they promise high profits.
- The same holds for today's investors.
- Although most involved people intuitively see that a modular approach provides a better effective generation process and easier

support management, most of these people forget that without a suitable modules market the modules are too expensive and too scarce to make the assumption true.

- A modular system generation approach has no sense when it does not include an integrated and well functioning modules market. This also means that a system of web based and local repositories that contain the specifications of modules and specifications must be involved as well.
- Given enough resources, even a tiny group of determined software experts can design and construct a working version of a modular software generation system that includes all essential parts.

## Conclusions

This world is not good at creating new standards. However, we are good in accepting default standards. Large electronic firms seem incapable of creating a suitable software generation system. Understandably, the existing software industry appears not willing to give up the profits that they retrieve from the current deplorable way of software generation.

## Way out

There still exists a possible way to get out of this misery. When a small group of enthusiastic software developers and venture capital investors start with a project that establishes a working version of a modular software generation system that includes all ingredients to get a successful result, then they may cause the seed that will extend like an oil drop and smother the current way of software generation.

In a world where such a system exists the complex software assemblies are no longer created by a genial system architect and hundreds of man years of expensive programmers but instead by a creative modular system assembler that uses automated tools to construct his target in a fraction of the time, with a fraction of the resources and with a fraction of

the costs compared to his present-day colleague. He retrieves his modules from a modules market and he may also design and produce some missing modules. In a later phase he may decide to offer these new modules on the market.

His present-day colleague produces software systems, whose structure resembles a layered set of patchwork blankets. Even the most ingenious architect cannot oversee the details of this complex architecture. Therefore the system cannot be completely described properly. Thus, it cannot be tested fully and nobody can guarantee its proper functioning. Modular systems are inherently less complex. Especially its system configuration is orders of magnitude less complex. This results in a better manageability of the complexity and a higher robustness. On its turn it results in a better chance to be able to guarantee its proper functioning.

## Discussion

The modules market is very democratic. Everybody that owns an appropriate modules development system can participate and fill a niche of the modules market. The modules market is a good replacement of the market for open source software. It has the advantage that the module developers can earn money for the intellectual property that they invested in the design and construction of the module. Still the products stay very affordable. In contrast the open source software community is non-democratic. In many cases the community forbids the contributing software developers to earn money from their intellectual property investments.



# Part six

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**Managing the software generation process**

# Managing the software generation process

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## Abstract

The current software generation process is rotten. This paper analyses why that is the case and what can be done about it.

## Software complexity

It is no secret that the generation of complex software poses great problems for its producers. The cost is growing exponentially with their size and the time from conception to finalization grows likewise. The resulting products are fragile and force the vendors to reserve sufficient resources to cope with future warranty and damage claims. Buyers are aware of this situation but without reasonable alternative they are ready to live with the situation. The source of the misery is the complexity of the software and this complexity is mainly due to the relational complexity of its constituents. A radical modular approach as is applied in hardware system generation would cure the problem, but that requires a completely different way of software generation and software marketing.

## Introduction

First the factors that hamper efficient system generation are treated independent of the application area. Then the solutions for eliminating these factors are given. Next the differences between the hardware area and the software area are shown in an historic view. Finally a possible improvement of the software case is sketched.

## Managing complexity

### Breaking level

Managing simple projects hardly ever poses problems. However, a situation in which complexity surpasses the boundary where a quick view no longer reveals potential problems requires special methods. These measures compensate or cure the lack of overview. The level of the boundary depends on the number of items involved in the process and on the nature of the relations between these items.

### Measure of complexity

The number of potential relations between the items involved in the process explains a close to quadratic growth of potential complexity with the number of items involved. Between  $N$  items exist  $N \times (N-1)$  potential relations. Usually only a small percentage of the potential relations are truly relevant relations. Dynamically relevant relations are the potential carriers of communication and control signals. They carry the activity and determine the capabilities of the considered system. It takes expertise knowledge to decide whether a potential relation is dynamically relevant. Gaining this expertise takes time and other resources. This explains why all potential relations have a direct impact on manageability. For that reason, the number of potential relations may act as a rough measure of potential complexity. Similarly the number of dynamically relevant relations may act as a rough measure of the actual complexity of the system. More precise measures will also consider the type of the relations. The type of the relation determines how that relation must be treated.

Procedures such as modularization of the system and categorization and grouping of the interrelations into interfaces significantly reduce the actual complexity of system design and creation. Each interface represents a well defined group of dynamically relevant relations. Well known interfaces contribute significantly to the reduction of complexity. They reduce a set of interrelations to a single relation. Modules can be assembled into systems by connecting them via compatible interfaces.

Both the modules and the interfaces that couple these modules are of crucial importance for managing the complexity of system generation.

### **Extreme complexity**

Very high degrees of complexity may introduce secondary effects that impair manageability far more severely than can be explained by the number of potential relations between the items involved in the system generation process. This occurs when it becomes humanly impossible to properly specify the activity of all dynamically relevant relations.

The inability to specify the product, implicates the inability to test it and as a consequence it implicates the inability to guarantee the proper functionality of the system. The implications of the lack of resources that are required to cope with complexity and the inability to specify the situation in sufficient detail can easily raise costs in an exponentially increasing way. Apart from causing unacceptably growing costs, the system generation process yields fragile results. The resulting product may even endanger the environment where it is applied. This requires reserving resources to insure resistance against future claims.

## **The modular approach**

### **Modularization**

The reasons why modularization significantly improves manageability of the generation process are manifold. For example, it may be possible to delegate the design or the creation of modules to other parties. Potential reuse of existing modules or their design is another important reason. However, the most important reason for applying modularity is the fact that proper encapsulation of the modules and the use of well known interfaces significantly reduce the number of dynamically relevant relations.

A simple example may explain this. A monolithic system consisting of 1000 items contains 999.000 potential relations. Its relational complexity

can be characterized by this number. A comparable modular system that consists of ten modules contains far less potential relations. Let the modules be coupled by well-known interfaces and let part of the interfaces be similar. Not every module connects to every other module. Let the largest module contain 200 items and let the total number of interfaces between any pair of modules be less than 5. The largest module has a potential relational complexity of 39.800. The complexity of the other modules is less. Thus the relational complexity met by the module designers is less than 40.000 and for most modules the relational complexity is less than 10.000. Between modules the interfaces take the role of the relations that are the internal members of these interfaces. The system designer is confronted with a relational complexity that is less than 100. The benefits of the reuse of interfaces and the advantages of the possible reuse of modules should also be considered. Thus compared to the monolithic case there is an increase in manageability of several orders of magnitude. Modularization of larger systems may offer benefits that are much higher. Diminishing relational complexity translates directly in lower man costs and in shorter time to realization. Further it has a very healthy effect on the robustness and reliability of the end product.

### **Modular system design**

The system designer gets the strongest benefit from the modularization. Modularization simplifies system assembly significantly. This opens the possibility to automate the system integration process.

Modularization reaches its highest effectiveness when the design and creation process enables the assembly of modules out of other modules. In this way the microelectronics industry reaches the exponential growth of the capabilities of integrated components that is known as Moore's law.

### **Interfaces**

In the design of a system the introduction of an interface increments the number of potential relations. However, because the interface encapsulates a series of dynamically relevant relations, the total relational

complexity will decrease. The new relation will only play a dynamic role when the corresponding modules are coupled or decoupled. This coupling can be done at system assembly time or during the operation of the system. At instances where no coupling or decoupling is performed the new relation acts as a static relation. It relays the communication and control signals to the dynamically relevant relations that are members of the interface. In the count for complexity a well known interface replaces the combined contributions of its members. In that view, it can be considered as a single dynamically relevant object.

Dynamically relevant relations are carriers of information or control signals. Depending on the direction of the control signal the corresponding interface member belongs to the require part of the interface or to the provide part of the interface. In the first case it acts as the sender of control signals. The require part of the interface contains members that belong to the current client module. If the interface member acts as the receiver of control signals, then the interface member belongs to the provide part of the interface. In return the interface member causes the module to deliver corresponding services. The provide part of the interface belongs to the module that acts as the current server. In order to become active the require part of the interface of the client module must be connected to the provide part of the interface of the module that acts as the server.

A module may act as a server at one instance and it may act as a client at other instances. In each of its roles it will use the appropriate provide or require interface parts. Multitasking modules may provide parallel actions.

In the assembly the coupling of the require interface part and a corresponding provide interface part may be stationary or it may be temporally. The provide interface part of an interface may serve one or more other interfaces. The service may be presented in parallel or in sequential order. The specifications of the provide interface part must at

least cover the requirements of each of its customers. With respect to its potential capabilities, the provide interface part may offer more than is requested by a coupled require interface part. The specification of the provide interface part must be in accordance with the specification of the require interface part, but this only holds for the part that covers the services that the require interface part may demand.

In many cases the trigger of a provide interface member by a connected require interface member will not only result in an action of the server module. It may also cause the return of a response via the same connection. The response can be used for synchronization purposes and it may contain requested information.

In general an interface may contain both a provide part and a require part and the partition may change dynamically. It is difficult to understand and handle such mixed interfaces. When manageability is strived for, then mixed interfaces must be avoided. An exception exists when the communication requires a handshaking process. Preferably pure interfaces should be used. A pure interface contains either a require part or a provide part but not both. In the simplest case the specification of a require interface closely matches the specification of the corresponding provide interface.

### **Proper modules**

Proper modules are properly encapsulated. A proper module hides its internals. Securing the intelligent property that went in its design is one of the reasons for this strict measure. Preventing unwanted access to the module is another reason. Proper modules can only be accessed through publicly known and well specified provide interfaces. A module is a part of an actual system or it is targeted as a part of one or more possible future systems. Proper modules take care that each access through an interface keeps the functional integrity of the module intact. An exception may be that the module signals to its environment that it is no longer in a

valid state. The environment may then decide to ignore the module in future actions or it may reset the module to a valid state.

