Gravitational Time Dilation, Relativistic Gravity Theory, Schwarzschild's Physically Sound Original Metric and the Consequences for Cosmology

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Abstract It is natural to assume that the expanding universe was arbitrarily compact in the sufficiently remote past, in which state gravitational time dilation strongly affected its behavior. We first regard gravitational time dilation as the speed time dilation of a clock falling gravitationally from rest. Energy conservation implies that this depends solely on the the Newtonian gravitational potential difference of the clock trajectory's ends. To extend this to the relativistic domain we work out relativistic gravity theory. The metric result it yields for gravitational time dilation is consistent with our Newtonian gravitational potential result in the Newtonian limit. However the Robertson-Walker metric form for the universe implies complete absence of gravitational time dilation. Since we assume the universe was once arbitrarily compact, we turn instead to the metric for a static gravitational point source, but find that its textbook form puts a sufficiently compact universe inside an event horizon. This is due to transformation of the three radial functions which describe a static, spherically-symmetric metric into only two before inserting that now damaged metric form into the Einstein equation. Schwarzschild's original metric solution, which isn't in textbooks, involved no such transformation and therefore is physically sound; we obtain from it a picture of a universe which had an outburst of star and galaxy formation in the wake of its inflation.

1. Gravitational time dilation equals the speed time dilation of a gravitationally falling clock

The gravitational potential energy of a test particle of mass m located at height h above the earth's surface can be taken as mgh, where $g = 9.8\,\mathrm{m/s}^2$, the acceleration of gravity at the earth's surface, *provided that the height h is very much less than the earth's radius* $r_E \approx (40,000/(2\pi))\,\mathrm{km} = 6.37\times 10^6\,\mathrm{m}$.

Thus a test particle *at rest at height h_1 has energy mgh_1*, and if it falls from height h_1 *to a lesser height* h_2 $(h_2 < h_1)$, it acquires kinetic energy $\frac{1}{2}m|\dot{\mathbf{r}}|^2$, which, by conservation of total energy, satisfies,

$$mgh_1 = mgh_2 + \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 \text{ so, } \tfrac{1}{2}m|\dot{\mathbf{r}}|^2 = mgh_1 - mgh_2, \tag{1.1a}$$

and therefore,

$$|\dot{\mathbf{r}}|^2 = 2(gh_1 - gh_2). \tag{1.1b}$$

If a clock is embedded in this gravitationally falling test particle, then that clock is observed to tick at a slightly lesser rate at $h_2 < h_1$, where it has squared speed $|\dot{\mathbf{r}}|^2$, than the rate at which it ticked at h_1, where it was *at rest,* because of the well-known squared-speed Lorentz time dilation tick-rate reduction factor,

$$\sqrt{1-\left(|\dot{\mathbf{r}}|^2/c^2\right)} = \sqrt{1-\left((2(gh_1-gh_2))/c^2\right)} \approx \Big[1-\left((gh_1-gh_2)/c^2\right)\Big]. \tag{1.2a}$$

Because the test particle's gravitational mass m, which occurs in its gravitational potential energy mgh, is equal to its inertial mass m, which occurs in its kinetic energy $\frac{1}{2}m|\dot{\mathbf{r}}|^2$, the test particle's mass m *doesn't appear at all* in the Eq. (1.2a) factor for the change in the tick rate of the test particle's embedded clock when it falls gravitationally from rest at height h_1 to the lesser height h_2. Thus it is very convenient to define a gravitational potential function $\phi(h) \stackrel{\text{def}}{=} gh$, which is independent of the test particle's mass m and gives the test particle's gravitational potential energy mgh when it is multiplied by the test particle's mass m (this is analogous to the electrical potential function $\phi(\mathbf{r})$, which is independent of the test particle's charge q and gives the test particle's electrical potential energy $q\phi(\mathbf{r})$ when it is multiplied by the test particle's charge q). In terms of the gravitational potential function values $\phi(h_2)$ and $\phi(h_1)$, the dimensionless Eq. (1.2a) gravitational tick-rate change factor (GTRCHF) for the clock which falls gravitationally from rest at height h_1 to the lesser height h_2 is,

$$\mathrm{GTRCHF}(h_2; h_1) = \sqrt{1+\left((2(\phi(h_2)-\phi(h_1)))/c^2\right)}, \tag{1.2b}$$

which, aside from the universal constant c, depends *exclusively* on the gravitational potential function values $\phi(h)$ at the two particular heights $h = h_1$ and $h = h_2$. This *very strongly suggests* that a clock's gravitational tick-rate change factor $\mathrm{GTRCHF}(h_2; h_1)$ *doesn't depend on the particular process whereby the clock changes its height from h_1 to the lesser height h_2.*

Consider, for example, the case where the mass-m test particle with embedded clock is lowered from height h_1 to the lesser height h_2 *arbitrarily slowly by a correspondingly arbitrarily-slow lowering device.* In

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that case *work* $mg(h_1 - h_2) = mgh_1 - mgh_2$ *is done on the arbitrarily-slow lowering device, so energy* $(mgh_1 - mgh_2)$ *is removed from the test particle by that lowering device,* resulting in the test particle being virtually *motionless* at the lesser height h_2 (which of course *is the reason* for *using* the arbitrarily-slow lowering device) *instead* of having the kinetic energy $\frac{1}{2}m|\dot{\mathbf{r}}|^2 = mgh_1 - mgh_2$ at the lesser height h_2 that *it would have* if the arbitrarily-slow lowering device *wasn't used* (see Eq. (1.1a)). Eq. (1.2b), however, says that the gravitational tick-rate change factor $\mathrm{GTRCHF}(h_2; h_1)$ for the mass-m test particle's embedded clock *depends solely on* $(\phi(h_2) - \phi(h_1)) = -((mgh_1 - mgh_2)/m)$ regardless of whether the mass-m test particle *sheds the gravitational potential energy* $(mgh_1 - mgh_2)$ by converting it to kinetic energy or by doing that amount of work on the arbitrarily-slow lowering device. In other words, the Eq. (1.2b) gravitational tick-rate change factor $\mathrm{GTRCHF}(h_2; h_1)$ for the mass-m test particle's embedded clock *is independent* of whether the test particle *falls gravitationally* from height h_1 to the lesser height h_2 or *is lowered arbitrarily slowly* from height h_1 to the lesser height h_2.

