Hilbert and Pólya conjecture, dynamical system, prime numbers, black Holes, quantum mechanics, and the Riemann hypothesis

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Abstract : In mathematics, the search for exact formulas giving all the prime numbers, certain families of prime numbers or the n-th prime number has generally proved to be vain, which has led to contenting oneself with approximate formulas [8]. The purpose of this article is to give a simple function to produce the list of all prime numbers. And then I give a generalization of this result and we show a link with the quantum mechanics and the attraction of black Holes. And I give a new proof of lemma 1 which gave a proof of the Riemann hypothesis [4].

Finally another excellent new proof o f the Riemann hypothesisis given and I deduce the proof of Hilbert Polya's conjecture

Keywords : Prime Number, number theory, distribution of prime numbers, the law of prime numbers, the Gamma function, the Mertens function, quantum mechanics, black Holes, holomorphic function, Hilbert-Polya's conjecture, the Riemann hypothesis.

In memory of the great professor, the physicist and mathematician, Moshé Flato

I- INTRODUCTION, RECALL, NOTATIONS AND DEFINITIONS

Prime numbers [See 4, 5, 6, 7, 8] are used especially in information technology, such as public-key cryptography which relies on factoring large numbers into their prime factors. And in abstract algebra, prime elements and prime ideals give a generalization of prime numbers.

In mathematics, the search for exact formulas giving all the prime numbers, certain families of prime numbers or the n-th prime number has generally proved to be vain, which has led to contenting oneself with approximate formulas [8].

Recall that Mills' Theorem [8] : "There exists a real number A, Mills' constant, such that, for any integer n > 0, the integer part of A^{3^n} is a prime number" was demonstrated in 1947 by mathematician William H. Mills [8], assuming the Riemann hypothesis [4, 5, 6,7] is true. Mills' Theorem [8] is also of little use for generating prime numbers.

The purpose of this article is to give a simple function to produce the list of all prime numbers : more precisely if ψ is the function defined on $\mathbb{N}\cap[3,+\infty[$ by : $\psi(p)=\psi_p[\ ]=\Theta_{k=1}^{k=\infty}\delta(\frac{\Gamma(p+2k)+1}{p+2k})(p+2k)+(1-\delta(\frac{\Gamma(p+2k)+1}{p+2k}))\times[\ ]$, then $\{2,\psi^i(3);i\in\mathbb{N}\}$ is the list of prime numbers.

With the notations : If the u_i are functions, denote by $\Theta_{k=1}^{k=\infty}u_i=u_1\circ u_2\cdots$.

And δ the definite function from $\mathbb{R}$ on $\{0,1\}$ by $\delta(x)=1\Longleftrightarrow x\in\mathbb{N}$

In this article I suppose known the function Gamma $\Gamma : z\mapsto\int_0^{+\infty}t^{z-1}\,e^{-t}\,dt$ and its properties (See [4,6]).

Finally in the last paragraph we give a generalization of this result and we show a link with the quantum mechanics and the attraction of black Holes. And I give a new proof of lemma 1 which gave a proof of the Riemann hypothesis [4]. Finnaly Another new proof is given.

Recall that the original Hilbert and Pólya conjecture [2] is the assertion that the non-trivial zeros of the Riemann zeta function can be the spectrum of a self-adjoint operator and that Hilbert-Pólya conjecture implies the Riemann hypothesis. In this article i give a proof of the hilbert polya conjecture.
It should be noted that we so hoped to demonstrate Hilbert Polya's conjecture to demonstrate Riemann's hypothesis but ultimately it was the opposite that happened and it was the proof of Riemann's hypothesis that demonstrated Hilbert Polya's conjecture

II- STATEMENT AND PROOF OF THE RESULT :

Theorem 1 (A function generating the prime numbers) *:*
Let ψ *be the function defined on* $\mathbb{N} \cap [3, +\infty[$ *by :* $\psi(p) = \psi_p[\,] = \Theta_{k=1}^{k=\infty} \delta(\frac{\Gamma(p+2k)+1}{p+2k})(p+2k) + (1 - \delta(\frac{\Gamma(p+2k)+1}{p+2k})) \times [\,]$.
If p *is a prime number, then* $\psi(p)$ *is the prime number following* p*. And* $\{2, \psi^i(3); i \in \mathbb{N}\}$ *is the list of prime numbers.*

Proof : It follows from Proposition 1.