A proper module must be able to perform one or more actions. These actions may be controlled via one or more of its provide interfaces. Purely static objects are never considered as proper modules.

### **Properties and actions**

Each proper module has a set of properties that together describe its status. Besides of that, each proper module provides a series of actions. Each module interface offers indirect access in order to control the members of a well defined and ordered subset of these actions. The properties cannot be accessed directly. However, a given action may enable the reading of the value of a property or it may enable the direct or indirect setting of one or more properties.

### **Costs of modularization**

Modularization has its price. The design and generation of modules and the organization of compatible interfaces is relatively expensive. Only extensive reuse of modules may render modularization economic. Reuse of modules and the availability of compatible well known interfaces between modules may significantly improve the manageability of the design and creation of complex systems. However, reuse implicates standardization and it asks for actions that promote availability, accessibility and diversity. These requirements are best provided by a healthy and lively modules market and media that publish the specification of the characteristics of available modules and interfaces. An open market may ensure a healthy price to quality ratio. It also stimulates the continuous improvement of the quality of the modules that become available. Preparing modules for an open market requires the hiding of the intellectual property that is invested in the design and creation of the module. On the other hand the specification of provide interfaces must be publicly known. Promoting other uses of the provide interfaces and the require interfaces that are applied in a given module will in its turn

promote the use of that module. It will increase the chance that other modules will become compatible with the considered module.

## Abuse

Modularization can also be abused. Wrong access to a module may rupture its integrity. In that case the module is no longer trustworthy. A proper modularization technology must prevent improper access to modules. It means that access that bypasses the official interfaces of the module must be prevented. Clients of a module may be systems or other modules. During its actions a module may run through a sequence of states. A client of a module must only access the module while the module is in a state that is known to be safe for this access. Properly created modules will then take care that their integrity will not be impaired. If the state of a module is not known, then the client may decide to reset the module to a safe known state.

Abuse of modularity is stimulated by the misuse of the terms 'module' and 'component'. It often occurs that a system part is called 'component' or 'module' while it is far from properly encapsulated. Such system parts are not designed to preserve their integrity. People that do not have sufficient expertise may fall into this trap and may think that by assembling such improper components a similar reduction of complexity can be achieved as can be achieved with proper modules.

## Modularization success cases

The success of modularization is widely demonstrated in the design and generation of hardware. Electronic appliances, autos, buildings, clothes, in fact most assembled products are not affordable without the fact that they are constructed from components. Many of the constituting components are themselves assembled from components. More important, the price, quality, diversity and availability of these components depend strongly on the corresponding lively components markets. The beneficial effects of the open market depend strongly on trustable specification of the

characteristics of the components and on media that report on availability and quality of these products.

Even nature relies on modularization. Most living creatures contain organs and are constituted from multiple cells. Human communities use modularity in the hierarchical structure of their organizations. This is best shown in a town hall or a post office where dedicated counters belonging to corresponding departments offer publicly known services to their customers.

The application of modularization in the software industry is far from a great success. Proper software modules exist, but their application is sparse. The current software development tools do not support the assembly of systems from modules. The software components rely on the support that is offered by the operating system that embeds these components. Most software components are designed to operate as singles in a larger non-modular environment. Generally, these modules do not couple with other modules. Currently the software industry does not offer a technology that enables the construction of modules out of other modules.

### **Requirements for success**

When applied properly, modularization may significantly improve the system design and creation process. Keywords are the standardization, the diversity and the availability of modules and interfaces and the ease of the system integration process. The existence of a lively and effective modules market is also a very important aspect. System integration may be automated, but this requires the proper tuning of component specification, the system design tools and the matching components market. The technology must enable the construction of modules out of simpler modules. Using these preconditions the microelectronics industry provides very complex and tremendously capable integrated circuits.

With a proper automated design and assembly organization in place the modular system creation time will shrink to a small fraction of the time required by the manual non-modular equivalent. Where manual design and assembly of a complex monolithic target requires a genius as the system architect, a creative human operator may burn far less resources and achieve a similar or even better result by using an appropriate automated modular approach. Automation of the system design and creation process puts high demands on trustworthy and machine readable specifications of modules and interfaces.

## **Difficulties posed by modularization**

The requirements posed by modularization are also the reasons why modularization is never a straightforward solution.

### **Diversity**

The requirement of a high degree of diversity is in direct conflict with the requirement of sufficient standardization. An interface has both static and dynamic aspects. Dynamic requirements may ask for different interfaces that have similar static characteristics but different dynamic behavior. Environmental requirements may ask for specially adapted interfaces. Interfaces may be replaced by other interfaces that have a wider scope or a better performance. Similar considerations hold for modules. In order to increase market profits, to simplify component discovery and to ease system integration the diversity of similar interfaces must be kept within sensible bounds. The same holds for modules.

### **Compatibility**

In order to enable successful assembly, the selected modules must be mutually compatible. This translates to the requirement that the interfaces that couple the modules must be compatible. Provide interfaces must cover the demands of the coupled require interfaces. The requirements include both static and dynamic characteristics.

Real time behavior of modules may require measures that prevent or cure deadlock and race conditions. The design tools must enable the installation of these measures. Other measures must prevent that the system runs out of essential resources. The modules must be designed to support these measures. When all relevant data of the constituting modules are known, then the system design tools can help the system designer to implement sufficient resources and to take the appropriate measures.

## Platforms

Components may be designed for different application areas. For example software may be designed for desktop purposes, for servers or for embedding in electronic appliances. In each of these cases there exists a choice of hardware platforms. Electronic hardware platforms require adapted software components and will certainly influence the dynamical characteristics of the interfaces of the software components. Mechanical modules may target automotive systems, avionics, nautical systems, stationary instruments or other mechanical systems. Each application area and supporting platform may require its own range of modules and interfaces. Each application area requests an adapted components market and an adapted system assembly technology.

## Hiding intellectual property

In some application areas the hiding of the intellectual property that went into the design and the creation of modules is provided by their physical form or by market conditions such as a patent system. However, some application areas currently lack sufficient means to hide the design of the components. Without proper IP hiding a component's creator can never make profit in an open components market. In the past, this fact has certainly prevented that the software industry developed a healthy and lively software components market. This does not say that it is impossible to generate an effective IP hiding system for software modules.

## Availability

Availability is assured when several suppliers exist for popular modules. An easily accessible publication organization must promote and enable the discovery and the selection of existing modules.

## Specification

The specification must be accurate and complete. The specification must contain sufficient details such that the system integrator can determine how the considered module can be assembled with other modules into a target system. Automated assembly asks for a machine readable and therefore well standardized specification format. This requires a dedicated XML format. The format can be defined in an XSD document. For humans, an XML document is not easily readable. The XML document can be made readable for humans via one or more XSL documents. The specification of the static characteristics of an interface is well established. Currently there exists much less support for standardized specification of the dynamical characteristics of interfaces.

## Hardware versus software

### History

The hardware industry booked far more success with the application of modularization than the software industry. Partly the volatile nature of software is responsible for this fact. However, the differences in the evolution of the corresponding design and creation technologies had more influence on the success of modularization.

Long before the birth of electronic computers, modularization took its position in hardware industry. Computer hardware became affordable through far reaching application of modularization. The early computer programmers used machine code as the language to communicate with the computers. Soon the burden of inputting all these codes separately was eased by an assembly compiler that translated assembly terms into corresponding machine code sentences. Program parts could become

reusable routines. Libraries of these routines became products that could be applied in different programming projects. The next step was the introduction of the third generation languages. These tools offered a better readable and much more flexible coding of the functionality that the programmers had to write. Powerful compilers translated the source code and combined it with the precompiled library members that were called by the written program.

## **Basic architecture trends**

Up to so far this was no more than easing the process of producing machine code. The growing complexity of the programs demanded software development tools that enable a better overview of the architecture of the design. At this point two trends developed.

### **Functional analysis**

The first trend, phrased 'structural analysis' created a split between the handling of properties and the handling of the actions that influence these properties. The methodology collected properties in 'data stores', actions in 'processes', data messages in 'data flows' and control messages in 'control flows'. The graphical representation of the result of the analysis was called a 'data flow diagram'. In advance, the approach proved very successful. It led to the introduction of several important software development items such as, routine libraries, file systems, communication systems and data bases. Most third generation programming languages and the early software development tools supported the 'structural analysis' approach.

### **Abstract data types**

The second trend promoted the modular approach. It used 'abstract data types' introduced by David Parnas as its modules. In design phase the 'abstract data type' acted as an individual. It was well encapsulated and could only be accessed through one or more interfaces. In the seventies of the last century the complexity of most software projects did not enforce a modular approach. For that reason this modular design methodology was

not well supported by programming languages and by corresponding modular software development tools.

### **Object orientation**

In a later phase the complexity of the software design increased such that a more modular approach became necessary. Instead of taking the proper modular approach of the 'abstract data type' the main software development turned to object orientation. Here the objects resemble 'abstract data types', but the objects are not properly encapsulated. Access via interfaces is possible, but the client of the object may also access the actions of the objects more directly. More severely, often the internal properties of the object can be altered directly by external actors. The possibility to inherit functionality from an object with a simpler design was given much more attention. The result was the development of libraries of classes of objects with a deep inheritance hierarchy.

Currently, object orientation is well supported by software languages and software development tools. Pity enough, current object oriented software development tools do not promote the use of popular interfaces.

Object orientation has some severe drawbacks. Without sufficient precautions, classes taken from different class libraries cannot be combined in programs. A class library with a deep inheritance hierarchy may become obsolete when its top classes contain services that are no longer up to date with current technology.