In a celebrated experiment two hyper-accurate atomic clocks were attached to a wall, one clock 33 cm above the other. According to Eq. (1.2b), the clock below runs slower than the clock above by the factor $\sqrt{1 + \big((2(\phi(h_2) - \phi(h_1)))/c^2\big)} \approx \big[1 - \big((g(h_1 - h_2))/c^2\big)\big]$, which is equal to $\big[1 - \big((9.8\ \mathrm{m/s^2})(0.33\,\mathrm{m})/(3 \times 10^8\ \mathrm{m/s})^2\big)\big] = \big[1 - 3.59 \times 10^{-17}\big]$. After five days (120 hours, or 4.32×10^5 seconds), the clock below is therefore supposed to record 15.5×10^{-12} seconds (15.5 picoseconds) less than the clock above records. Remarkably, certain atomic clocks are actually accurate for such fantastically short time intervals. As a precaution against systematic errors, the positions of the two clocks were swapped and the experiment was repeated.

We next consider the gravitational time dilation of a clock on the earth's surface at the equator relative to a clock in a satellite directly overhead *whose circular orbit lies in the equator's plane, has a period of 24-hours and travels in the direction of the earth's rotation.* If such a satellite is directly overhead the clock on the earth's surface at the equator, it of course *remains fixed directly overhead that clock,* just as the upper clock on the wall in the experiment described above *remains fixed directly above the lower clock.* Most satellites *contrariwise move at high speed relative to any clock on the earth's surface,* so for most satellites it is *necessary* to take into account squared-speed Lorentz time dilation *in addition* to gravitational time dilation. Even the famed GPS satellites have 12-hour orbit periods instead of 24-hour orbit periods, so squared-speed Lorentz time dilation *must be taken into account* for GPS satellites *in addition* to gravitational time dilation.

Since the height h_{GS} above the earth's surface of a geosynchronous 24-hour circular orbit in the equator's plane is *several times* the earth's radius $r_E \approx 6.37 \times 10^6$ m, we must abandon the approximation implicit in Eq. (1.1a) that the acceleration of gravity *is fixed at* $g = 9.8\ \mathrm{m/s^2}$, and switch to full Newtonian gravity.

The earth's Newtonian gravitational force $\mathbf{F}$ on a mass-m test particle is $-GmM\mathbf{r}/|\mathbf{r}|^3$, where G is the universal gravitational constant, M is the earth's mass and $\mathbf{r}$ is the vector from the earth's center to the test particle, provided that $|\mathbf{r}| \geq r_E$. Furthermore, Newton's Second Law states that $\mathbf{F} = m\ddot{\mathbf{r}}$, so,

$$\ddot{\mathbf{r}} = -GM\mathbf{r}/|\mathbf{r}|^3, \tag{1.3a}$$

regardless of the specific nonzero value m of the test particle's mass. The result of taking the norms of the vectors on both sides of Eq. (1.3a) is,

$$|\ddot{\mathbf{r}}| = GM/|\mathbf{r}|^2. \tag{1.3b}$$

In the special case that $\mathbf{r}$ lies on the earth's surface, so that $|\mathbf{r}| = r_E$, Eq. (1.3b) becomes,

$$|\ddot{\mathbf{r}}|_{|\mathbf{r}|=r_E} = GM/(r_E)^2. \tag{1.3c}$$

The entity $|\ddot{\mathbf{r}}|_{|\mathbf{r}|=r_E}$ is the norm of the acceleration vector of a gravitational test particle at the earth's surface, which of course is equal to $g = 9.8\,\mathrm{m/s^2}$. Eq. (1.3c) thus implies that,

$$GM = g(r_E)^2, \tag{1.3d}$$

so we can rewrite Eq. (1.3b) as,

$$|\ddot{\mathbf{r}}| = g(r_E/|\mathbf{r}|)^2, \tag{1.3e}$$

and we can likewise rewrite Eq. (1.3a) as,

$$\ddot{\mathbf{r}} = -g(r_E)^2\big(\mathbf{r}/|\mathbf{r}|^3\big). \tag{1.3f}$$

We now reexpress Eq. (1.3f) as, $\ddot{\mathbf{r}} + g(r_E)^2\big(\mathbf{r}/|\mathbf{r}|^3\big) = \mathbf{0}$, and we then take the dot product of both sides of that equation with $\dot{\mathbf{r}}$ to obtain $(\ddot{\mathbf{r}}\cdot\dot{\mathbf{r}}) + g(r_E)^2\big((\dot{\mathbf{r}}\cdot\mathbf{r})/(\mathbf{r}\cdot\mathbf{r})^{3/2}\big) = 0$. This last equation can be rewritten, $\frac{d}{dt}\big(\frac{1}{2}(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}) - g(r_E)^2\big(1/(\mathbf{r}\cdot\mathbf{r})^{1/2}\big)\big) = 0$, which expresses a conservation law that is more neatly written as,

$$\tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(r_E/|\mathbf{r}|)\big) = 0. \tag{1.4a}$$

We digress to derive the very useful approximation to Eq. (1.4a) for those cases where the mass-m test particle's distance h of above the surface of the earth, $h = \big(|\mathbf{r}| - r_E\big)$, satisfies $0 \le h \ll r_E$,

$$0 = \tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(r_E/|\mathbf{r}|)\big) = \tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(r_E/(r_E+h))\big) = \tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(1/(1+(h/r_E)))\big) =$$
$$\tfrac{d}{dt}\Big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E + gh + g\,r_E\,O\big((h/r_E)^2\big)\Big) = \tfrac{d}{dt}\Big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 + gh\big(1 + O\big((h/r_E)\big)\big)\Big) \approx \tfrac{d}{dt}\big(\tfrac{1}{2}|\dot{\mathbf{r}}|^2 + gh\big), \tag{1.4b}$$

where we have used the fact that $\frac{d}{dt}(-g\,r_E) = 0$, since g and r_E are constants. The Eq. (1.4b) result, which holds when $0 \le h \ll r_E$, as well implies that $\frac{d}{dt}\big(\frac{1}{2}m|\dot{\mathbf{r}}|^2 + mgh\big) \approx 0$, which of course *underlies* Eq. (1.1a).

