WHEN MATH MEETS NEUROSCIENCE: RELATIONSHIPS BETWEEN ELLIPTIC CURVES AND SCALP EEG WAVE FRONTS

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ABSTRACT

Elliptic curves are fundamental mathematical objects obeying two-variable equations in which the square of one variable relates to the third power of another. Wondering whether elliptic curves might stand for the abstract counterpart of biological activities, we identified a correspondence between the two-dimensional plots of various elliptic curves and the electric waves detectable on the cortical surface by scalp EEG techniques. Specifically, the patterns described by the intersection of elliptic curves' three-dimensional contour plots and the two-dimensional projective plane overlap in a few terms of milliseconds with the patterns displayed by two-dimensional cortical EEG wave fronts. Hence, elliptic curve-like mathematical structures might correspond to the dynamical shapes produced by the real brain activity recorded from subjects performing various tasks. We suggest that, apart from the usual applications in physics and cryptography, elliptic curves might theoretically disclose possible equations subtending the anatomical and functional neural routes endowed in the central nervous system.

KEYWORDS: Weierstrass Equation; genus 1; cubic polynomial; Raven tasks; visual stimuli.

INTRODUCTION

In mathematics, an elliptic curve (henceforward EC) is a non-singular, plane curve enclosed in a two-dimensional finite algebraic field (Ciss and Moody, 2017) and defined by polynomial cubic equations in two variables:

$$y^2 = f(x),$$

where the square of one variable y relates to the third power of another variable x and f(x) is a cubic polynomial with no repeated roots (Knapp 1992).
With a suitable change of variables, every EC with real coefficients can be written in the standard form:

$$y^2 = x^3 + Ax + B$$

with some constants A and B meeting a few straightforward conditions.
A more technical definition states that an EC (over a field k) is a smooth projective curve of genus 1 (defined over k) with a distinguished (k-rational) point. Unlike ellipses, that are curves of genus 0, ECs display genus 1 (Heath-Brown, 2004). They are characterized by two-dimensional paths devoid of either cusps or intersections, except at the saddle point, where a transition occurs from a closed to an open curve. ECs are curves and finite groups at the same time (Washington 2008) and can be defined in terms of points, integer numbers and rational numbers (Gebel et al., 1994). Different types of EC do exist, including Weierstrass curves, Huff curves, Edwards curves, general quartics curves, and so on (Ciss and Moody, 2017). To provide an example, an elliptic curve E over a Fq (finite field) is given by the general Weierstrass equation (Zimmer 1989):

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

where $a_1, a_3, a_2, a_4, a_6 \in Fq$.
ECs have many practical applications. They are used to tackle computational problems involving group laws, such as algorithms for factoring large integers. Further, EC throws a bridge between Monster groups, number theory and physics, given its relationships with the Pariah group (Duncan e tal. 2017). Analogies between quantum gravity and DNA/RNA packings have also been proposed (Planat et al. 2021). In cryptography, the EC Diffie-Hellman key exchange protocols require that two parties combine their messages with a shared secret, allowing microprocessors to safely determine shared secret keys (Jiang et al., 2020).