### **Current software components**

The software industry also came with more proper software modules. Examples are Microsoft's COM components and the Java Beans. COM components are supported by some operating systems and Java Beans are supported by the Java virtual machine.

The support for COM in software languages and in software development tools is small. The design of the architecture of the COM skeleton

prevents trustworthy memory garbage collection management in cases where the module can be removed dynamically. COM is supported on some embedded systems that use UNIX or an operating system that supports POSIX.

Both Java Beans and COM components are not designed to construct components from components and need the support of an operating system or a virtual machine.

There exists a small open market for these software components. Most of them target desktop applications.

### **State of affairs**

At this moment the software industry does not apply modularization in a serious way. There exists no theoretical reason why modularization in software system generation can not be as successful as the current modularization in hardware system generation currently is. However, effective modular software generation asks for a completely different way of software generation than is accomplished by the present software development industry.

Implementing proper modularization will offer chances to parties that are now excluded by the power of companies that control software development tools and software development processes. With the appropriate services in place, everybody who has access to a software component development environment can produce products that fill a market need. Future institutions that support software component development and component based system assembly will help the component developer in marketing the created components. In that case the current powers in the software industry will endanger losing market control. It is to be expected that they will battle to stay in control.

# **Coupling the market and the design and creation of software modules and interfaces**

## **Standardization and marketing**

Modularization asks for a dedicated and powerful standardization of specifications, interfaces and coupling procedures. A globally accessible service must support the distribution of the public documents. For example, dedicated web based repositories may contain standardized and categorized specification documents that can be discovered by an appropriate search mechanism. The development tools must be able to access the specification contents contained in these documents. Another globally accessible service must support the gathering, the sale and the delivery of the corresponding components. Both services must cooperate.

The tools and the services must intimately interact to enable the quick and efficient design of interfaces, components and target systems. At the same time the services must ensure that the intelligent property that is invested in the uploaded components keeps hidden from the public world. It must also be guaranteed that the component designers will get their rightful fee. It is very difficult to organize a properly controllable pay per copy of the components binary. It is suggested that the customers pay per project for each used binary.

## **Designing and generating components**

The component designer collects the required interfaces from web based or local repositories or he designs one or more new interfaces. Then he designs and creates one or more components. He must test these thoroughly. When ready he uses the components for local system design or he packs one or more components into a package and sends this together with the appropriate documents to the institute that will market his products. The institute checks the contributions and after a positive conclusion the institute puts the binaries and documents in its banks. The institute will put the documents in the appropriate repositories where they become publicly accessible. Users of the components may buy the

components from the institute. The institute will ensure the payment of the developer that has put the product in the bank.

### **Versions and diversity**

Versions and diversity of components both impede and support the manageability of the system integration process. Therefore the number of versions must be limited. Diversity of components must be made manageable by reducing the number of supported platforms and by limiting the number of supported environments. Development and creation of close copies of existing components must be avoided. Breaking these rules can easily destroy the advantages of modular system design.

### **Hiding intelligent property**

Hiding intelligent property that is invested in the design of the component is one of the most difficult points of software component technology. It can be arranged by power: excluding customers from future membership when they offend the 'rules'. Or it can be ensured by a combination of encryption and recompilation supported by a hardware decryption. Every project gets its own encryption key. It must be ensured that a system designer can still use components that he himself has designed and created.

### **Automating system integration**

The system integrator starts with collecting the required application components and with creating the necessary connection and scheduling scripts. The components are put in packages and a project document defines the target. Because of the fact that at the start of the system integration practically all relevant data are known, the system integration tool can automatically add a dedicated supporting operating system that includes automatic memory garbage collection. The retrieved component specifications suffice to enable the construction of skeleton systems. After linking, these skeleton components can already be tested. However, the 'empty' components do not produce much activity. During system development the skeleton components can be replaced step by step by fully operational binaries.

## Publishing

Publications related to modularization comprise specifications, market promotion media and product quality comparison reports. The internal code of components is normally hidden. If the institution that designed the component wants this, it is possible to make this code public as part of the component specification.

## A fully fledged software components industry

### Sketch

There exists no theoretical reason why proper modularization cannot be achieved for software as it is done for hardware. The realization of some aspects will be easier while the achievement of other aspects will be harder. It is easier to send software products over internet. It is easy to search the document repositories of the component shops for interesting components and compatible interfaces. Using XML it becomes feasible to automate the design and creation process that makes use of these web based repositories, which contain machine readable specification documents that describe components and interfaces. A local file based equivalent of such a repository may store retrievals and new designs and serve both the system designer and the components developer. The repositories contain a search machine that looks for categorization terms that classify the specification documents for specific application areas. New designs can be uploaded to a central service that will check the information and store it in the worldwide accessible repositories. A webservice that acts as a dedicated web based shop may offer the corresponding modules. In the background of the webservice, binary banks will hold the binaries of the modules. The webservice will use a dedicated money bank to support the financial part of its activity. Via the webservice the component designers may upload their results to the central institution that will then market their products. Component development tools and system assembly tools interact with the

repositories and the webservices to implement an integrated design, assembly and marketing environment.

## The demo

This is a very sketchy view of a possible implementation of an integrated software components creation and marketing system. In order to investigate the feasibility of this sketchy picture a demonstration system is built that contains working versions of all important constituents.

The demonstration system supports:

- Embedded software and desktop software<sup>1</sup>
- Provide interfaces
- Require interfaces<sup>2</sup>
- Memory mapped hardware interfaces
- Streaming interfaces
- Notification interfaces<sup>3</sup>
- Package<sup>4</sup> of a coherent set of components.
- Components<sup>5</sup> that consist of simpler components.
- Automatic creation of the supporting operating system from dedicated modules<sup>6</sup>
- Stepwise system build-up from a mix of skeleton components, partially functional components and fully functional components
- Automatic memory management
- System modes<sup>7</sup>

<sup>1</sup>In embedded software the generated system interacts directly with the hardware. The system assembly tool adds the HAL.

<sup>2</sup>Require interfaces are implemented as placeholders for special types that represent a reference to a provide interface.

<sup>3</sup>Notification interfaces accept hardware triggers.

<sup>4</sup>A package is a library of a coherent set of components. A component supplier will preferably deliver his products in the form of packages. A system designer will save his subsystems in the form of packages.

<sup>5</sup>A composed component is a dedicated package accompanied by a dedicated (fixed) connection scheme and a dedicated (fixed) scheduling scheme.

<sup>6</sup>In embedded software the system integration tool generates operating system modules in C++ source code. In desktop software the system design tool generates a layer that interacts with the virtual machine. This layer is generated in source code that corresponds with that virtual machine (C# or java).

<sup>7</sup>System modes are controlled by connection schemes and scheduling schemes. Dynamic removal or creation of modules should be restricted to the instances where the system mode changes. Memory management is also restricted to these instances.

A standard RTOS schedules threads by stopping and starting routines. In a component based environment the real time scheduler must stop, reset and start modules. Eventually the modules must be reconnected according to the currently valid connection scheme.

The demonstration system consists of the following components:

- An example of a web based repository
  - This repository exists of a hierarchy of directories that contain
    - XML documents, which contain structured specifications. Each document contains a series of categorization tags.
    - XSD documents, which define the structure of the specifications

- XSL documents, which help convert XML documents into humanly readable documents
  - The repository has a hierarchical structure. Components and interfaces are assembled in separate directories.
  - The repository is publicly accessible. Using the XSL files the XML documents are humanly readable via a modern web browser.
  - The repository contains a search machine that uses the attached category tags to find corresponding documents.
- An example of a local file based repository
  - This repository exists of a hierarchy of directories and has the same structure as the web based repository. This includes the search capability.
  - The local repository contains a larger variety of documents than the web based repository.
  - It acts as a local store for information that is retrieved from one or more web based repositories.
  - It acts as a local store for documents that are prepared to be send to a general institute that may put these documents on a web based repository.
  - The XML documents specify:
    - Component
    - Interface
      - Require interface
      - SW/SW
      - HW/SW
      - Streaming
      - Notification
    - Types
      - Plain type
      - Enum type
      - Interface type
      - Sequence type
      - Structure type

- Package description
  - Connection scheme
  - Scheduling scheme
  - State chart
  - Project description
- An example of a webservice that may act as the representative of a central institute. This institution serves the community that creates or uses software components. Components may appear as packages of simpler components.
  - The institute owns a local repository that contains all specifications of interfaces that exist in the domain of the webservice.
  - The institute owns a binary database that holds the binaries of all available software components.
  - The institute owns a local repository that contains all specifications of software components that exist in the domain of the webservice.
  - The webservice uses the binary databases and the local repositories to automatically serve the customers of the institute. Customers have no direct access to these stores.
  - The webservice helps partners of the central institute to distribute documents to their specialized web based repositories.
  - The webservice helps customers in buying software components and retrieving the corresponding binaries from the binary bank
  - The webservice helps software component developers to upload the binaries and corresponding specifications of their products.
  - The central institute takes care that the software component developers get paid for products that are downloaded via the webservice.
- A repository browser tool

- The tool helps with searching local or web based repositories for existing interfaces and components. Selected documents can be transferred from the web based repository to the local repository.
- An interface and component design tool
  - The tool helps with specifying new interfaces. This includes:
    - Software-software interfaces
    - Software-hardware interfaces
    - Streaming interfaces
    - Notification interfaces
  - The tool helps in specifying other design documents that go into the repositories.
  - The tool helps with searching local or web based repositories for existing interfaces.
  - The tool helps designing and creating the skeleton of a software component
  - The tool helps with filling the skeleton with dedicated code
  - The 'internal' code is normally hidden. However, it is possible to make this code public with the rest of the specification.
- A system assembly tool
  - The tool helps with searching local or web based repositories for existing software components. It can retrieve the corresponding binaries from web based or local binary banks.
  - The tool can work with components that are still in skeleton form.
  - The tool can check whether components can fit together.
  - The tool assembles selected components and adds a dedicated component based operating system.