Returning now to Eq. (1.4a), we *deduce* from it that *the squared speed* $|\dot{\mathbf{r}}|^2$ of a test particle with an embedded clock *which has fallen gravitationally to the earth's surface* (namely to the radius $|\mathbf{r}| = r_E$) *from initial rest* (namely the initial speed $|\dot{\mathbf{r}}| = 0$) *and the initial radius* $|\mathbf{r}| = r_{\mathrm{GS}}$, *where* r_{GS} *is the orbit radius of the geosynchronous satellite, satisfies,*

$$-g\,r_E(r_E/r_{\mathrm{GS}}) = \tfrac{1}{2}|\dot{\mathbf{r}}|^2 - g\,r_E(r_E/r_E) \text{ so, } |\dot{\mathbf{r}}|^2 = 2g\,r_E(1 - (r_E/r_{\mathrm{GS}})). \tag{1.5}$$

Due to the Eq. (1.5) squared speed $|\dot{\mathbf{r}}|^2$ which the embedded clock gains as a result of its embedding test particle's gravitational fall from rest at the radius r_{GS} to the earth's radius r_E, the embedded clock's tick rate is slowed at the end of that fall by the Lorentz time dilation tick-rate reduction factor,

$$\sqrt{1 - |\dot{\mathbf{r}}/c|^2} = \sqrt{1 - \big(\big(2g\,r_E(1 - (r_E/r_{\mathrm{GS}}))\big)/c^2\big)} \approx \Big[1 - \big(\big(g\,r_E(1 - (r_E/r_{\mathrm{GS}}))\big)/c^2\big)\Big]. \tag{1.6}$$

To evaluate the Eq. (1.6) gravitational reduction factor for the tick rate of a clock on the earth's surface at the equator relative to the tick rate of an identical clock directly overhead in a geosynchronous satellite, *we need the value of* r_{GS}, the radius of the geosynchronous orbit. If we take the equator's plane to be the x-y plane, so the earth's axis of rotation is the z-axis and the earth's center is located at $x = y = z = 0$, a circular test-particle orbit in the equator's plane, whose center is the earth's center, and which has fixed radius $|\mathbf{r}|$ and period T has the form,

$$\mathbf{r} = |\mathbf{r}|\Big(\cos\big(2\pi(t/T) + \delta\big),\ \pm\sin\big(2\pi(t/T) + \delta\big),\ 0\Big), \tag{1.7}$$

where the $\pm$ sign determines whether the test particle travels in the direction of the earth's rotation or in the opposite direction. *Two crucial properties of the* Eq. (1.7) *circular orbit* $\mathbf{r}$ *are that* $(\dot{\mathbf{r}}\cdot\mathbf{r}) = 0$, *which implies that* $|\mathbf{r}|^2$ *is independent of time, and that* $\ddot{\mathbf{r}} = -(2\pi/T)^2\,\mathbf{r}$, which implies that $\mathbf{r}\times\ddot{\mathbf{r}} = \mathbf{0}$, an attribute shared by Eq. (1.3f), which is the property of *orbital angular momentum conservation* because $d\mathbf{L}/dt = d(\mathbf{r}\times(m\dot{\mathbf{r}}))/dt = m(\mathbf{r}\times\ddot{\mathbf{r}})$. Insertion of $\ddot{\mathbf{r}} = -(2\pi/T)^2\,\mathbf{r}$ into Eq. (1.3f) and application of the fact that $|\mathbf{r}|$ is independent of time *completely determines* $|\mathbf{r}|$ *in terms of* T *as,*

$$|\mathbf{r}| = \Big(g(r_E)^2\big(T/(2\pi)\big)^2\Big)^{\frac{1}{3}} \text{ or equivalently, } |\mathbf{r}| = \Big((g/r_E)\big(T/(2\pi)\big)^2\Big)^{\frac{1}{3}}\,r_E. \tag{1.8}$$

Inserting the geosynchronous period T = 24 hours $= 8.64\times10^4$ s into Eq. (1.8) together with $(g/r_E) = 1.54\times10^{-6}\,\mathrm{s}^{-2}$ yields that the radius r_{GS} of the geosynchronous orbit equals $6.63\,r_E$. Therefore the height above the earth's surface of the geosynchronous orbit is $h_{\mathrm{GS}} = \big(r_{\mathrm{GS}} - r_E\big) = 5.63\,r_E \approx 35{,}800$ km.

Inserting $(r_E/r_{\mathrm{GS}}) = (1/6.63)$ together with $\big(g\,r_E/c^2\big) = 6.93\times10^{-10}$ into Eq. (1.6) yields that a clock on the earth's surface at the equator ticks at a rate which is slower by a factor of $\big[1 - 5.89\times10^{-10}\big]$ than an identical clock in a geosynchronous satellite directly overhead. This minute deviation from equality of the two clocks' tick rates is vastly greater however than the deviation described by the tick-rate reduction factor $\big[1 - 3.59\times10^{-17}\big]$ we previously noted for a clock on a wall 33 cm below an identical clock on that wall. These examples show that although terrestrial gravitational time dilation is exceedingly small, atomic-clock technology has nevertheless confirmed its existence and physical systematics beyond any reasonable doubt.

If we reexpress Eq. (1.4a) as $\frac{d}{dt}\big(\frac{1}{2}|\dot{\mathbf{r}}|^2 + \phi(|\mathbf{r}|)\big) = 0$, where $\phi(|\mathbf{r}|) = -g\, r_E(r_E/|\mathbf{r}|)$ is the Newtonian gravitational potential function for the earth, which is valid when $|\mathbf{r}| \geq r_E$, then *the squared speed* $|\dot{\mathbf{r}}|^2$ of a clock which falls gravitationally from rest at $|\mathbf{r}|_>$ to $|\mathbf{r}|_<$, where $|\mathbf{r}|_> \geq |\mathbf{r}|_< \geq r_E$, *follows from the conservation relation* $\phi(|\mathbf{r}|_>) = \frac{1}{2}|\dot{\mathbf{r}}|^2 + \phi(|\mathbf{r}|_<)$, *which yields that the clock's tick-rate reduction factor is,*

$$\sqrt{1 - |\dot{\mathbf{r}}/c|^2} = \sqrt{1 - \big(2\big(\phi(|\mathbf{r}|_>) - \phi(|\mathbf{r}|_<)\big)/c^2\big)} \approx \Big[1 - \big(\big(\phi(|\mathbf{r}|_>) - \phi(|\mathbf{r}|_<)\big)/c^2\big)\Big]. \tag{1.9}$$

Eq. (1.9) *captures the essence* of the foregoing Eqs. (1.6) and (1.2a).

We next extend our understanding of gravitational time dilation to relativistic gravity theory, which we work out from the principles of Lorentz covariance and equivalence, and the validity of Newtonian gravity for a nonrelativistic test particle in a static gravitational potential which is much weaker than c^2.