Proposition 1 (The sghiar's function and the prime numbers) *:*
Let $\mathcal{S}(z) = \frac{\Gamma(z)+1}{z}$.
if $z \in \mathbb{N}^*$ *then* $\mathcal{S}(z) \in \mathbb{N}^* \Longleftrightarrow z$ *is a prime number*

Proof

It follows from Wilson's theorem [3] - which assures that p is a prime number if and only if $(p-1)! \equiv -1 \mod p$

III- GENERALIZATION OF THE RESULT AND A LINK WITH QUANTUM MECHANICS AND BLACK HOLES

Theorem 2 *:*

let μ be a function from $\mathbb{R}$ to $\{0,1\}$

If E is a subset of $\mathbb{N}$ such that $E = \mu^{-1}(1)$ and p_0 is the first element of E.

Let ψ be the function defined on $\mathbb{N}$ by : $\psi(p) = \psi_p[\ ] = \Theta_{k=1}^{k=\infty} \mu(p+k)(p+k) + (1-\mu(p+k)) \times [\ ]$.

If p is one element of E, then $\psi(p)$ is the element of E that follows p. And $\{\psi^i(p_0); i \in \mathbb{N}\} = E$

Notes :

1- Contrary to appearances, ψ is well defined and is very easily calculated by a computer algorithm.

2- Interpretation of elemental forces : $\mu(p+k)(p+k) + (1-\mu(p+k)) \times [\ ]$:

- Either $\mu(p+k)(p+k) + (1-\mu(p+k)) \times [\ ]$ is the identity, therefore leaves invariant any particle of space.

- Either $\mu(p+k)(p+k) + (1-\mu(p+k)) \times [\ ]$ is the force which attracts any particle of space towards the point p+k : thus p+k acts like a black hole.

3 - The trajectory of p_0 under the action of ψ passes through any point of E because at each step $\psi^i(p_0)$ is attracted by the following black hole.

4- So if the prime number $\psi^i(p_0)$ is considered as a particle, under the action ψ, $\psi^i(p_0)$ can only be found at $\psi^{i+1}(p_0)$ prime location. Recall that a link has been established between the prime numbers, the zeros of the Riemann zeta function and the energy level of various quantum systems [see 1 and 2]

IV-THE RIEMANN HYPOTHESIS

I give a new proof of lemma 1 which gave a proof of the Riemann hypothesis [4].

Lemma 1 *(second proof)*

$$0 < Re(z) < 1 \Longrightarrow |\int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt| \neq 0$$

I will simplify the proof of Lemma 1 which allowed us to give a proof of the Riemann Hypothesis. :

It suffices to prove that $Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt) \neq 0$ or $Im(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt) \neq 0$ for $0 < Re(z) < 1$ and $Im(z) \geq 0$

Let $z = x + iy$, by change of variable, and by setting $t^{x-1} = e^u$, we deduce :

$$-Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt) = \int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} - 1} cos(y\frac{u}{x-1}) \frac{1}{x-1} e^{\frac{u}{x-1}} du$$

$$-Im(\int_0^{+\infty} \frac{t^{z-1}}{e^t - 1} dt) = \int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} - 1} sin(y\frac{u}{x-1}) \frac{1}{x-1} e^{\frac{u}{x-1}} du$$

If $-Re(\int_0^{+\infty} \frac{t^{z-1}}{e^t-1} dt) = 0$,then we deduce that :

$0 = \int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} -1} (1 - 2sin^2(\frac{1}{2}y\frac{u}{x-1})) \frac{1}{x-1} e^{\frac{u}{x-1}} du$

And consequently :

$\int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} -1} e^{\frac{u}{x-1}} du = \int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} -1} 2sin^2(\frac{1}{2}y\frac{u}{x-1}) e^{\frac{u}{x-1}} du$