Some hard rules must be obeyed.

- All components and all interfaces have a globally unique identifier.
- Any binary and any specification document that is uploaded to the central institute and that is accepted by this institute must never be changed or removed.
- New versions of an item are related to the previous version via a relation document that is attached to the specification document.
- The number of new versions of an item must not surpass 4.
- Close copies of items, that are not new versions, will not be accepted.

## Software availability

The project that is discussed in this paper produced a large amount of software. The author of the paper owns this software and makes the C# source code available for free access at:

<https://docs.google.com/leaf?id=0B8ZNO nNHFrbrODU3YmNkNzQtMDE0ZC00MzNkLWJiZTk tN2U2MGV kOTAzYTk w&hl=en>



# Part seven

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## Physics and religion

# Physics and religion

I am not knowledgeable in religions. However, I know something of physics and of mathematics. It is easy for me to find a creator that is supported by the laws of physics and the theorems of mathematics. I soon give you a sketch. It is much more difficult if not impossible to find the thing that created and controls the creator. I will also explain why.

## **Universe as a creator**

Let us first take the easy part and start with the habits of multi-dimensional Fourier transforms. A Fourier transform converts something that is very localized into something wide spread and vice versa. At the same time the parameter space turns into its canonical conjugate equivalent. Now take the universe as a subject and do the Fourier transform. It may be necessary to do some coordinate transformations in order to get a proper Fourier transform, but that is a secondary consideration. The universe is something that is very wide spread in its position coordinate system. Thus in the corresponding canonical conjugate (momentum) coordinate system the result of the transformation is very much localized. You could view it as an individual located in momentum space. If you do the same thing for the time domain then finally you get an individual in the momentum-energy domain.

We know that universe is evolving and during that evolution very complex items are created. Thus, universe as an individual is its own creator. A mechanism called

modularization achieves that when conditions are favorable, the result is that very complex systems arise. If modules can be generated out of modules this process can go very far. It even achieves the generation of intelligent species. Those specimens can use their intelligence to speed up the evolution process. What in the beginning was a purely stochastic process then turns into an intelligence driven creation process. Thus nature builds conglomerates; starting from pure energy it creates particles, then atoms and molecules, then living species, then intelligent species and then God knows what. Nature appears to possess a built in tendency to reduce complexity and at the same time it builds more and more complicated systems.

## **The upper creator**

Now comes the hard part. Consider the creator of the discussed creator. He is the creator of the universe. He tolerates that the first creator uses a very harsh way to achieve his goal. The evolution process is very hard for the subjects that fall off in the creation process. They are torn apart and the debris is used as resource for new creations. We humans do not see that as a big problem as long as it concerns dead stuff, but as soon as it concerns living species and especially when it concerns animals or humans, then we feel pity for the victims.

Religions nearly all suggest that this upper creator of the primary creator shows concern with what happens to the living species. At least he must take care of the intelligent species. However, if he does that, then he must intervene in the first

creation process. With other words, he must change the laws that govern the first creation process. Despite the fact that many humans believe that this is possible, physics does not show that such things actually happen. Which law of nature has built in controls such that the upper creator can control those laws?

However, laws are one part of the story. What actually happens is not only controlled by the laws of nature. It is also controlled by (boundary) conditions. A much larger amount of influential boundary conditions exists than the number of laws of nature. The upper creator can control the boundary and starting conditions of the primary creation process. That is a huge task, but he is almighty. It means that from the moment that it started, everything is already set. So it has little sense to pray to this almighty upper creator, then he already planned everything. He even planned your prayer.

Still religion can be good. It keeps communities together. It can give their members a happy and secure feeling. On the other hand religions can also be intolerant with respect to other religions. Wars are fought for economic reasons but they are also fought for reasons of religious nature. That is not good. Animals and even plants fight wars for economic reasons. Only humans fight for religious purposes, which if you think about it, is pure madness. Intelligent species should know better.





<http://www.crypts-of-physics.eu> ; (Deep cellar, Nesle, France, <http://www.croxculture.com> )



# Part eight

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The Hilbert book model



# The Hilbert book model

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## Introduction

Every time when I read an article about the phenomena, which occur far from us in the universe, I'm surprised about the attention that this Farawayistan gets compared to the phenomena in the world of the smallest. Everything that happens there is dismissed with collective names such as “quantum mechanics” and “field theory”. Rarely or never the treatise goes deeper. In this sub-nano-world spectacular images, such as appear in stories about the cosmos are not available.

## What's playing?

Still, this is part of our environment is at least as interesting and mysterious as the cosmos. What makes it even more interesting is that the fundamentals of physics can largely be found in this area. This gets enforced by the growing awareness that our knowledge of these foundations contains a lot of gaps.

## Quantum Theory

Quantum mechanics and the corresponding quantum field theory have been developed mainly in the beginning of the last century. This development occurred fairly violently and in many cases, scientists were already happy with a limited understanding that nevertheless brought enough usable formulas so that one could analyze quantum phenomena and could construct useful applications.

## History

In the early days of quantum mechanics the approach was based on adapting equations of motion that were in use in classical mechanics. These equations were quantified via an intuitive process. In Schrödinger's approach the time dependence is placed in the state function of the particle. The approach of Heisenberg positions the time dependence in the operators that act on the state function. This difference in approach ultimately makes no difference for the properties of the physical particles. That means that the state function and the operators only play a background game. In contrast, the properties of the particles play the foreground act. This indifference also means that time does not belong to the properties of a particle. With respect to the state functions and the operators time only plays the role of a parameter. Apparently it does not matter whether you place this role in the state functions or in the operators. This parameter characterizes the progress of the dynamics<sup>174</sup>. On the other hand, position belongs to the properties of a particle. This indicates a fundamental difference between the role of space and the role of time.

The biggest confusion arose when it became clear that the smallest things could behave both as a particle and as a wave package. This confusion continues because it also means that nature is unpredictable in the behavior of its smallest parts. Many are unable or unwilling to accept this fundamental property.

## Clarification

Already early on in the last century some solid explanations were given. Garret Birkhoff and John von Neumann showed that nature is not complying with the laws of classical logic. Instead nature uses a

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The role of time becomes clear in the paragraph about the Hilbert book.<sup>174</sup>

logic in which exactly one of the laws is weakened when it is compared to classical logic. As in all situations where rules are weakened, this leads to a kind of anarchy. In those areas where the behavior of nature differs from classical logic, its composition is a lot more complicated. That area is the site of the very small items. Actually, that area is in its principles a lot more fascinating than the cosmos. The cosmos conforms, as far as we know, nicely to classical logic. In scientific circles the weakened logic that is mentioned here is named traditional quantum logic.

## Hilbert model

Birkhoff and von Neumann went a step further. They discovered the fact that a mathematical structure, which more than a century earlier was discovered by mathematician David Hilbert, is in many respects similar to the structure of this quantum logic. This structure is a space with infinitely many dimensions. A position in this Hilbert space can be specified by using numbers. For each position that must be done with infinitely many numbers. Fortunately, that what is happening in the Hilbert space can also be specified with functions. Luckier wise a lot was already known about functions that suit this purpose.

## Numbers

The numbers that can be used, need not be limited to the real numbers, which we use in order to measure our three dimensional living environment. Constantin Piron found that these numbers at least must be members of a so-called division ring. There are only three division rings: the real numbers, the complex numbers and the quaternions. Virtually no one still knows the quaternions. William Rowan Hamilton discovered quaternions already in the nineteenth

century. They are hyper complex numbers with a one-dimensional real part and a three-dimensional imaginary part.

## Hilbert operators

Here you see appear an immediate reason for our three dimensional world. It also delivers a mystery, because the structure of Einstein's space-time differs from the structure of the quaternions. However, there are more puzzles. Although the Hilbert space has an infinite number of dimensions, this infinity is countable. Countable means that in principle, a label with an ascending integer can be attached to each dimension. The set of real numbers is uncountable, but the set of rational numbers is countable and the set of rational quaternions is that too. So, to each dimension of the Hilbert space a rational quaternion can be attached. Mathematicians use the name operators for the mathematical things that can do this. The real numbers describe a continuum and the set of quaternions does that too. But the set of rational quaternions does not do this. This means that it is impossible to accurately describe smooth phenomena with the model obtained so far.

## Graininess

The reality is even worse. There is increasing evidence that in its smallest form nature is grainy. So-called Planck units exist. These are unit sizes for time, place, action and entropy. It is basically impossible to measure the corresponding quantities more accurately than these Planck unit sizes indicate. It is as if within these limits the world does not exist or else, that nature steps over these regions.

## GPS

Now suppose that we want to design a three-dimensional GPS system for nature by using the three-dimensional part of quaternions. This system would have to take into account the graininess of length. However, this is a great problem. A lattice consisting of a tightly packed collection of grains is afflicted with preferential directions. Such directions appear in nature in solids but they are not omnipresent in the universe. Therefore we need to find a different solution for the customized GPS system. This solution must not use multidimensional collections of grains, because that would pose the same problem.

## Grain chains

A potential solution is a GPS that works with one-dimensional chains of grains. The chains represent paths. Not actual paths, but hypothetical paths. They can freely move in 3D space. There is one grain in the chain that represents the current position on this path. Only the direct environment of this grain corresponds to an actual path. Now remains the problem to give each grain in the chain its own position.

In addition to the Hilbert space with countable dimension the mathematicians developed a Gelfand triple. As a kind of sandwich the two outer parts of this triple attach to the previously described separable Hilbert space. Because this triple directly associates to the separable Hilbert space, this sandwich is also known as a “rigged Hilbert space”. In fact this name is incorrect because the triple is not a proper Hilbert space. Fortunately, the rigged Hilbert space has an uncountable number of dimensions and can easily deliver a GPS system that can act as a continuum background coordinate system. The grain chains also have an equivalent in this rigged Hilbert space

and this fact can be used to attach a position in the background coordinate system to each of the grains of a selected chain.