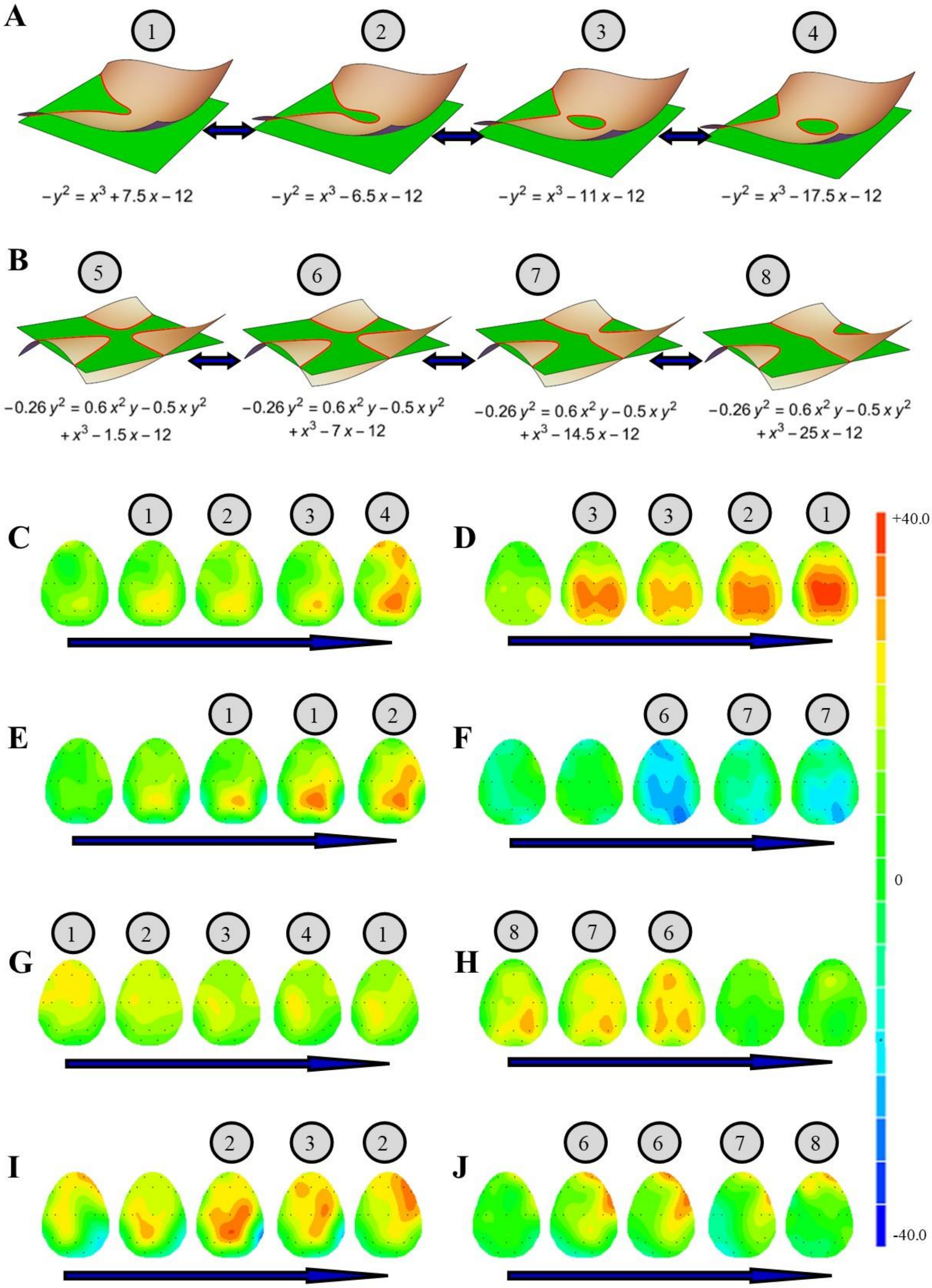
Here we investigate whether a mathematical object like EC can be correlated with the dynamics and/or functions of biological systems. In particular, we explore the possibility that EEG traces from subjects performing different tasks might exhibit the peculiar features of ECs. In this proof-of-concept study, we report that the cortical electric activity which is generated during various metal activities matches the curves described by different ECs. Our approach suggests feasible applications of ECs in the context of the neuroscientific investigation. We conclude that the links between abstract mathematical concepts and biological activities are worth exploring.

MATERIALS AND METHODS

We utilized ECs from the Wolfram Demonstration Project, where intersection curves are illustrated in the real space (Cinkir 2011). These ECs are generated by the intersection of the surface $z = x^3 - a y^2 + b x y^2 + c x^2 y$ with the plane $z = A x + B$. Two types of ECs with their three-dimensional contour plots intersecting the two-dimensional plane are illustrated in **Figures A and B**. Four frames are provided for each of the two ECs.
In **Figure A**, the plane is
$z = Ax + B$,
where A (rotation) is a varying parameter and B (translation) = 12.
The surface is
$z = x^2 - a y^2 + b x y^2 + c x^2 y$,
where a = -1; b = 1; c = 1.
Therefore, the surface is described by $z = x^3 + y^2$.
The parameter A (rotation) varies among the four frames, while the other parameters are kept constant.

In **Figure B**, the plane is
$z = Ax + B$,
where A (rotation) is a varying parameter and B (translation) = 12.
The surface is
$z = x^2 - a y^2 + b x y^2 + c x^2 y$,
where a = -0.26; b = -0.5; c = 0.6.
Therefore, the surface is described by $z = 0.6 x^2 y - 0.5 x y^2 + x^3 + 0.26 y^2$.
The parameter A (rotation) varies among the four frames, while the other parameters are kept constant.