And : $\int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} -1} cos^2(\frac{1}{2}y\frac{u}{x-1}) e^{\frac{u}{x-1}} du = \int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} -1} sin^2(\frac{1}{2}y\frac{u}{x-1}) e^{\frac{u}{x-1}} du$

Let $u = v + \frac{\pi(x-1)}{y}$

As $\int_{-\infty}^{+\infty} \frac{e^u}{e^{e^{\frac{u}{x-1}}} -1} cos^2(\frac{1}{2}y\frac{u}{x-1}) e^{\frac{u}{x-1}} du = \int_{-\infty}^{+\infty} e^{\pi\frac{x}{y}} \frac{e^v}{e^{e^{\frac{\pi}{y}+\frac{v}{x-1}}} -1} sin^2(\frac{1}{2}y\frac{v}{x-1}) e^{\frac{v}{x-1}} dv$

We deduce that : $\int_{-\infty}^{+\infty} (e^{\pi\frac{x}{y}} \frac{e^v}{e^{e^{\frac{\pi}{y}+\frac{v}{x-1}}} -1} - \frac{e^v}{e^{e^{\frac{v}{x-1}}} -1}) sin^2(\frac{1}{2}y\frac{v}{x-1}) e^{\frac{v}{x-1}} dv = 0$ But $e^{\pi\frac{x}{y}} \frac{e^v}{e^{e^{\frac{\pi}{y}+\frac{v}{x-1}}} -1} - \frac{e^v}{e^{e^{\frac{v}{x-1}}} -1} \lneqq 0$ (Easy to see) . Hence the result.

V- ANOTHER EXCELLENT PROOF OF THE RIEMANN HYPOTHESIS

Theorem 3 *The real part of every nontrivial zero of the Riemann zeta function is 1/2.*

The link between the function ζ and the prime numbers had already been established by Leonhard Euler with the formula [5], valid for $Re(s) > 1$:

$$\zeta(s) \ = \ \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}} = \frac{1}{\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\cdots}$$

where the infinite product is extended to the set $\mathcal{P}$ of prime numbers. This formula is sometimes called the Eulerian product.
And since the Dirichlet eta function can be defined by $\eta(s) = (1-2^{1-s})\,\zeta(s)$ where : $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$
We have in particular :

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^z}$$

for $0 < Re(z) < 1$,
Let : $s = x + iy$, with $0 < Re(s) < 1$
$\zeta(s)\zeta(\overline{s}) \ = \ \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}\frac{1}{1-p^{-\overline{s}}} = \prod_{p\in\mathcal{P}} \frac{1}{(1-e^{-xln(p)}cos(yln(p)))^2+(e^{-xln(p)}sin(yln(p)))^2}$
But : $\prod_{p\in\mathcal{P}} \frac{1}{(1-e^{-xln(p)}cos(yln(p)))^2+(e^{-xln(p)}sin(yln(p)))^2} \geq \prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-xln(p)})^2+(e^{-xln(p)})^2}$
If $\zeta(s) = 0$, then $\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-xln(p)})^2+(e^{-xln(p)})^2} = 0$ and since the non-trivial zeros of ζ are symmetric with respect to the line $X = \frac{1}{2}$ because the zeta function satisfies the functional equation [4,6] : $\zeta(s) = 2^s\pi^{s-1}\ \sin\left(\frac{\pi s}{2}\right)\ \Gamma(1-s)\ \zeta(1-s)$
then $x = \frac{1}{2} + \alpha$, and if $s' = \frac{1}{2} - \alpha + iy$, then $\zeta(s') = 0$

But the function $\frac{1}{(1+e^{-tln(p)})^2+(e^{-tln(p)})^2}$ is increasing in $[0,1]$, so $\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-tln(p)})^2+(e^{-tln(p)})^2} = 0 \ \forall t \in [\frac{1}{2}-\alpha, \frac{1}{2}+\alpha]$.