## Anchor Points

The grains of the chains that occupy the current position in the chain's "path" are in fact anchor points of elementary particles. Depending on its type an elementary particle has one or more of these anchor points. According to Schiller's strand theory<sup>175</sup>, photons use only one anchor point and electrons have three anchor points. Per time step the anchor point can **at the utmost** take one space step. If it does that, then it lands in the next grain of the chain. That is why the chain represents a kind of path. The ratio between space step and time step is fixed and is equal to a constant  $c$ . That number equals the speed of a freely moving light particle. In each time step a photon invariably takes a space-step. It also means that no particle can go faster than such a freely moving light particle.

## Fields

The chains are not allowed to move arbitrarily. There is something that ensures that the chain keeps its smooth shape. This is provided by a probability distribution that is associated with the anchor point. In fact, it's a hyper complex function whose squared modulus equals the mentioned probability distribution. This function has quaternions as its function values and accepts quaternions as a parameter. The three-dimensional imaginary part of the hyper complex parameter may indicate a position. In that case, the probability distribution gives the probability that the next grain will be located at the value of the parameter. The form of the probability distribution ensures that only minimal changes occur. The quaternionic function

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<sup>175</sup> See References

contributes to the local field. It is the part of the field that corresponds to the considered grain chain.

## Private Fields

An elementary particle can have one or more anchor points. In this way the corresponding hyper complex functions together form the private field of the elementary particle. This private field has the same properties as the wave function of the particle. Quantum mechanical scholars use this wave function in order to describe the behavior and the properties of the elementary particle.

Together, all the private fields of particles form a joint covering field that, like the separate private fields, covers the whole Hilbert space. In our model, this joint covering field is part of the physical fields in our environment.

## Field Theory

According to field theory each static field can be split in a rotation free (longitudinal) part and a solenoidal (divergence free, transverse) part. Due to the configuration of the field, this split may run along curved lines. This defines a local curvature. The curvature value can be used to define a new field. It is derived from the joined covering field. We can call that new field the curvature field. It has all the aspects of the gravitational field. We can take the part of the curvature field that belongs to a particle as its private curvature field. From this private curvature field the mass of the particle can be computed. Physicists usually apply this relationship in the reverse order.

## **The field model**

The field model, which is applied here, differs significantly from the common field model. Usually the electromagnetic fields and the gravitational field are assumed to be independent of each other and the gravitational field is assumed to cause a curvature in the coordinate system that must be taken into account in the treatment of the electromagnetic fields.

In this new model the cause of the local curvature is laid down in the properties and the configuration of the covering field, which consists out of the superposition of all fields except the gravitational field. The covering field also contains the fields that match the wave functions of particles. The curvature field is then derived from the local curvature. In other words, in this new model the gravitational field is a derived field. This approach causes an immediate unification of field theories.

## **Hilbert sandwich**

The Hilbert space itself has no place for fields. Each private field covers the whole Hilbert space. However, in the same manner as described above for the Gelfand triple, it is possible to expand the aforementioned sandwich with three additional layers, which respectively represent the two decomposition parts of the covering field and the curvature field. Therefore, all in all, the expanded Hilbert sandwich consists of six layers.

## **Hilbert book**

Each sandwich describes a static condition. Thus, this combination can still not describe any dynamics. This lack can be solved by putting

a whole series of these sandwiches in an ordered sequence. In this way, a Hilbert book can be formed, in which each page represents a Hilbert sandwich. Glancing through this book then gives a picture of the dynamics of our universe. The page number acts as a progression step counter. This counter is not our common notion of time, but it has certainly something to do with it.

## Discussion

What is described here is only a model. It is not more than a reflection of reality. The events we see in the cosmos are largely determined by the curvature field. The new GPS operator knows an outside horizon beyond which no chains exist. That operator also has internal horizons inside of which no chains exist. We know these internal horizons as the exterior of black holes.

## References

References: <http://www.crypts-of-physics.eu/Cracksofphysics.pdf> and <http://www.motionmountain.net/research.html>

In this book of Christoph Schiller, he uses the name **strand** for the equivalents of our chains. Please notice, strands are no strings! The strands model has little to do with string theory.



# Part nine

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A tall quantum tale

# A tall quantum tale

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I state you a proposition  
and that proposition indicates  
how the world works



## Story

### Prelude

A group of elderly Magi sit in a circle and discuss what happens around them. That is not much. The youngest of them gets bored and starts

considering their discussion. The chat appears regulated, because if they start from a false proposition they will be able to draw any inference, whether true or not true, and then the conversation ends only in balderdash ad infinitum.

After some time, he has collected the rules. These rules prevent the conversations from getting out of control. He proposes these rules to his companion discussers. They are very pleased. From this moment on, every conversation runs fluently. The inventor writes his finding in a book and calls that book "Logic".

However, in their environment still little occurs that is worth a proper discussion. Since the talks no longer get out of control, most of the time passes in silence. The inventor feels bored again and therefore he tries to invent something else. He realizes that if he changes the rules in his book a little, then as a result, the discussions could be become much more interesting. He writes a new book that contains the changed rules. Next he changes the forest that exists in their neighbourhood in order to reflect the discussion rules.

After finishing this book and the forest, the situation has completely changed. Continuously, things appear in the forest around them that keep their conversations for ever alive. The writer calls the second book "Quantum Logic" and he renames his first book "Classical Logic". The toolkit that he uses to create the new structure of the forest also has a name. It is called "Mathematics".





**M**



**S**

## The encounter

An old, very experienced senior meets a young curious guy, which is full of questions about the things that he has observed during his trip through his world. The youngster asks the elder whether he can ask him a few of his most urging questions. The senior reacts positively by nicking shortly. However, already the first question of the studious guy startles him:

**S:** Mister, can you explain me how the world works?

The elder thinks a while very deeply and comes then with his answer:

**M:** That would be a hell of a job, but I can at least give it a try. Please, sit down on that stone, because this will take some time.

The lad sits down and looks expectantly to his narrator. The old man takes a breath and starts:

**M:** This can be done in the form of a tale. It could be done better in the form of a truck load of formulas, but I doubt that you would understand these formulas. Do you accept that I pack the story in a tale?

**S:** Well I like a tale much better than a truck load of formulas. I probably would not understand one of them. So please start with your tale.

The elder takes a breath and starts his tale.

**M:** The world is governed by a book of laws. It must conform to these laws. There is no punishment in not following the laws, but the world cannot do anything else then operate according to the rules that are written in the book of laws.

**S:** Where is that book and how is it called?

**M:** It is in the possession of the governor of Hilbert's bush. The book's name is "The rules of quantum logic".

**S:** What is in that book?

**M:** The book contains a small set of rules that regulate what the relations are between propositions that can be made about things that live in our world.

**S:** What things?

**M:** Well, anything that has an identity and that stores the condition it is in. Let us call such a thing an item or a particle and let us use the name state for the condition it is in. Mostly the concerned things are very small. However, these things can be very large.

**S:** What is different with that logic? I know only one kind of logic.

**M:** You know the kind of logic that humans base their reasoning on. They use the rules of logic in their discussions when they start with truth and want to stay with truth. Nature uses a kind of logic that has a much richer structure. However, in that logic only one rule is different.

**S:** How many rules contains the book and what do these rules mean?

**M:** The book contains somewhat more than twenty rules and they specify the structure of the relations between the allowable propositions.

**S:** There are not much rules in the book! How can that book rule the world?

**M:** You are right about this, but these rules are very powerful.

**S:** Please explain that.

**M:** Well, the structure of the propositions is reflected in the structure of Hilbert's bush. Hilbert's bush is a huge and dense forest and is connected to our world. Via these connections Hilbert's bush controls how the world works.

**S:** Thus, if I visit Hilbert's bush, then I can see how the world works?

**M:** No, if you visit Hilbert's bush, then you can see how the world is controlled.

**S:** How, can I visit Hilbert's bush?

**M:** Well, you can join me on a virtual trip to Hilbert's bush. I will be your guide.

**S:** Fine. How does Hilbert's bush look?

The man describes a very strange environment. The chap follows the old man in his mind and shows astonished. However, in advance his guide warned that he would present a tale. So, he must believe what the man tells.

**M:** It is like a huge forest of poles. All poles have the same length and the feet of all poles are hooked at the same point in the centre of the bush. In this way the poles form an enormous sphere.

**S:** Where do these poles stand for?

**M:** The poles are the axes of a multidimensional cube that has an enormous dimension. First think of a three dimensional cube. Take a corner of it and take the three axes at that corner. You can identify the position of all points in the cube by three positions on rulers that are taken along the three axes.

Now, as in an umbrella, fold these axes together, such that they form a small bundle. Next add a large amount of axes to that bundle. Give every axis a unique label in the form of one or more numbers. Add a ruler to each of these axes. You can still define the position of each point in the multidimensional cube by stating the corresponding positions on the rulers. Next increase the number of dimensions until it reaches infinity. The axes now form a dense ball and they all are numbered with a unique label. Finally unfold in your imagination the “umbrella” again until all axes are again perpendicular to each other. You can start counting the dimensions of the cube, but you will never finish counting.

**S:** Thus the poles are a plain set of axes.

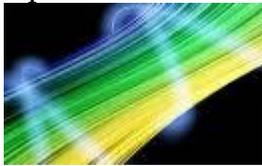
**M:** Yes, but the space between the perpendicular axes can also be filled with poles. In this way several sets of mutually perpendicular axis poles can be found.

