Deriving the Pythagorean Theorem Using Physics Conservation Laws

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Abstract

The Pythagorean Theorem is derived using conservation of energy and linear momentum that involves an elastic collision between two equal masses.

Although the Pythagorean Theorem is used extensively in physics, many people are not aware that physics conservation laws require that the Pythagorean Theorem be true. This work provides a derivation of the Pythagorean Theorem from conservation of energy and linear momentum. Consider two objects of equal mass that undergo an elastic or energy conserving collision, as shown in Fig. 1. The kinematics is very similar to a pool ball driven into a single stationary target pool ball, if we neglect friction and rotational inertia of the pool balls. In Fig. 1 a spherical object moves at velocity $\mathbf{V_{i1}}$ towards a similar object that is stationary with zero initial velocity. Assuming each object has mass, M, the conservation of energy states that the initial energy is equal to the final energy, as shown in eq. (1), where $\mathbf{V_{i1}}$ and $\mathbf{V_{i2}}$ are the initial velocities of the objects. $\mathbf{V_{f1}}$ and $\mathbf{V_{f2}}$ are the final velocities of objects after the collision.

$$\frac{1}{2}MV_{i1}^2 + \frac{1}{2}MV_{i2}^2 = \frac{1}{2}MV_{f1}^2 + \frac{1}{2}MV_{f2}^2 \tag{1}$$

Since V_{i2} equals zero, eq. (1) is reduced to eq. (2).

$$V_{i1}^2 = V_{f1}^2 + V_{f2}^2 \tag{2}$$

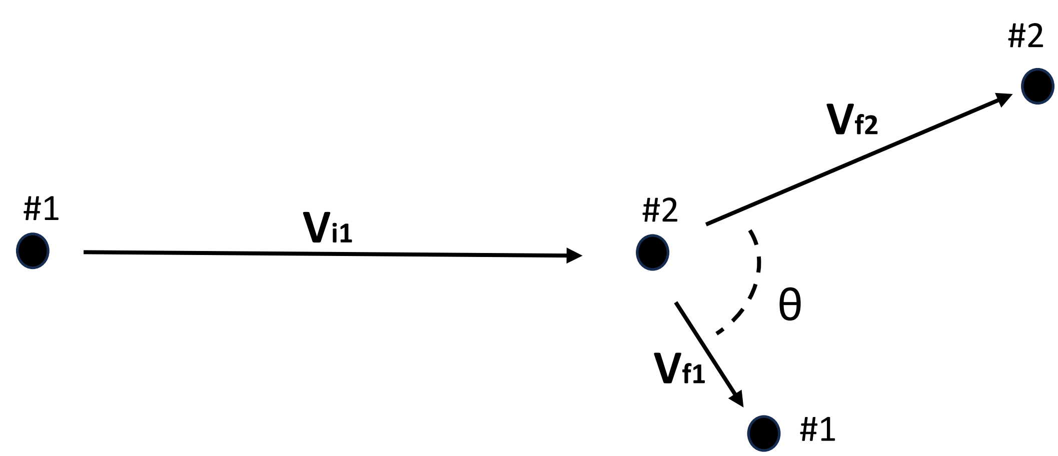

Fig. 1. Elastic collision diagram of object #1 colliding with stationary object #2.

Fig 2. is the momentum vector diagram, which shows that the initial momentum vector, which is the longest side of the triangle, appears as the sum of the final momentum vectors represented by the other two sides of the triangle.

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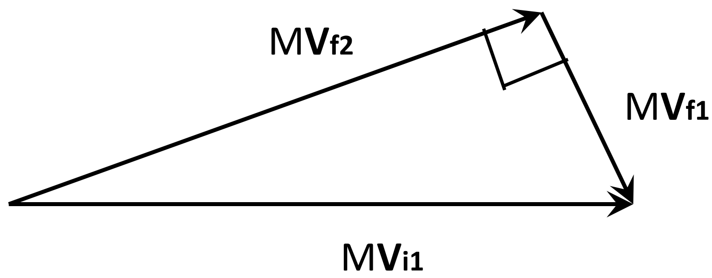

Fig. 2. Momentum vector diagram illustrating the Pythagorean Theorem.

Eq. (3) represents the conservation of linear momentum associated with Fig. 2. The mass can be divided out of both sides of eq. (3) to obtain eq. (4).

$$M\boldsymbol{V}_{\boldsymbol{i1}} + M\boldsymbol{V}_{\boldsymbol{i2}} = M\boldsymbol{V}_{\boldsymbol{f1}} + M\boldsymbol{V}_{\boldsymbol{f2}} \quad (3)$$

$$\boldsymbol{V}_{\boldsymbol{i1}} + \boldsymbol{V}_{\boldsymbol{i2}} = \boldsymbol{V}_{\boldsymbol{f1}} + \boldsymbol{V}_{\boldsymbol{f2}} \quad (4)$$

Since $\mathbf{V}_{\mathbf{i2}}$ equals zero, eq. (4) is simplified to eq. (5).

$$\boldsymbol{V}_{\boldsymbol{i1}} = \boldsymbol{V}_{\boldsymbol{f1}} + \boldsymbol{V}_{\boldsymbol{f2}} \quad (5)$$

Both sides of eq. (5) can be squared to obtain the associate scalar equation, as shown in eq. (6), where θ is the angle between $\mathbf{V}_{\mathbf{f1}}$ and $\mathbf{V}_{\mathbf{f2}}$ after the collision, as shown in Fig.1.

$$V_{i1}^2 = V_{f1}^2 + V_{f2}^2 + 2V_{f1}V_{f2}\cos(\theta) \quad (6)$$

Using $(V_{i1})^2$ from eq. (6) in eq. (2), one obtains eq. (7), which indicates two solutions: $V_{f1} = 0$ or θ equals 90 degrees.

$$V_{f1}V_{f2}\cos(\theta) = 0 \quad (7)$$

If the final velocity of object #1 is zero, object #2 obtains the energy and linear momentum that object #1 had before the collision occurred, i.e., $M\mathbf{V}_{\mathbf{f2}} = M\mathbf{V}_{\mathbf{i1}}$. If V_{f1} is not zero in eq. (7), then θ must be 90 degrees. Since θ= 90 degrees, Fig. 2 is a right triangle with hypotenuse $M\mathbf{V}_{\mathbf{i1}}$ and sides $M\mathbf{V}_{\mathbf{f1}}$ and $M\mathbf{V}_{\mathbf{f2}}$. Using eq. (7) in eq. (6) yields eq. (8), which states that the square of the hypotenuse of a right triangle is equal to the sum of the squares of the other two sides. Since no restrictions have been placed on the shape of the right triangle, eq. (8) holds for a right triangle of any shape. Therefore, eq. (8) is the Pythagorean theorem, which completes the desired proof. QED

$$V_{i1}^2 = V_{f1}^2 + V_{f2}^2 \quad (8$$

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