2. Relativistic gravity from Lorentz covariance, equivalence and its Newtonian precursor

Lorentz covariance asserts that for every given gravitational field a test particle of positive rest mass m is subject to *a Lorentz-covariant version of Newton's Second Law,*

$$\frac{d^2x^\lambda}{d\tau^2} = F^\lambda/m, \tag{2.1}$$

where τ is Lorentz invariant and *both* the trajectory $x^\lambda(\tau)$ and the four-force F^λ *are Lorentz four-vectors.*

The equivalence principle asserts that for every given gravitational field and test-particle space-time trajectory $x^\mu(\tau)$ *in that gravitational field, there exists a one-to-one transformation* $r^\alpha(x^\mu)$ *of space-time onto itself such that the transformed trajectory* $R^\alpha(\tau) \stackrel{\text{def}}{=} r^\alpha(x^\mu(\tau))$ *manifests zero proper acceleration, i.e.,*

$$\frac{d^2R^\alpha}{d\tau^2} = 0, \tag{2.2a}$$

where,

$$d\tau \stackrel{\text{def}}{=} (dR^0/c)\sqrt{1 - |d\mathbf{R}/dR^0|^2} = \big(\sqrt{(dR^0)^2 - |d\mathbf{R}|^2}\,\big)/c = \big(\sqrt{\eta_{\alpha\beta}dR^\alpha dR^\beta}\,\big)/c, \tag{2.2b}$$

where, of course,

$$\eta_{00} = +1, \quad \eta_{11} = \eta_{22} = \eta_{33} = -1 \quad \text{and} \quad \eta_{\alpha\beta} = 0 \ \text{ if } \ \alpha \neq \beta. \tag{2.2c}$$

Eq. (2.2b) implies that,

$$\eta_{\alpha\beta}\frac{dR^\alpha}{d\tau}\frac{dR^\beta}{d\tau} = c^2, \tag{2.2d}$$

and it also implies that,

$$(c\,d\tau)^2 = \eta_{\alpha\beta}dR^\alpha dR^\beta. \tag{2.2e}$$

The insertion of $R^\alpha(\tau) \stackrel{\text{def}}{=} r^\alpha(x^\mu(\tau))$ into Eq. (2.2a) enables us to extract an equation of motion for $x^\lambda(\tau)$, the trajectory of the test particle in the gravitational field, that is of the form of Eq. (2.1), where the four-acceleration F^λ/m is independent of the test particle's mass m, but depends on its proper velocity $dx^\mu/d\tau$ and on partial derivatives of the transformation $r^\alpha(x^\mu)$ and its inverse $x^\lambda(r^\alpha)$,

$$0 = \frac{d^2R^\alpha}{d\tau^2} = \frac{d}{d\tau}\Big(\frac{d}{d\tau}\big(r^\alpha(x^\mu(\tau))\big)\Big) = \frac{d}{d\tau}\Big(\frac{\partial r^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}\Big) = \frac{\partial r^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2 r^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau}. \tag{2.3a}$$

Because the transformation $r^\alpha(x^\mu)$ maps space-time one-to-one onto itself, it has the unique inverse $x^\lambda(r^\alpha)$ (with the same property), and the partial derivatives of $x^\lambda(r^\alpha)$ and $r^\alpha(x^\mu)$ satisfy the well-known identity,

$$\frac{\partial x^\lambda}{\partial r^\alpha}\frac{\partial r^\alpha}{\partial x^\mu} = \delta^\lambda_\mu. \tag{2.3b}$$

We now multiply the expression in Eq. (2.3a) which follows its rightmost equal sign by $\frac{\partial x^\lambda}{\partial r^\alpha}$, sum over the index α and then apply the Eq. (2.3b) identity to obtain,

$$0 = \frac{\partial x^\lambda}{\partial r^\alpha}\frac{\partial r^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial x^\lambda}{\partial r^\alpha}\frac{\partial^2 r^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\nu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{d^2x^\lambda}{d\tau^2} + \frac{\partial x^\lambda}{\partial r^\alpha}\frac{\partial^2 r^\alpha}{\partial x^\mu\partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}. \tag{2.3c}$$

Therefore,

$$\frac{d^2x^\lambda}{d\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{2.3d}$$

where,

$$\Gamma^{\lambda}_{\mu\nu} \stackrel{\text{def}}{=} \frac{\partial x^{\lambda}}{\partial r^{\alpha}} \frac{\partial^2 r^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}, \tag{2.3e}$$

is called *the affine connection.* Eqs. (2.3d) and (2.3e) show that the four-acceleration F^{λ}/m of Eq. (2.1) is equal to $-\Gamma^{\lambda}_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}$, which is independent of the test particle's mass m, but depends on its proper velocity $dx^{\mu}/d\tau$ and on partial derivatives of the transformation $r^{\alpha}(x^{\mu})$ *and its inverse* $x^{\lambda}(r^{\alpha})$. *A critically important point here* is that the Eq. (2.3e) affine connection $\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^{\lambda}}{\partial r^{\alpha}} \frac{\partial^2 r^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$ *is ill-defined if the transformation* $r^{\alpha}(x^{\mu})$ *doesn't have the well-defined inverse* $x^{\lambda}(r^{\alpha})$.

Just as we inserted $R^{\alpha}(\tau) \stackrel{\text{def}}{=} r^{\alpha}(x^{\mu}(\tau))$ into Eq. (2.2a), which is the equation of the non-accelerating motion of R^{α}, to obtain Eqs. (2.3d) and (2.3e), the equations of the gravitationally-accelerating motion of x^{λ}, we shall now insert $R^{\alpha}(\tau) \stackrel{\text{def}}{=} r^{\alpha}(x^{\mu}(\tau))$ into Eq. (2.2d) to obtain the generalization of $\eta_{\alpha\beta}$ which corresponds to the gravitationally-accelerating motion of x^{μ},

$$c^2 = \eta_{\alpha\beta} \frac{dR^{\alpha}}{d\tau} \frac{dR^{\beta}}{d\tau} = \eta_{\alpha\beta} \frac{dr^{\alpha}(x^{\mu}(\tau))}{d\tau} \frac{dr^{\beta}(x^{\nu}(\tau))}{d\tau} = \eta_{\alpha\beta} \frac{\partial r^{\alpha}}{\partial x^{\mu}} \frac{dx^{\mu}}{d\tau} \frac{\partial r^{\beta}}{\partial x^{\nu}} \frac{dx^{\nu}}{d\tau} = \left(\eta_{\alpha\beta} \frac{\partial r^{\alpha}}{\partial x^{\mu}} \frac{\partial r^{\beta}}{\partial x^{\nu}}\right) \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau}. \tag{2.4a}$$

Therefore,

$$g_{\mu\nu} \frac{dx^{\mu}}{d\tau} \frac{dx^{\nu}}{d\tau} = c^2, \tag{2.4b}$$

where,

$$g_{\mu\nu} \stackrel{\text{def}}{=} \eta_{\alpha\beta} \frac{\partial r^{\alpha}}{\partial x^{\mu}} \frac{\partial r^{\beta}}{\partial x^{\nu}}, \tag{2.4c}$$

is called *the metric tensor.* Eq. (2.4b) is the generalization of Eq. (2.2d) to the gravitationally-accelerating motion of x^{μ}, and it obviously implies that,

$$(c\, d\tau)^2 = g_{\mu\nu} dx^{\mu} dx^{\nu} \quad \text{and} \quad d\tau = \left(\sqrt{g_{\mu\nu} dx^{\mu} dx^{\nu}}\,\right)/c, \tag{2.4d}$$

which are the generalizations of Eqs. (2.2e) and (2.2b) to the gravitationally-accelerating motion of x^{μ}.