**S:** What is the function of these axis poles?

**M:** The axis poles have colours. Some axis poles are green poles. Together they form a base in which the position of all other poles can be expressed. Another set of axis poles are red. Also they form a base. Some of the poles are silver white. They are not necessarily axis poles. The silver white poles appear in bundles.

**S:** That is a strange kind of forest!

**M:** Indeed, but it is not the only thing that is strange about Hilbert's bush. Let me tell more about the silver white poles. The bundles of white poles represent and at the same time control the items in our world.



**S:** How is that arranged?

**M:** The items in our world are reflections of the bundles of white poles in Hilbert bush. What happens to the bundles will happen to the items.

The student tries to imagine the strange situation. Apparently two worlds exist. One in which he lives and one from where his life is controlled. He visualizes the forest in his brain.

**S:** What is the function of the green and red poles?

**M:** At their top these other poles contain a data store in the form of a label. The data stores of the green poles contain position data. They are a kind of kilometre indications that you find along our roads. Instead of a single number the stores contain all three coordinates. It works like a kind of primitive GPS system.

**S:** With some trouble I can understand what you paint for me.

**M:** The data stores of the red poles contain speed data, or better said momentum data. In this way a bundle of silver white poles can determine the current position and the momentum of the moves of its pupil in the real world.

**S:** Why are there two types of data poles?

**M:** The governor arranged it that way. In this way the bundle cannot determine both types of data at the same time. It is another detail of how the governor models our world. The stores of the poles contain the values of the properties of the type observation to which the pole belongs. Mathematicians call these values eigenvalues and the corresponding poles eigenvectors. With this trick the governor leaves us uncertain about our exact condition.

**S:** What are mathematicians?

**M:** Mathematicians are scientists that amongst other things study the mechanisms, which determine the structure and behaviour of Hilbert's forest. The creator of the forest used mathematics to give it its functionality.

**S:** Can white poles read data?

**M:** No, in fact a shepherd that takes care of the silver white bundle does that. The forest is very dense. So, the shepherd can walk on top of these poles and guard his herd of sheep. From now on, I will call the silver white poles the shepherd's sheep.

**S:** How does the shepherd read the data?

**M:** The shepherd must turn to the data pole in order to read its data. If he is close to a green pole, then he is rather far from a red pole. In fact he

may be at nearly the same distance from a series of red poles. He will usually read the nearest data pole. The same holds when the shepherd looks at other colours. Thus, the governor plays a strange trick with our world.

For the insiders: This is the source for the existence of Heisenberg's uncertainty principle. It is the cause of the quantum behaviour of small particles.

**S:** I must say, that is a strange situation!

**M:** Yes, let me proceed. It will become even much stranger.

**S:** Please, go on.



**M:** The shepherd drives his sheep through Hilbert's bush. He does that guided by the smells that he receives from other silver white bundles. The smells are mixtures of perfumes that are attractive and perfumes that are repellent. The shepherd reacts on these smells.

**S:** What is causing these smells?

**M:** These smells are caused by the properties of the sheep. They hang as a blurring mist around each white pole, thus around each individual sheep. The sheep may also move inside the scope of the herd. That movement may also be caused by the influence of the emitted smells.

**S:** How does the shepherd keep his sheep together?

**M:** Well, that happens in a particular way. The bush is so dense, that it is impossible to let the poles move. Instead at each of his steps the shepherd redefines the poles that belong to his herd. These poles turn silver white. The poles that get outside of the herd obtain their original green or red colour. The smells create a tendency to minimize action of the cheap. Further there exists another mechanism, which is called inertia.

**S:** What is inertia?

**M:** The smells invoke a sticky resistance of the system of all herds against change. Inertia represents the combined influence of all other herds. The most distant herds together form the largest part of the set of herds. So, they have the largest effect. The influence of each individual herd decreases with distance. However, the number of herds increases faster with distance. The difference between the distant herds averages away. As a consequence the distant herds form a uniform background influence.

**S:** What is the effect of inertia on a herd?

**M:** Locally the inertia produces an enormous smell pressure. A smooth uniform movement does not disturb this potential. When the herd accelerates it stirs the perfumes and in this way the inertia produces a smell that goes together with this movement.

**S:** I understand now how position is treated. What about time?

**M:** The shepherd owns a simple clock. That clock counts his steps. His steps are all the same size. When he drives his sheep around, he follows a track in Hilbert's bush. All shepherds take their steps in synchrony. In fact at each of their steps the complete forest is redefined. In this process the smells act as a guide. They store the current condition of the forest and these represent the preconditions for the new version of the forest. You can say that the smells represent potential versions of the forest. This includes potential versions of sheep. These potential sheep are virtual sheep.

**S:** So, compared to space, time is handled quite differently.

**M:** You understand it quickly and perfectly! You understand it better than the physicists of the last few centuries. Most of them were wrong with this subject. They think that time and space belong in one inseparable observable characteristic.

**S:** How many of these herds exist?

**M:** As many as there are particles in our world. So, there exist an enormous number of herds, but they are still countable. They can all be identified. All shepherds take their own track through Hilbert's bush.

**S:** That must make Hilbert's bush very large!

**M:** It is. Let me proceed. It must be obvious now that the herds influence each other's movements via their smells.

The lad reflects and pictures the forest in his mind as an enormous sphere. On top of that sphere a large number of shepherds push their own herd of silver white lights forward on curving tracks that are determined by the smells that other herds produce. At each of the shepherd's steps Hilbert's

forest is reconfigured. The old man must have a strange image of the world. Nonetheless, he must have his reasons.

**S:** So, the shepherds play a crucial role!

**M:** Yes, they manipulate their own herd. However, the smells of their sheep influence for other shepherds the observation of the position and momentum of other herds.

**S:** How do the smells influence that observation?

**M:** They give the data that are transmitted in the smell an extra turn. It means that other shepherds do not get a proper impression of the position and momentum data that are sent by other herds.

**S:** Is there a good reason for this confusing behaviour?

**M:** No, there is no reason. It is just a built in habit of all sheep. On the other hand, the governor established that habit when he designed mathematics. He designed mathematics such, that Hilbert's bush and its inhabitants behave according to the rules in his book.

**S:** What is the consequence of this strange behaviour?

**M:** The consequence is that the particles in the world get the wrong impression of the position and momentum of other items. For them it appears that there exists a maximum speed. And these items think that they live in a curved space.

For the insiders: This is the source of the existence of relativity as it was discovered, but not explained by Einstein.

**S:** Do they think that?

**M:** For them, it is the truth!

**S:** So, I live in a curved space and for me there exists a maximum speed.

**M:** That is right. You properly understand how the world is controlled. As long as you do not interpret that maximum speed as the limit set by your local police officer.

**S:** What happens inside a herd?

**M:** The sheep inside a well-shaped herd perform rhythmic movements. You could say that they are dancing. Physicists call it harmonic movements. These dances occur under the control of the shepherd. He considers them as his own possession.

**S:** What do you mean with a well-shaped herd?

**M:** A well-formed herd represents in our world a well-formed object, such as an atom.

**S:** Why is everything set up in such a strange way?

**M:** The governor of Hilbert's bush is very intelligent, but also very lazy. He does not want to create many rules, so that he does not have to write much in his law book. That is why he invented Hilbert's bush. He builds the consequences of all his rules into the structure and the dynamics of Hilbert's bush. That structure is in principle very simple. The same holds for the dynamics. In this way he does not have to take care on how the world evolves. However, this leaves an enormous freedom for what happens in the world that is controlled by Hilbert's bush. That on itself results in an enormous complexity of the world we live in. That renders the governor very, very smart and very, very lazy.

**S:** How did Hilbert's bush get its name?

**M:** Hilbert was the first human that discovered the governor's bush. So people give it his name.

**S:** Can everybody visit Hilbert's bush?

**M:** In principle yes. Everybody that possesses sufficient imagination can visit Hilbert's bush. There exist two guides. A mister Schrödinger tells the story as we did. He tells the story as if the bundle of silver white poles moves through the bush of green and red poles. The other guide, mister Heisenberg tells the story as if the bundle of white poles is stationary and the bush of green and red poles moves around. For the world it does not matter what moves. It only senses the relative motion.

**S:** How did intelligent creatures like us enter that world?

**M:** The governor installed a tendency to reduce complexity by means of modularization into his forest. When more compatible modules become available it becomes easier to construct more capable modules and more capable items from these modules. Given enough time, more and more capable items are created, which finally result in intelligent creatures. Scientists call this process evolution. It is a chaotic process, but it possesses a powerful tendency.

**S:** Uch. Can I tell this to my friends?

**M:** Yes, you can. And if you have learned to read formulas and work with them you can come back and I will tell you the same story in a cart load of formulas.

**S:** Thanks. I will come back when I am grown up. Can I still ask a final question?

**M:** You are a sauce-box, but you are smart. Go ahead.

**S:** What are you going to do after this?

**M:** I will visit a very old and very wise scientist, called Mendel. He claims that he has a cohesive explanation for all smells that shepherds react to.



**S:** Why is that important?

**M:** If his claim is right, then he has found the Holy Grail of physics.

**S:** Gosh!

After this the boy departs. Later he will become a good physicist.



## Interpretation

The book of laws contains a number of axioms that define the structure of traditional quantum logic as an orthomodular lattice.

Hilbert's bush stands for an infinite dimensional separable Hilbert space that is defined over the number field of the quaternions. The set of the closed subspaces of the Hilbert space has the same lattice structure as traditional quantum logic.

The green poles represent an orthonormal base consisting of eigenvectors of the normal operator  $Q$ . This operator represents an observable quantity, which indicates the location of the item in space.

The red poles represent an orthonormal base consisting of eigenvectors of the normal operator  $P$ . This operator is the canonical conjugate of  $Q$  and represents an observable quantity, which indicates the momentum of the item.