As $\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-zln(p)})^2+(e^{-zln(p)})^2}$ is holomorphic : because :

$\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-zln(p)})^2+(e^{-zln(p)})^2} = \prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}\frac{1}{1-B/p^z}$ with $A = i-1$ and $B = -i-1$, and both $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}$ and $\prod_{p\in\mathcal{P}} \frac{1}{1-B/p^z}$ are holomorphic in $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$ as we have :

$$\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z} = \prod_{p\in\mathcal{P}} (1+f_p(z))$$

with $f_p(z) = \frac{1}{(p^z/A)-1}$

$$\mid f_p(z) \mid \leq \frac{1}{\mid p^z/A \mid -1} = \frac{1}{(p^{\Re(z)}/\sqrt{2})-1} \leq k\frac{1}{p^{\frac{1}{2}}}$$

where k is a positive real constant.

So :

$$\mid \sum_{p\in\mathcal{P},p=N}^{\infty} f_p(z) \mid \leq k \mid \sum_{n=N}^{\infty} \frac{1}{n^{\frac{1}{2}}} \mid = k \mid \zeta_N(\frac{1}{2}) \mid$$

.

But(see Lemma 1 [6]) : $\zeta_N(\frac{1}{2}) = o_N(1)$

We deduce that the series $\sum_p \mid f_p \mid$ converges normally on any compact of $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$ and consequently $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}$ is holomorphic in $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$. In the same way $\prod_{p\in\mathcal{P}} \frac{1}{1-B/p^z}$ is holomorphic in $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$

If $\alpha \neq 0$, then the holomorphic function $\prod_{p\in\mathcal{P}} \frac{1}{(1+e^{-zln(p)})^2+(e^{-zln(p)})^2}$ will be null (because null on $]\frac{1}{2}, \frac{1}{2}+\alpha]$), and it follows that $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z}$ or $\prod_{p\in\mathcal{P}} \frac{1}{1-B/p^z}$ is null in $\{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$. Let's show that this is impossible :

If $\prod_{p\in\mathcal{P}} \frac{1}{1-A/p^z} = \prod_{p\in\mathcal{P}}(1+f_p(z)) = 0$ with $f_p(z) = \frac{1}{(p^z/A)-1}$ $\forall z \in \{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$. So for the same reason as above, the application :

$\circledS : X \longmapsto \prod_{p\in\mathcal{P}} \frac{1}{1-X/p^z}$ is holomorphic in the open quasi-disc $\mathcal{D} = \{X \in \mathbb{C}, 0 <\mid X \mid< \sqrt{2}\}$ with $z \in \{z \in \mathbb{C}\backslash\{1\}, \Re(z) \geq \frac{1}{2}\}$

Let's extend the function $\circledS$ by setting :

For $z \in \{z \in \mathbb{C}\backslash\{1\}, \Re(z) \gneqq \frac{1}{2}\}$ and $\forall s \in \mathbb{R},$ with $s \leq 0$, such as $\Re(s+z) \geq 0$

$\circledS(C/q^s) = \prod_{p\in\mathcal{P}} \frac{1}{1-C/(q^s p^z)}$ (where q is a prime number, and C is such that $\mid C \mid = \sqrt{2}$)

In particular we have :

$\circledS(A/q^s) = \prod_{p\in\mathcal{P}} \frac{1}{1-A/(q^s p^z)}$ (where q is a prime number)

But for $z \in \{z \in \mathbb{R}\backslash\{1\}, z \gneqq \frac{1}{2}\}$ we have :

$$\prod_{p\in\mathcal{P}} \mid \frac{1}{1-A/(q^s p^z)} \mid \leq \prod_{p\in\mathcal{P}} \mid \frac{1}{1-A/(p^z)} \mid$$

It follows that :

$$\circledS(A/q^s) = 0$$

So :

$$\circledS(X) = 0, \forall X \in \mathcal{D}$$

And consequently :

$$\circledS(1)(z) = \zeta(z) = 0$$

$\forall z \in \{z \in \mathbb{C}\backslash\{1\}, \Re(z) \gneqq \frac{1}{2}\}$

which is absurd, so $\alpha = 0$, hence the Riemann hypothesis.

VI - The Hilbert-Polya conjecture

we know George Polya [2] suggested a physical approach to prove the Riemann hypothesis.

Conjecture (Hilbert-Polya conjecture). If the non-trivial zeroes of the Riemann zeta function are written in the form $\frac{1}{2}+it_n$ then the t_n correspond to the eigenvalues of a Hermitian operator

Proof of the conjecture :

Note : It is easy to see that the Hilbert-Polya conjecture implies the Riemann hypothesis.