EEG traces. We assessed digital EEG traces recorded by Norbert Jausovec, who made available his archive to us before his untimely end. The recording procedures and the examined subjects are described in Jaušovec and Jaušovec (2010) and Tozzi et al. (2021). The EEG activity was monitored according to the Ten-twenty Electrode Placement System of the International Federation over nineteen scalp locations. The analysis system (SynAmps) had a bandpass of 0.15-100.0 Hz and voltage gain of –6dB at cutoff frequencies, resolution of 084 μV/bit in a 16 bit A to D conversion. The traces were recorded using the following parameters: rate = 1000 Hz, HPF = 0.15 Hz, LPF = 100 Hz, Notch = off. We extracted two-dimensional ovals from the EEG traces depicting the brain areas detectable on the scalp (**Figures C-J**). The sequence of ovals was taken at a temporal distance of 10 ms from each other.
The **Figure** illustrates the electric activity of subjects undergoing easy Raven tasks (**C-F**), verbal analogy tasks (**G-H**), visual stimuli tasks (**I-J**).



Figures A-B. Three-dimensional plots of two different elliptic curves with varying parameter A (modified from Cinkir 2011). Note that the EC plots are bidirectional, i.e., they may proceed on the four illustrated frames in both the ways. **Figures C-J**. Unidirectional two-dimensional wave fronts of electric activity extracted from EEG traces, with ovals temporally spaced 10 ms from each other. The bar on the right depicts the electric waves amplitudes. The numbered grey disks point out the similarities between the shapes of the ECs and the electric wavefronts. For further details, see the main text.

RESULTS

We found that the mathematical patterns described in **Figures A-B** match the real electric activity portrayed in the oval frames of the **Figures C-J**. The patterns described by the intersection of ECs' three-dimensional contour plots and the two-dimensional plane match the wave fronts generated by the electric activity detected by scalp EEG. The numbered grey disks highlight the patterns in common between the mathematical ECs and the real cortical findings. Note that the EEG wave fronts can follow both the forward and backward ECs profile and can change their shape in a few terms of milliseconds. Summarizing, our data suggest that different kinds of ECs can be found in EEG traces recorded during various tasks.

CONCLUSIONS

We evaluated the possibility to detect the occurrence of ECs in the brain activity and found a correspondence between the two-dimensional paths predicted by the mathematical theory of EC and the electric wave fronts detectable by scalp EEG. The results of our proof-of-concept study pave the way for new methodological developments to investigate elusive brain functions. We found a matching between EEG waves and ECs by investigating the relatively short timescale of 10 ms, but our results could theoretically be extended also to longer temporal frames like, e.g., the slow waves underlying the spontaneous activity of the brain. We examined the electric wave front of EEG traces, but our approach could be used to investigate also the neuronal dynamics provided by other neuro-techniques. To provide and example, it has been suggested that diffusion tensor imaging and diffusion tensor tractography (Tsai 2018; Alizadeh et al., 2019) designate anatomic nervous structures such as tracts, commissures, fasciculi, radiations that are arranged as arcs roughly resembling ECs (Tozzi and Mariniello, 2022). Therefore, ECs might stand for the abstract counterpart of the anatomical and functional neural projections endowed in the central nervous system.

What would be the practical benefit for neuro-researchers when assessing the brain activity in the mathematical terms of ECs? The abelian EC are equipped with intrinsic symmetries that are hidden at first sight (Kühne 2021). Further, their oscillating patterns display scale invariance (He et al, 2022; Zubrilina 2023). The occurrence of EC's hidden symmetries and scale invariance in the brain could explain the long-range, simultaneous activation of neurons located in distant brain areas and detected via pairwise comparison (Volgushev et al., 2006; Keren and Marom, 2016). A great deal of modern cryptography is based upon ECs, guaranteeing authentication mechanisms with low computational cost (Nita and Mihailescu, 2023). Since two points on the curve intercept the curve at a third point and these points can be plotted on cartesian coordinates, two users could arrive at a shared secret by moving around an EC (Blake et al, 2005). The same methodology employed by elliptic-curve cryptography could theoretically be used to detect hypothetical, hidden cortical codes. Yet, ECs could provide the possibility to use simple geometries with plane curves for the study of brain dynamics. For instance, it is well-known that the solutions of the cubic ECs are confined to spatial regions that are topologically equivalent to a torus. The fact that EC datapoints can be easily mapped onto the surface of the torus suggests that the anatomical and functional nervous pathways crossed by the brain electric activity could be methodologically evaluated in terms of trajectories taking place inside manageable toroidal manifolds (Tozzi and Peters 2016).