We have seen that *it is critically important* for the inverse $x^{\lambda}(r^{\alpha})$ of the transformation $r^{\alpha}(x^{\mu})$ *to be well-defined,* otherwise *the affine connection is ill-defined.* Therefore we can be sure that the partial-derivative identities such as $\frac{\partial x^{\lambda}}{\partial r^{\alpha}} \frac{\partial r^{\alpha}}{\partial x^{\mu}} = \delta^{\lambda}_{\mu}$ of Eq. (2.3b) hold, so the 4×4 partial-derivative matrix $\frac{\partial r^{\alpha}}{\partial x^{\mu}}$ *definitely has a matrix inverse.* Therefore the Eq. (2.4c) *definition* of the metric tensor, $g_{\mu\nu} \stackrel{\text{def}}{=} \eta_{\alpha\beta} \frac{\partial r^{\alpha}}{\partial x^{\mu}} \frac{\partial r^{\beta}}{\partial x^{\nu}}$, *guarantees that it has a* 4×4 *matrix inverse* (the 4×4 matrix $\eta_{\alpha\beta}$ clearly is its own inverse).

Besides having a matrix inverse, the metric tensor is a symmetric matrix, so it has four eigenvalues, none of which can equal zero (or it wouldn't have an inverse). The four eigenvalues of the diagonal matrix $\eta_{\alpha\beta}$ are +1, -1, -1, -1, and the matrix form of the metric tensor $g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial r^{\alpha}}{\partial x^{\mu}} \frac{\partial r^{\beta}}{\partial x^{\nu}}$ is $g = D\eta D^T$, where $D_{\mu\alpha} \stackrel{\text{def}}{=} \frac{\partial r^{\alpha}}{\partial x^{\mu}}$; this matrix D *definitely has an inverse.* Therefore the symmetric metric tensor g *is a congruence transformation* of the diagonal matrix η, so *by the Sylvester's law of inertia theorem,* the *signs* of the eigenvalues of the metric tensor g are the *same* as the *signs* of the eigenvalues of η, namely +, -, -, -. *No exception to the rule that the signature of the metric tensor is +, -, -, - can be tolerated.*

The affine connection can be expressed as a linear combination of three partial derivatives of the metric tensor contracted into the metric tensor's *inverse.* Since $g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial r^{\alpha}}{\partial x^{\mu}} \frac{\partial r^{\beta}}{\partial x^{\nu}}$,

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \eta_{\alpha\beta} \frac{\partial^2 r^{\alpha}}{\partial x^{\mu} \partial x^{\lambda}} \frac{\partial r^{\beta}}{\partial x^{\nu}} + \eta_{\alpha\beta} \frac{\partial r^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 r^{\beta}}{\partial x^{\nu} \partial x^{\lambda}} = \eta_{\alpha\beta} \frac{\partial x^{\kappa}}{\partial r^{\gamma}} \frac{\partial^2 r^{\gamma}}{\partial x^{\mu} \partial x^{\lambda}} \frac{\partial r^{\alpha}}{\partial x^{\kappa}} \frac{\partial r^{\beta}}{\partial x^{\nu}} + \eta_{\alpha\beta} \frac{\partial r^{\alpha}}{\partial x^{\mu}} \frac{\partial r^{\beta}}{\partial x^{\kappa}} \frac{\partial x^{\kappa}}{\partial r^{\gamma}} \frac{\partial^2 r^{\gamma}}{\partial x^{\nu} \partial x^{\lambda}} =$$

$$\Gamma^{\kappa}_{\mu\lambda} g_{\kappa\nu} + g_{\mu\kappa} \Gamma^{\kappa}_{\nu\lambda} = g_{\mu\kappa} \Gamma^{\kappa}_{\nu\lambda} + g_{\nu\kappa} \Gamma^{\kappa}_{\mu\lambda}. \tag{2.5a}$$

Therefore,

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\lambda\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} = g_{\mu\kappa} \Gamma^{\kappa}_{\nu\lambda} + g_{\nu\kappa} \Gamma^{\kappa}_{\mu\lambda} + g_{\lambda\kappa} \Gamma^{\kappa}_{\nu\mu} + g_{\nu\kappa} \Gamma^{\kappa}_{\lambda\mu} - g_{\mu\kappa} \Gamma^{\kappa}_{\lambda\nu} - g_{\lambda\kappa} \Gamma^{\kappa}_{\mu\nu} = 2 g_{\nu\kappa} \Gamma^{\kappa}_{\lambda\mu}, \tag{2.5b}$$

where we have used the facts that $g_{\mu\nu} = g_{\nu\mu}$ and that $\Gamma^{\kappa}_{\mu\nu} = \Gamma^{\kappa}_{\nu\mu}$.

We know that the metric tensor *always has the signature* +, -, -, -, *and always has an inverse.* To extract the affine connection $\Gamma^{\sigma}_{\lambda\mu}$ from the Eq. (2.5b) result, *we need the inverse of the metric tensor* $g_{\nu\kappa} = \frac{\partial r^{\alpha}}{\partial x^{\nu}} \eta_{\alpha\beta} \frac{\partial r^{\beta}}{\partial x^{\kappa}}$, which in fact is $g^{\sigma\nu} = \frac{\partial x^{\sigma}}{\partial r^{\gamma}} \eta^{\gamma\lambda} \frac{\partial x^{\nu}}{\partial r^{\lambda}}$, where *the inverse* $\eta^{\gamma\lambda}$ *of* $\eta_{\gamma\lambda}$ *is* $\eta_{\gamma\lambda}$ *itself,* since $\eta_{\gamma\lambda}$ is a diagonal matrix whose diagonal matrix elements are all either $+1$ or -1. To show *explicitly* that the $g^{\sigma\nu}$ written above *is indeed the inverse of the metric tensor* $g_{\nu\kappa}$, which is also written above, we now calculate,