The bundle of silver white poles and the herd of sheep represent a closed subspace of the Hilbert space that on its turn represents a particular quantum logical statement. This statement concerns a particle or a wave packet in our surroundings.  $Q$  describes the thing as a particle.  $P$  describes the thing as a wave packet.

The shepherd represents a complicated operator  $U_t$  that pushes the subspace, which is represented by his herd, around in the Hilbert space. The operator  $U_t$  may be seen as a trail of infinitesimal unitary operators. It is a function of the trail progression parameter  $t$ . The progression parameter differs from our common notion of time, which is the coordinate time.

Traditional quantum logic defines only the stationary structure of what happens in Hilbert's bush. The dynamics are introduced by the shepherds that react on the smells.

The smells correspond to physical fields. The fields transport information about the conserved quantities that characterize the movements of the item and its elements. Each type of preserved quantity has its own field type. The operators  $U_t$  react on these fields. Inertia shows how these operators reflect the actions of the fields. Any acceleration of the item goes together with a reconfiguration of the fields.

The operator  $U_t$  transforms the observation operators  $Q$  and  $P$  into respectively

$$Q_t = U_t^{-1} \cdot Q \cdot U_t$$

and

$$P_t = U_t^{-1} \cdot P \cdot U_t$$

.This distorts the correct observation and ensures that the observer experiences a speed maximum and a curved space.

The eigenvalues of  $Q$  and  $P$  and the trail progression parameter  $t$  characterize the space-time in our live space. As already indicated  $t$  is not the same as our common coordinate time.

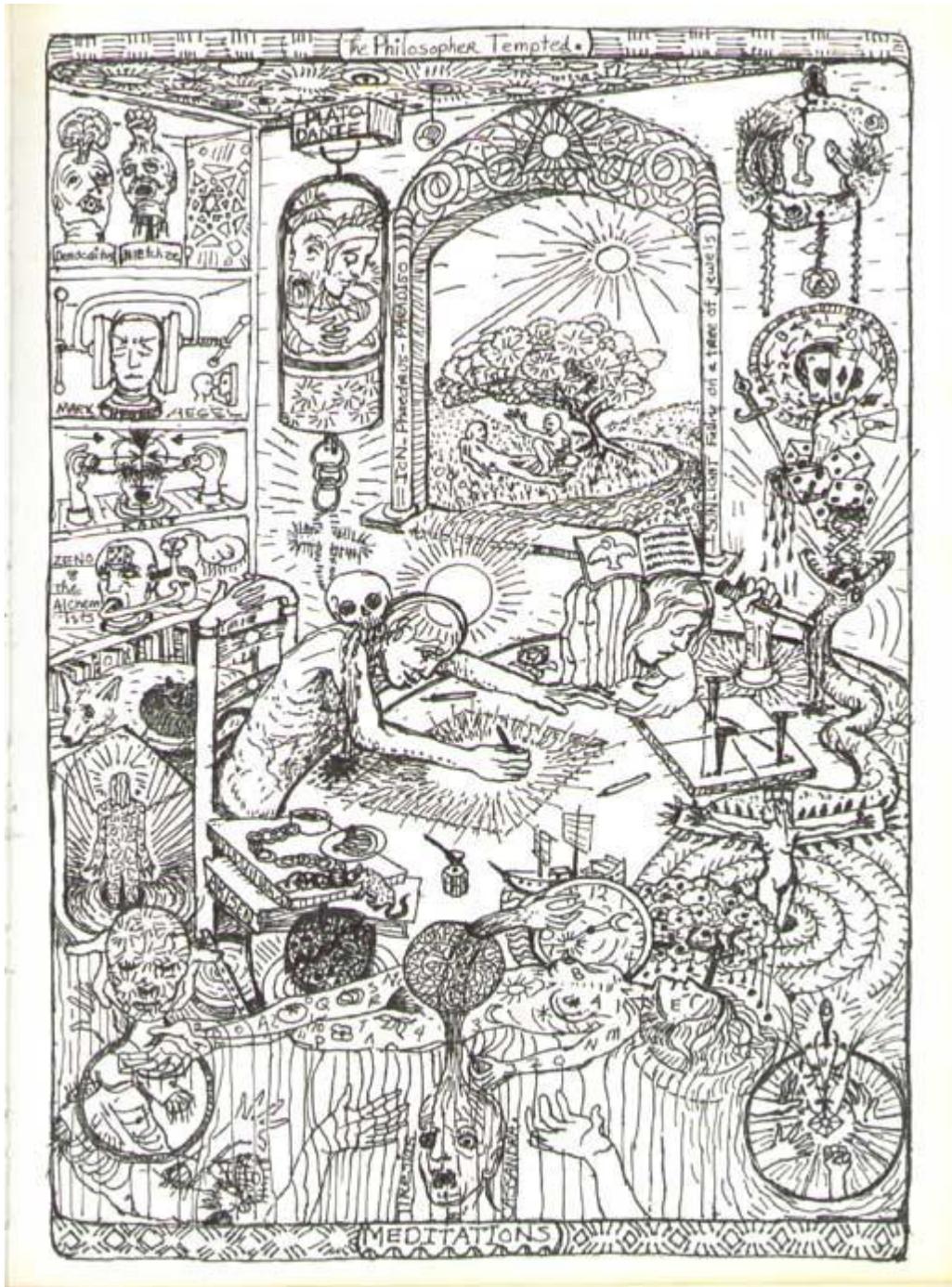
De eigenfunctions of  $U_t$  control the (harmonic) internal movements of the particles.

The sheep represent the elements/properties of the particle.

The effect of modularization is treated in <http://www.crypts-of-physics.eu/ThereExistsATendencyInNatureToReduceComplexity.pdf>;  
part four of this book



HvL



## INDEX

- 2<sup>n</sup>-on construction, 213, 287
- 2<sup>n</sup>-ons, 67, 120, 288
- 3 sphere, 283
- abstract data types, 385
- acquired indications, 73
- action, 223
  - unit of action, 111
- action along the live path, 256
- adjoint, 301
- affine geometry, 253
- affine space, 144
- anchor points, 412
- angular momentum, 123
- architecture trends, 385
- atomic predicate, 89
- availability, 384
- background coordinates, 30
- balance equation
  - continuity equation, 185
- base transforms, 162
- base vectors, 300
- basic constituent, 127
- binomial process, 77
- blur, 122, 126
  - blur interpretation, 124
- blur function, 123
- blurring function, 121
- boson, 246
- bosons, 91
- boundary, 99
- bra vectors, 298
- brain
  - brain waves, 346
  - pre-processing, 343
  - processing, 343
- bra-ket, 93
- bras and kets, 297
- calculus of variations, 253
- canonical conjugate, 97, 100, 139
- canonical coordinates, 35
- Christoffel symbols, 255
- closure, 307
- clouds of quanta, 78
- coherent state, 39, 159
- colinear, 301
- compatibility, 382
- complex Fourier transform, 140
- complexity, 352
  - extreme complexity, 375
  - managing complexity, 374
  - measure of complexity, 374
  - potential complexity, 353
  - software complexity, 373
- components
  - designing and generating components, 388
- conservation laws, 195, 329
- conservation of angular momentum, 198
- conservation of energy, 195
- conservation of linear momentum, 196
- continuity equation, 184, 246
  - continuity equation for charges, 184
  - continuity equation for field energy density, 195
- coordinate distance, 98
- coordinate time, 239
- coordinate time step, 230
- coordinates

- quaternion coordinates, 282
- coordinates
  - polar coordinates, 282
- cosmological limits, 337
- countable, 125
- coupling, 356
- covering field, 34
- cracks in the fundamentals, 28
- creator
  - universe as a creator, 399
  - upper creator, 400
- crossing switch, 124
- curvature field, 34
- curved space, 75
- decomposition
  - Helmholtz, 127
  - Hodge, 127
- demo, 391
- Detective Quantum Efficiency, 345
  - DQE, 76, 124, 344
- differentiable distribution, 194
- differential potential equations, 190
- differentiation, 137
- Dirac delta function, 120, 122
  - multi-dimensional, 145
- Dirac equation, 249
- discrete distribution, 190
- distance, 299
- distribution
  - Gaussian distribution, 123
  - Poisson distribution, 123
- diversity, 382
  - versions and diversity, 389
- dreams, 349
- dynamics, 28, 41, 74
  - time and dynamics, 103
- eigenspace, 304
- eigenvalue, 304

- eigenvector, 304
- electromagnetic field theory, 71
- elementary Hilbert distribution, 136
- elementary types, 91
- entropy, 321
  - unit of entropy, 111
- equations of motion, 55, 246
- eth, 177
- Euler Lagrange equation, 309
- example proposition, 87
- expectation value, 209
- extended Hilbert space, 34
- extended quantum logic, 34, 75, 228
- fermions, 91
- Feynman path integral, 235
- field, 32, 249, 412
  - basic field constituent, 32
  - covering field, 130
  - curvature field, 134
  - field model, 414
  - field theory, 413
    - private fields, 413
- field angular momentum, 198
- field energy current density, 195
- field energy density, 195
- field equations
  - Hilbert field equations, 181
- field linear momentum, 196
- field linear momentum
  - field linear momentum density, 196
- field linear momentum flux tensor, 197
- flux vector, 195
- Fourier mode, 320
- Fourier transform, 36, 97, 123
  - alternative approach, 147
  - alternative definition, 149