The theoretical occurrence of EC could allow researchers to investigate the cortical activity extracted from neurodata with the mathematical weapons of algebraic geometry, complex analysis, number theory, integral-differential algorithm. This would allow the assessment of brain functions through a large family of algorithms that can generate different series of identities. To provide an example, an integral-differential algorithm has been introduced to describe elliptic real period functions featuring dihedral symmetry (Klee 2019):

$$T(\alpha) = {}_2F_1\left(\frac{1}{2}, \frac{s-1}{s}; 1; \alpha^2\right), \; s = 3, 4, \text{ or } 6$$

with signatures $s \in \{2, 3, 4, 6\}$ as in the Ramanujan theory of elliptic functions that involves integral period functions K_1, K_2 and K_3 (Ramanujan 1914; Shen 2017). It is noteworthy that the Hamiltonian can be associated with the period function $T(\alpha)$ which satisfies a special case of the hypergeometric differential equation (Klee 2019):

$$4(s-1)\,\alpha\,T(\alpha) - s^2\left(1-3\alpha^2\right)\partial_\alpha T(\alpha) - s^2\left(1-\alpha^2\right)\alpha\,\partial_\alpha^2 T(\alpha) = 0, \; s = 3, 4, \text{ or } 6.$$

In the context of neuroscience, this means that the cortical electric activity could be evaluated in terms of elliptic Hamiltonians. When the local cortical electric potential reaches a minimum, it might be hypothesized that the phase curves become loops, each one characterized by the period function $T(\alpha)$ (Klee 2019). This leads to the occurrence of elliptic hypergeometric functions (Olde Daalhuis 2010) that could be useful in the study of the cortical harmonic oscillators (Aravanis 2010). The above-mentioned set of elliptic period functions satisfies an easily calculable ordinary differential equation, namely, the Picard-Fuchs equation (henceforward PF) (Klee 2018). The use of PF for the assessment of elliptic matrices extracted from neuroscientific recordings might allow the evaluation of integrals without resorting to direct integration techniques. Yet, PF could be useful in the investigation of brain's thermodynamic contraints, entropy and energetic demands, since this versatile equation can encompass the energy as a parameter (Kreshchuk and Gulden, 2019).

In conclusion, an EC-framed account of the cortical electric activity could suggest the occurrence of mathematical functional structures underlying nervous routes.

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REFERENCE

1) Alizadeh, M.; Kozlowski, L.; Muller, J.; Ashraf, N.; Shahrampour, S.; Mohamed, F.B.; Wu, C.; Sharan, A. Hemispheric Regional Based Analysis of Diffusion Tensor Imaging and Diffusion Tensor Tractography in Patients with Temporal Lobe Epilepsy and Correlation with Patient outcomes. Sci. Rep. 2019, 9, 215. https://doi.org/10.1038/s41598-018-36818-x.
2) Aravanis, Christos T. 2010. Hermite polynomials in Quantum HarmonicOscillator. B.S. Undergraduate Mathematics Exchange, Vol. 7, No. 1 (Fall 2010) 27-30.
3) Blake, Ian F.; Gadiel Seroussi, And Nigel P. Smart. 2005 Advances in elliptic curve cryptography, London Mathematical Society Lecture Note Series, vol. 317, Cambridge University Press, Cambridge, 2005.
4) Cinkir, Zubeyir. 2011. Real Elliptic Curves. http://demonstrations.wolfram.com/RealEllipticCurves/ Wolfram Demonstrations Project.
5) Ciss, Abdoul Aziz; Moody, Dustin. 2017. Geometric progressions on elliptic curves. Glas Mat. 2017;52(1):1-10. doi: 10.3336/gm.52.1.01.
6) Duncan, John F R; Michael H Mertens 2, Ken Ono. 2017. Pariah moonshine. Nat Commun. 2017 Sep 22;8(1):670. doi: 10.1038/s41467-017-00660-y.
7) Gebel, J.; Pethő, A.; Zimmer, H.G. Computing Integral Points on Elliptic Curves. Acta Arith 1994, 68, 171–192.
8) He, Yang-Hui; Kyu-Hwan Lee, Thomas Oliver, Alexey Pozdnyakov. 2022. Murmurations of elliptic curves. arXiv:2204.10140.
9) Heath-Brown, D.R. The average analytic rank of elliptic curves. Duke Math. J. 2004, 122, 591–623. https://doi.org/10.1215/S0012-7094-04-12235-3.
10) Jaušovec N, Jaušovec K. 2010. Emotional Intelligence and Gender: A Neurophysiological Perspective. In A. Gruszka, G. Matthews, & B. Szymura (Eds.), Handbook of Individual Differences in Cognition (pp. 109–126). New York, NY: Springer New York. Retrieved from http://www.springerlink.com/index/10.1007/978-1-4419-1210-7_7.
11) Jiang, Yi; Shen, Y, Zhu Q. 2020. A Lightweight Key Agreement Protocol Based on Chinese Remainder Theorem and ECDH for Smart Homes. Sensors (Basel). 2020 Mar 2;20(5):1357. doi: 10.3390/s20051357.
12) Jong, Mathijs de; Seijo, Luis; Meijerink, Andries; Rabouw, Freddy T. (2015). "Resolving the ambiguity in the relation between Stokes shift and Huang–Rhys parameter". Physical Chemistry Chemical Physics. 17 (26): 16959–16969. doi:10.1039/C5CP02093J.
13) Keren, H., Marom, S. 2016. Long-range synchrony and emergence of neural reentry. Sci Rep 6, 36837 (2016). https://doi.org/10.1038/srep36837
14) Klee, Brad. 2018. Deriving Hypergeometric Picard-Fuchs Equations. http://demonstrations.wolfram.com/DerivingHypergeometricPicardFuchsEquations/ Wolfram Demonstrations Project.
15) Klee, Brad. 2019. A Few More Geometries after Ramanujan. http://demonstrations.wolfram.com/AFewMoreGeometriesAfterRamanujan/ Wolfram Demonstrations Project.
16) Knapp, Anthony W. 1992. Elliptic curves, Mathematical Notes, vol. 40, Princeton University Press, Princeton, NJ.
17) Kreshchuk, Michael; Tobias Gulden 2019 J. Phys. A: Math. Theor. 52 155301. DOI 10.1088/1751-8121/aaf272
18) Kühne, L. Equidistribution in Families of Abelian Varieties and Uniformity. arXiv 2021, arXiv:2101.10272.
19) Nita, Stefania Loredana; Marius Iulian Mihailescu. 2023. Elliptic Curve-Based Query Authentication Protocol for IoT Devices Aided by Blockchain. Sensors (Basel). 2023 Jan 26;23(3):1371. doi: 10.3390/s23031371.