$$g^{\sigma\nu}g_{\nu\kappa} = \frac{\partial x^\sigma}{\partial r^\gamma}\eta^{\gamma\lambda}\frac{\partial x^\nu}{\partial r^\lambda}\,\frac{\partial r^\alpha}{\partial x^\nu}\eta_{\alpha\beta}\frac{\partial r^\beta}{\partial x^\kappa} = \frac{\partial x^\sigma}{\partial r^\gamma}\eta^{\gamma\lambda}\delta^\alpha_\lambda\eta_{\alpha\beta}\frac{\partial r^\beta}{\partial x^\kappa} = \frac{\partial x^\sigma}{\partial r^\gamma}\eta^{\gamma\alpha}\eta_{\alpha\beta}\frac{\partial r^\beta}{\partial x^\kappa} = \frac{\partial x^\sigma}{\partial r^\gamma}\delta^\gamma_\beta\frac{\partial r^\beta}{\partial x^\kappa} = \frac{\partial x^\sigma}{\partial r^\beta}\frac{\partial r^\beta}{\partial x^\kappa} = \delta^\sigma_\kappa. \quad (2.5c)$$

Thus multiplying both sides of Eq. (2.5b) by $\frac{1}{2}g^{\sigma\nu}$ and summing over ν yields the affine connection $\Gamma^\sigma_{\lambda\mu}$ as,

$$\Gamma^\sigma_{\lambda\mu} = \tfrac{1}{2}g^{\sigma\nu}\left[\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\lambda}}{\partial x^\nu}\right]. \quad (2.5d)$$

We next insert the Eq. (2.5d) result for the affine connection in terms of the metric tensor into the Eq. (2.3d) dynamical equation for a test particle in a gravitational field. We relate that to Newtonian gravity by assuming that the speed $|\dot{\mathbf{x}}|$ of the test particle is much less than c, and that the metric tensor $g_{\mu\nu}$ is static, with a deviation $h_{\mu\nu} \stackrel{\text{def}}{=} (g_{\mu\nu} - \eta_{\mu\nu})$ from $\eta_{\mu\nu}$ that is much smaller in norm than unity.

For $|\dot{\mathbf{x}}| \ll c$, $d\tau = dt\sqrt{1-(|\dot{\mathbf{x}}|/c)^2} \approx dt$, so $dx^0/d\tau \approx dx^0/dt = c \gg |\dot{\mathbf{x}}| \approx |d\mathbf{x}/d\tau|$. Therefore, for $|\dot{\mathbf{x}}| \ll c$, the Eq. (2.3d) dynamical equation for a test particle in a gravitational field, namely,

$$\frac{d^2x^\sigma}{d\tau^2} + \Gamma^\sigma_{\lambda\mu}\frac{dx^\lambda}{d\tau}\frac{dx^\mu}{d\tau} = 0, \quad (2.6a)$$

is well approximated by,

$$\frac{d^2x^\sigma}{d\tau^2} + c^2\,\Gamma^\sigma_{00}\left(\frac{dt}{d\tau}\right)^2 = 0. \quad (2.6b)$$

We next insert into Γ^σ_{00}, as it is given by Eq. (2.5d), the metric tensor $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, *where* $h_{\mu\nu}$ *is assumed to be static,* and contributions to Γ^σ_{00} which are second-order or higher in $h_{\mu\nu}$ are discarded,

$$\Gamma^\sigma_{00} = \tfrac{1}{2}g^{\sigma\nu}\left[\frac{\partial g_{0\nu}}{\partial x^0} + \frac{\partial g_{0\nu}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^\nu}\right] \approx -\tfrac{1}{2}\eta^{\sigma\nu}\frac{\partial h_{00}}{\partial x^\nu}, \quad (2.6c)$$

which, upon insertion into Eq. (2.6b), yields,

$$\frac{d^2x^\sigma}{d\tau^2} = \tfrac{1}{2}\,c^2\left(\frac{dt}{d\tau}\right)^2\eta^{\sigma\nu}\frac{\partial h_{00}}{\partial x^\nu}. \quad (2.6d)$$

Because h_{00} is *static,* $\frac{\partial h_{00}}{\partial x^0} = 0$, so the $\sigma = 0$ component of Eq. (2.6d) yields,

$$\frac{d^2(ct)}{d\tau^2} = 0, \quad (2.6e)$$

which implies that,

$$\frac{dt}{d\tau} \text{ is } \textit{constant.} \quad (2.6f)$$

The $\sigma = 1$, 2 and 3 *components of* Eq. (2.6d) yield the *three-vector* equation,

$$\frac{d^2\mathbf{x}}{d\tau^2}\Big/\left(\frac{dt}{d\tau}\right)^2 = -\nabla_{\mathbf{x}}(\tfrac{1}{2}\,c^2\,h_{00}), \quad (2.6g)$$

which, together with $\frac{dt}{d\tau}$ being constant, implies that,

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla_{\mathbf{x}}(\tfrac{1}{2}\,c^2\,h_{00}). \quad (2.6h)$$

The corresponding *Newtonian* gravitational acceleration equation of course is,

$$\frac{d^2\mathbf{x}}{dt^2} = -\nabla_{\mathbf{x}}\,\phi, \quad (2.6i)$$

where *a typical example* of such a Newtonian gravitational potential ϕ is,

$$\phi = -GM/|\mathbf{x}|, \quad (2.6j)$$

which applies specifically *when the gravitational source is a static point mass* M *located at* $\mathbf{x} = \mathbf{0}$.

Comparison of the Eq. (2.6h) metric equation with the Eq. (2.6i) Newtonian equation shows that,

$$h_{00} = 2\phi/c^2, \text{ so } \; g_{00} = \eta_{00} + h_{00} = 1 + 2\phi/c^2. \quad (2.6k)$$

Therefore, *in the Newtonian limit* the gravitational metric matrix element g_{00} is,

$$g_{00} = 1 + 2\phi/c^2. \quad (2.6l)$$

We next turn to the matter of the time recorded by a clock embedded in a test particle which is in arbitrary motion in an arbitrary gravitational field versus the time which that clock records when the test particle is at rest in *zero* gravitational field. The metric tensor $g_{\mu\nu}$ turns out to be at the core of this matter; using the proper differential time $d\tau = \left(\sqrt{g_{\mu\nu}dx^\mu dx^\nu}\,\right)/c$ that is given by Eq. (2.4d), we obtain,

$$d\tau/dt = \sqrt{g_{\mu\nu}(x)(dx^\mu/dx^0)(dx^\nu/dx^0)}\,, \tag{2.7a}$$

and if the test particle with the embedded clock *is at rest* in the gravitational field,

$$d\tau/dt = \sqrt{g_{00}(x)}\,. \tag{2.7b}$$

The *most interesting consequence of* Eq. (2.7b) is that clocks *at rest at different space-time points in a gravitational field* will in general *tick at different rates,*

$$dt(x_1)/dt(x_2) = (d\tau/dx(x_2))/(d\tau/dx(x_1)) = \sqrt{g_{00}(x_2)/g_{00}(x_1)}\,, \tag{2.7c}$$

which implies that,

$$[(\text{clock tick rate }(x_2))/(\text{clock tick rate }(x_1))] = \sqrt{g_{00}(x_2)/g_{00}(x_1)}\,. \tag{2.7d}$$