- convolution, 181
- differentiation, 181
- Fourier transform properties, 180
- functions invariant under Fourier transform, 150
- multidimensional Fourier transform, 132
- Fourier transform invariant, 180
- Fourier transform split, 144
- fourth dimension, 101
- functional analysis, 385
- Functional analysis, 385
- functions and fields, 120
- fundament, 54
- fundamental measures and units, 110
- fundamental principle of the strand model, 264
- general relativity, 75, 254, 257, 332
- generalized Trotter formula, 305
- geodesic, 119, 223
  - geodesic equation, 119
- gluon, 246
- GMS
  - Hilbert space GMS, 101
- GPS, 411
  - background GPS coordinate system, 106
  - generating a Hilbert space GPS, 98
  - GPS coordinate operator, 96
  - GPS-like operator, 96
- GPS coordinates, 66
- grain chain, 411
- graininess, 410
- granular GPS, 79
- granularity, 29
- ground state, 157, 158
- group structure, 286
- half ordered set, 278
- halo, 263
- Hamilton-Jacobi, 327
- hardware versus software, 384
- harmonic functions
  - spin-weighted harmonic functions, 178
- Heisenberg picture, 231
- Heisenberg's uncertainty, 141
- Helmholtz decomposition, 52, 59, 60, 221, 229, 250
- Hermitian, 303
- Hilbert book, 414
- Hilbert book model, 41, 43, 54, 407
  - discussion, 415
- Hilbert distribution, 40, 120
  - elementary Hilbert distribution, 121, 127
- Hilbert field, 40, 56, 121
  - sampled Hilbert field, 122
- Hilbert function, 56, 104, 121
- Hilbert model, 409
- Hilbert operator, 410
- Hilbert sandwich, 414
- Hilbert space, 96
  - quaternionic Hilbert space, 297
  - separable Hilbert space, 297
- history, 27
  - history of quantum physics, 408
- history of quantum logic, 277
- Hopf coordinates, 285
- humans, 347
- image intensification, 344
- imaging quality characteristics, 344
- inertia, 218

- inertia and progression step, 241
  - rotational inertia, 224
- infinitesimal number transformation, 215
- influence, 218
- information association, 345
- inner product, 298
- intellectual property
  - hiding intellectual property, 383
- intelligent property
  - hiding intelligent property, 389
- intensified imaging, 75
- interactions, 247
- interfaces, 357, 376
- interpretation in logical terms, 228
- inventions of the human mind, 348
- item, 92
  - representation of items, 94
- item type definition, 90
- ket vector, 297
- Lagrangian, 119, 250, 328
- Laplace operator, 138
- lattice, 278
  - atomic lattice, 280
  - complementary lattice, 279
  - distributive lattice, 279
  - modular lattice, 280
  - orthocomplemented lattice, 279
  - weak modular lattice, 280
- lattice isomorphic, 53, 56, 58
- lepton, 91, 247
- limitedness, 105
- localizer, 136
- locator, 93
- logic, 87
  - classical logic, 55, 87, 281
  - dynamic quantum logic, 75, 89
  - extended quantum logic, 75
  - predicate logic, 281
  - propositional logic, 281
  - quantum logic, 55, 87
  - traditional quantum logic, 55, 281
- logic of quantum actions, 273
- Lorentz force density, 197
- Lorentz power density, 195
- matrices, 116
- Maxwell equations, 193
- measurement, 282, 326
  - measurement preparation, 326
- Minkowski signature, 230
- modular system design, 354
- modularization, 354
  - abuse, 380
  - abusing modularization, 359
  - costs of modularization, 379
  - difficulties posed by
    - modularization, 382
  - intelligent modularization, 356
  - modular approach, 375
  - modular system design, 376
  - power of modularization, 358
  - requirements for success, 381
  - standardization and marketing, 388
  - state of affairs, 387
  - success cases, 380
- Modulation Transfer Function
  - MTF, 259
- modules

- proper modules, 378
- momentum, 123
- nabla operator, 137
- nearby items, 224
- niners, 290
- Noether's theorem, 221
- Noether's theorem, 329
- noise filter, 346
- norm, 299
- number, 111, 409
  - construction, 116
- number waltz, 214
- numbers, 67, 211
- object orientation, 386
- observable, 282
- operators, 300
  - anti-Hermitian, 303
  - anti-unitary, 303
  - canonical conjugate operator, 308
  - continuous operator, 302
  - Hermitian operators, 303
  - ladder operator, 155, 306
  - linear operators, 300
  - normal operator, 301
  - normal operators, 302
  - orthogonal projection operators, 303
  - positive operators, 303
  - self adjoint operator, 301
  - unitary operators, 303, 305
- optical transfer function, 143
- Optical Transfer Function, 345
  - OTF, 76, 124, 259, 344
- optics, 40, 123, 259
- orbital angular momentum, 172
- Parseval's theorem, 180
- particle, 33
- particles, 246
- path
  - curve length, 118
  - path characteristics, 117
  - path equations, 118
  - reparameterization, 119
- Pauli equation, 248
- phase transfer function
  - PTF, 262
- photon, 246
- physical field, 28, 56, 60, 63, 64, 67, 74, 88, 212, 249, 260, 261, 273, 274, 358
- physical reality, 347
- Planck length, 98, 105, 111, 230
- Planck limits, 332
- Planck time, 111, 230
- Planck values, 266
- platforms, 383
- Point Spread Function
  - PSF, 259
- Poisson process, 44, 46, 77, 78, 123, 344
- poke, 247
- polar decomposition, 306
- position, 96
- position operator, 105
- potential, 187
- predicate, 281
- principle of minimal action, 358
- private field, 121, 127
- private Hilbert field, 136
  - private field, 40
- probability amplitude distribution
  - probability current, 186
  - probability density, 186
- probability density operator, 137

- probability distribution
  - probability amplitude
    - distribution, 123
  - probability density
    - distribution, 123
- probability measure, 208
- product rule, 113
- progression step, 79
- project, 69
- proper time, 230, 239
- properties and actions, 379
- proposition, 281
- publishing, 390
- quantum cloud, 46
- quantum logic, 277
- quantum tale, 418
  - encounter, 420
  - interpretation, 434
  - prelude, 418
- quantum Theory, 407
- quark, 91, 246
- quaternion, 73
- quaternion waltz, 214
- quaternionic conjugation, 112
- quaternionic function, 126
- quaternionic nabla, 183
- reasoning, 346
- redefiner, 242, 244
- redefinition, 93, 241
- regular quaternionic functions, 291
- Reidemeister move, 247
- relation
  - physical relations, 353
- relativity, 42
- religion, 401
- representation restriction, 74
- representing multiple fields, 330
- rigged Hilbert space, 29, 45
- rotational symmetry
  - Hankel transform, 165
- scalar product, 298
- Schrödinger equation, 247
- Schrödinger picture, 231
- science, 347
- separable, 125, 299
- separable Hilbert space, 29
- shot noise
  - quantum shot noise, 123
- sign selections, 111, 216
- single infinitesimal step, 235
- slide, 247
- software, 359
- software availability, 396
- software components, 386
- software components industry, 390
- space step, 230
- space time step, 230
- spacetime, 41
- specification, 384
- spectral theorem, 303
- spherical harmonics, 165
  - spherical harmonics expansion, 174
  - spin weighted spherical harmonics, 175
- spin, 200
  - spin and dyadic product, 317
- spinors
  - need for spinors, 217
- spread function, 123
- squeezing, 161
- state, 202
  - coherent state, 123
  - ground state, 123
  - item state, 205
  - pure state, 205

- state definition, 204
- static status quo, 41, 227
- strand, 71, 264
  - strand interpretation, 109
  - strands as type definitions, 91
- strand basics, 266
- strand model, 43
- strand operator, 30, 107
- strand table, 267
- symplectic decomposition, 287
- system, 320
  - isolated systems, 323
- system integration
  - automating system integration, 389
- system type definition, 91
- test proposition, 66
- the origin of dynamics, 227
- theories, 348
- thoughts, 317
- trail, 243
- trail of infinitesimal transforms, 232
- transverse plane
  - alternative transverse pane, 147
- twist, 247
- type definition, 90
- underpinning for strands, 30
- unique aspects of the model, 269
- unit sphere, 307
- unitary transform, 231
- universe of items, 218
- vacuum, 95
- vacuum expectation value, 181
- veiling glare, 263
- versor, 287
- vertebrate visual trajectory, 78
- vertebrates, 347
- vision of noisy images, 345
- W type boson, 246
- war against software complexity
  - analysis, 363
  - approach, 365
  - attack, 366
  - conclusions, 369
  - demo, 368
  - discussion, 370
  - history, 364
  - lessons, 368
  - prelude, 363
  - remnants, 367
  - set-back, 367
  - setting, 364
  - strategy, 365
  - way out, 369
  - what happened, 366
- wave function, 137, 205
- wave package, 318
- wave transfer function
  - WTF, 261
- Weil-Brezin transform, 149
- world's action, 330
- Z type boson, 246
- Zak transform, 149
  - Weil-Brezin-Zak transform, 313

## On the origin of physical dynamics

Author: Hans van Leunen

This book reports the current state of a running project that investigates the origin of physical dynamics.

It starts from the axioms of traditional quantum logic and extends this model such that it incorporates physical fields as well as dynamics.

It uses the isomorphism between the set of propositions of traditional quantum logic and the set of closed subspaces of an infinite dimensional separable Hilbert space that uses quaternions in order to specify its inner products.

The book finds solutions for the anomalies that are raised by the granularity of normal quaternionic operators. It also takes the consequence of the observation that all information about nature becomes available in the form of clouds of information carrying quanta.

The book unifies all fields, such that except for the curvature field, all fields including the wave functions are considered as probability amplitude distributions. The curvature field is derived from the curvature of the superposition of all these primary fields. The curvature follows from the decomposition of this covering field in rotation free and divergence free parts.

In order to implement dynamics, the developed model applies a sequence of extended quantum logics or equivalently a sequence of extended separable Hilbert spaces. Each of the members of the sequence represents a static status quo of the universe. This leads to a new model of physics:

### The Hilbert book model

Apart from this main subject the book contains a series of related papers.

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