Now by the theorem above, any non-trivial zero of ζ can be written as $\frac{1}{2}+ia_n$ with $a_n \in \mathbb{R}$, $\zeta(\frac{1}{2}+ia_n)=0$:

If $a_n > 0$:

Let us denote the function defined on $\mathbb{R}^+$ by : $\psi_{a_n}(t)=\sqrt{2a_n}e^{-a_nt}$

If $a_n < 0$:

Let us denote the function defined on $\mathbb{R}^+$ by : $\psi_{a_n}(t)=\sqrt{-2a_n}e^{a_nt}$

Consider the $\mathbb{R}$ -vector space $\mathcal{F}$ of the functions generated by the ψ_{a_n} :

If f and g are two functions of $\mathcal{F}$, then let : $<f,g>=\int_0^{+\infty} f.g(t)dt$

$\mathcal{F}$ is a Hilbertian space.

If $\mathcal{H}$ is the operator defined by $\mathcal{H}(f(t)) = -\frac{\partial f}{\partial t}$ if $f=\psi_{a_n}$ and $a_n > 0$. And $\mathcal{H}(f(t)) = \frac{\partial f}{\partial t}$ if $f=\psi_{a_n}$ and $a_n < 0$ Then $\mathcal{H}$ extends to $\mathcal{F}$ with $\mathcal{H}(\psi_{a_n}) = a_n\psi_{a_n}$ and a_n correspond to the eigenvalues of the Hermitian operator $\mathcal{H}$.

And the Hilbert-Polya conjecture is proven

VII-CONCLUSION

In articles 4, 5, 6 and 7 the functions γ of Euler, ζ of Riamann, and the function μ of Mertens, played an important role in the knowledge of the distribution of prime numbers by allowing the proof of the Riemann Hypothesis.

In this article, my Ⓢ function which is an extension of Riamann's ζ function has given an elegant proof of Riemann's hypothesis.
As for the function ψ of theorem 1, considered as an operator on the particles, made it possible to list all the prime numbers one after the other.
It should be noted that we so hoped to demonstrate Hilbert Polya's conjecture [2] to demonstrate Riemann's hypothesis but ultimately it was the opposite that happened and it was the proof of Riemann's hypothesis that demonstrated Hilbert Polya's conjecture

Euler-Riemann- Mertens-M.J.Sghiar

References :

[1] N. M. KATZ and P. SARNAK , Random Matrices, Frobenius Eigenvalues, and Monodromy. Colloq.Publ. 45, Amer. Math. Soc., Providence, 1999.

[2] N. M. KATZ and P. SARNAK ,zeroes of zeta functions and symmetry, Bulletin (New Series) of the american mathematical society, Volume 36, Number 1, January 1999, Pages 1–26.

[3] Roshdi Rashed, Entre arithmétique et algèbre : Recherches sur l'histoire des mathématiques arabes, journal Paris, 1984.

[4] M. Sghiar, The Special Functions and the Proof of the Riemann's Hypothesis. IOSR Journal of Mathematics(IOSR-JM), 2020, 16, Issue 3, Serie II, pp.10 - 12.

[5] M Sghiar. Des applications génératrices des nombres premiers et cinq preuves de l-hypothèse de Riemann. Pioneer Journal of Algebra Number Theory and its Applications, 2015, 10 (1-2), pp.1-31. http ://www.pspchv.com/content PJNTA-vol-10-issues-1-2.

[6] M. Sghiar. The Mertens function and the proof of the Riemann's hypothesis, International Journalof Engineering and Advanced Technology (IJEAT),ISNN :2249-8958, Volume- 7 Issue-2, December 2017. (Improved in https ://hal.science/hal-01667383)

[7] M. SGHIAR "La preuve de la conjecture de Birch et Swinnerton-Dyer. "IOSR Journal ofMathematics (IOSR-JM) 14.3 (2018) : 50-59

[8] William H. Mills, A prime representing function, Bull. Amer. Math. Soc., 1947, p. 604 et 1196