In the Newtonian limit g_{00} is static and differs only slightly from $\eta_{00} = 1$, i.e., $g_{00}(\mathbf{x}) = 1 + 2\phi(\mathbf{x})/c^2$ (see Eq. (2.6l)), so in the Newtonian limit Eq. (2.7d) implies that,

$$[(\text{clock tick rate }(\mathbf{x}_2))/(\text{clock tick rate }(\mathbf{x}_1))] = \sqrt{(1 + 2\phi(\mathbf{x}_2)/c^2)/(1 + 2\phi(\mathbf{x}_1)/c^2)} \approx \left[1 - \left(\left(\phi(\mathbf{x}_1) - \phi(\mathbf{x}_2)\right)/c^2\right)\right]. \tag{2.7e}$$

When $|\mathbf{x}_1| > |\mathbf{x}_2|$ and $\phi(\mathbf{x}_1) > \phi(\mathbf{x}_2)$, Eq. (1.9) *follows from* Eq. (2.7e) (as do Eqs. (1.6) and (1.2a)).

Although gravitational time dilation is perforce an extremely small effect in the Newtonian limit, *there is no reason* that its Eq. (2.7d) ratio form $\sqrt{g_{00}(x_2)/g_{00}(x_1)}$ *for the corresponding ratio of the clock tick rates* should *not* have deviated very strongly from unity in an early universe *which was sufficiently compact and dense.* The discovery in the late 1920's of *the cosmological red shift* very, very strongly suggests *that the universe is expanding,* which would make it a travesty of common sense to *not* suppose that the universe was arbitrarily compact and dense *in the sufficiently remote past* and, *because of that degree of compactness and denseness at that time,* manifested *very strong gravitational time dilation* at that time.

The part of the universe *which astronomers have been able to see,* however, gives a strong impression of large-scale homogeneity and isotropy, which in the mid-1930's led to a concerted effort to produce *a metric tensor form* that is consistent with a universe *which always has been, and always will be, homogeneous and isotropic,* a hypothesis that was dubbed the Cosmological Principle. A very prominent and quite astonishing feature of the Robertson-Walker metric form which emerged from those mid-1930's efforts is the requirement that $g_{00} = 1$ *at every point of space-time.* The requirement that $g_{00} = 1$ everywhere in space-time is consistent with the *absence* of a gravitational field, i.e., when $g_{\mu\nu} = \eta_{\mu\nu}$, but in the *presence* of a gravitational field, i.e., when $g_{\mu\nu}$ *is a tensor field,* the requirement that $g_{00} = 1$ *at every point of space-time* is *inconsistent* with $g_{\mu\nu}$ being a *Lorentz-covariant* tensor field.

At the beginning of this section we stipulated that gravity theory *is Lorentz-covariant,* just as electrodynamics *or any other sensible physical theory is required to be Lorentz-covariant.* The Einstein equation for the metric tensor $g_{\mu\nu}$ in terms of its stress-energy source $T_{\mu\nu}$ is actually *generally covariant* under *arbitrary* one-to-one transformations of space-time onto itself, *not just Lorentz transformations,* which *ensures that the equivalence principle is honored.* The Einstein equation, however, *by itself determines only six of the ten independent components of the metric tensor field* $g_{\mu\nu}$, so the Einstein equation by itself *definitely isn't the complete theory of the gravitational metric tensor* $g_{\mu\nu}$. (This is *entirely different* from the situation in classical electromagnetic theory, where six independent first-order Heaviside-Maxwell electromagnetic field equations *completely determine* the three components of the electric field **E** and the three components of the magnetic field **B**.) To *complete the determination of the gravitational metric tensor field* $g_{\mu\nu}$, the Einstein equation must be supplemented *by four additional physically cogent equations;* the equation $g_{00} = 1$, *which is incompatible with the Lorentz-covariance of* $g_{\mu\nu}$ *when a gravitational field is present,* is *the opposite* of physically cogent. One *immensely anomalous consequence* of requiring that $g_{00} = 1$ everywhere in space-time is that, according to Eq. (2.7d), *gravitational time dilation fails exist at all.* In order for gravitational time dilation *not to exist at all,* Eq. (2.7e) for the weak version of gravitational time dilation in the Newtonian gravitational limit tells us *that the constant c must go to infinity,* i.e., that g_{00} *cannot* be fixed to unity everywhere in space-time in the presence of a gravitational field *unless c is driven to infinity.* That this is indeed the consequence of fixing g_{00} to unity everywhere in space-time in the presence of a gravitational field, was discovered by the engineer and amateur Riemann geometer A. Friedmann in 1922. Friedmann was excited to find that the Einstein equation became solvable in closed form when he fixed g_{00} to unity everywhere in space-time; it became apparent in due course that that maneuver forced the Einstein equation to describe Newtonian gravity, the $c \to \infty$ limit of gravitational theory, *exactly and exclusively.* It is thus apparent that the Robertson-Walker metric form, which has $g_{00} = 1$ everywhere in space-time, is *incompatible* with a physically cogent *Lorentz-covariant* theory of gravity.