20) Olde Daalhuis, Adri B. (2010), "Hypergeometric function", in Olver, Frank W. J.; Lozier, Daniel M.; Boisvert, Ronald F.; Clark, Charles W. (eds.), NIST Handbook of Mathematical Functions, Cambridge University Press, ISBN 978-0-521-19225-5, MR 2723248.
21) Planat, Michel; David Chester, Raymond Aschheim, Marcelo M. Amaral, Fang Fang, Klee Irwin. 2021. Finite Groups for the Kummer Surface: The Genetic Code and a Quantum Gravity Analogy. Quantum Rep. 2021, 3(1), 68-79; https://doi.org/10.3390/quantum3010005Ramanujan S. 1914. Modular Equations and Approximations to π,". Quarterly Journal of Mathematics, 45, 1914 pp. 350–372.
22) Ratner, Schatz, Mark A., George C. (2001). Quantum Mechanics in Chemistry. 0-13-895491-7: Prentice Hall. pp. 90–91.
23) Shen, L. C. 2017. On Three Differential Equations Associated with Chebyshev Polynomials of Degrees 3, 4 and 6. Acta Mathematica Sinica, 33(1), 2017 pp. 21–36. doi:10.1007/s10114-016-6180-1.
24) Tozzi A, Peters JF. 2016. Towards a Fourth Spatial Dimension of Brain Activity. Cognitive Neurodynamics 10 (3): 189–199. doi:10.1007/s11571-016-9379-z.
25) Tozzi A, Bormashenko E, Jausovec N. 2021. Topology of EEG wave fronts. Cognit Neurodyn, 15, 887–896 https://doi.org/10.1007/s11571-021-09668-z.
26) Tozzi A, Mariniello L. 2022. Unusual Mathematical Approaches Untangle Nervous Dynamics. Biomedicines; 10(10):2581. https://doi.org/10.3390/biomedicines10102581
27) Tsai, SY. 2018. Reproducibility of structural brain connectivity and network metrics using probabilistic diffusion tractography. Sci Rep **8**, 11562. https://doi.org/10.1038/s41598-018-29943-0
28) Volgushev, Maxim; Sylvain Chauvette, Mikhail Mukovski, Igor Timofeev. 2006. Precise Long-Range Synchronization of Activity and Silence in Neocortical Neurons during Slow-Wave Sleep. J Neurosci. 2006 May 24; 26(21): 5665–5672. doi: 10.1523/JNEUROSCI.0279-06.2006
29) Washington, Lawrence C. 2008. Elliptic curves: Number theory and cryptography, 2nd ED. Discrete Mathematics and Its Applications, Chapman & Hall/CRC.
30) Zimmer Horst G. 1989 Computational aspects of the theory of elliptic curves, Number theory and applications (Banff, AB, 1988) NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 265,Kluwer Acad. Publ., Dordrecht, 1989, pp. 279–324.
31) Zubrilina, Nina. 2023. Murmurations. arXiv:2310.07681