Requirements for gravity theory which *are* physically cogent are that $g_{\mu\nu}$ *must in the presence of gravitation be a Lorentz-covariant tensor field, must have a matrix inverse and must have the signature* +, -, -, -. Thus *the Lorentz-invariant stipulation that* $\det(g_{\mu\nu}) \neq 0$, for example, is a physically cogent one. We need *four* physically cogent *equations* to supplement the *intrinsically incomplete* Einstein equation, however. *Four equations do follow* from the closely related (*but somewhat stronger than* $\det(g_{\mu\nu}) \neq 0$) Lorentz-invariant stipulation that,

$$\det(g_{\mu\nu}) = -1, \tag{2.8a}$$

which, besides being Lorentz-invariant *and ensuring that the metric tensor field* $g_{\mu\nu}$ *has a matrix inverse, dovetails with the case of no gravitational field,* $g_{\mu\nu} = \eta_{\mu\nu}$. The *four physically cogent Lorentz-covariant equations which* Eq. (2.8a) *implies are,* $\partial(\det(g_{\mu\nu}))/\partial x^\lambda = 0$. They are more elegantly expressed in terms of the affine connection as,

$$\Gamma^\sigma_{\sigma\lambda} = 0, \tag{2.8b}$$

since it is the case that,

$$\partial(\det(g_{\mu\nu}))/\partial x^\lambda = 2\det(g_{\mu\nu})\,\Gamma^\sigma_{\sigma\lambda}. \tag{2.8c}$$

Proving Eq. (2.8c) is lengthy and delicate; we divide it into proving two lemmas, (1) $\Gamma^\sigma_{\sigma\lambda} = \frac{1}{2}g^{\sigma\nu}(\partial g_{\nu\sigma}/\partial x^\lambda)$ and (2) $\partial(\det(g_{\mu\nu}))/\partial x^\lambda = \det(g_{\mu\nu})\,\mathrm{Tr}\big[g^{\mu\nu}\big(\partial g_{\mu\nu}/\partial x^\lambda\big)\big] = \det(g_{\mu\nu})\,g^{\sigma\beta}(\partial g_{\beta\sigma}/\partial x^\lambda) = 2\det(g_{\mu\nu})\,\Gamma^\sigma_{\sigma\lambda}$.

The first lemma follows from Eq. (2.5d),

$$\Gamma^\sigma_{\sigma\lambda} = \tfrac{1}{2}g^{\sigma\nu}\left[\tfrac{\partial g_{\sigma\nu}}{\partial x^\lambda} + \tfrac{\partial g_{\lambda\nu}}{\partial x^\sigma} - \tfrac{\partial g_{\sigma\lambda}}{\partial x^\nu}\right] = \tfrac{1}{2}g^{\sigma\nu}(\partial g_{\nu\sigma}/\partial x^\lambda), \tag{2.8d}$$

because $\big(\frac{\partial g_{\lambda\nu}}{\partial x^\sigma} - \frac{\partial g_{\sigma\lambda}}{\partial x^\nu}\big)$ is antisymmetric under interchange of σ and ν.

The second lemma calculates $\partial(\ln(\det(g_{\mu\nu})))/\partial x^\lambda$ in two different ways; the first way is very lengthy,

$$\begin{gathered}\partial(\ln(\det(g_{\mu\nu})))/\partial x^\lambda = (1/\delta x^\lambda)\big[\ln(\det(g_{\mu\nu} + \delta x^\lambda(\partial g_{\mu\nu}/\partial x^\lambda))) - \ln(\det(g_{\mu\nu}))\big] = \\ (1/\delta x^\lambda)\ln\big(\det(g_{\mu\nu} + \delta x^\lambda(\partial g_{\mu\nu}/\partial x^\lambda))/\det(g_{\mu\nu})\big) = (1/\delta x^\lambda)\ln\big(\det\big(\mathrm{I} + \delta x^\lambda g^{\mu\nu}(\partial g_{\mu\nu}/\partial x^\lambda)\big)\big) = \\ (1/\delta x^\lambda)\ln\big(1 + \delta x^\lambda \mathrm{Tr}\big[g^{\mu\nu}(\partial g_{\mu\nu}/\partial x^\lambda)\big]\big) = \mathrm{Tr}\big[g^{\mu\nu}(\partial g_{\mu\nu}/\partial x^\lambda)\big] = g^{\sigma\nu}(\partial g_{\nu\sigma}/\partial x^\lambda) = 2\,\Gamma^\sigma_{\sigma\lambda},\end{gathered} \tag{2.8e}$$

where the first lemma was applied in the final step. We next note that $\partial(\ln(\det(g_{\mu\nu})))/\partial x^\lambda$ *also yields,*

$$\partial(\ln(\det(g_{\mu\nu})))/\partial x^\lambda = \big(\partial(\det(g_{\mu\nu}))/\partial x^\lambda\big)/(\det(g_{\mu\nu})), \tag{2.8f}$$

which, combined with the result of Eq. (2.8e), implies that,

$$\partial(\det(g_{\mu\nu}))/\partial x^\lambda = 2\,(\det(g_{\mu\nu}))\,\Gamma^\sigma_{\sigma\lambda}, \tag{2.8g}$$

which proves Eq. (2.8c).

The Eq. (2.8b) *consequence* $\Gamma^\sigma_{\sigma\lambda} = 0$ of the Eq. (2.8a) physically-cogent Lorentz-covariant stipulation $\det(g_{\mu\nu}) = -1$ is somewhat reminiscent of the far better known *harmonic* stipulation,

$$g^{\mu\nu}\Gamma^\lambda_{\mu\nu} = 0, \tag{2.8h}$$

which is *also* Lorentz covariant. However, *unlike* the Eq. (2.8a) *physically-cogent* Lorentz-covariant stipulation $\det(g_{\mu\nu}) = -1$, the Eq. (2.8h) harmonic stipulation *doesn't ensure that the metric tensor field* $g_{\mu\nu}$ *has a matrix inverse.* Notwithstanding this benefit of Eq. (2.8a), it is still in principle *necessary to check whether the signature of the metric tensor field actually fulfills the requirement of being* +, -, -, -.

The Eq. (2.8a) physically-cogent Lorentz-covariant stipulation $\det(g_{\mu\nu}) = -1$ brings some welcome simplifications to gravity theory. The awkward factors of powers of $\sqrt{-\det(g_{\mu\nu})}$ which were needed to convert tensor densities to proper generally-covariant tensors *become unity identically.* The four-volume element becomes just d^4x for example. The first-order Heaviside-Maxwell electromagnetic field equations for the electric and magnetic fields for this reason also keep the same form in the presence of a gravitational field as they have in its absence (gravity of course *changes those fields' four-current source).*

Since an expanding universe presumably would have been arbitrarily compact in the sufficiently remote past, we next examine the gravitational time dilation implications of a gravitational point source.

3. The metric tensor field of a static gravitational point source

The metric tensor field $g_{\mu\nu}(x^\lambda)$ of a static gravitational point source located at the origin $\mathbf{x} = \mathbf{0}$ *must reflect spherical symmetry about the origin and insensitivity to time reversal,* so $(c\,d\tau)^2 = g_{\mu\nu}(x^\lambda)dx^\mu dx^\nu$ must perforce have the form,

$$(c\,d\tau)^2 = D(|\mathbf{x}|)(dx^0)^2 - G(|\mathbf{x}|)((\mathbf{x}\cdot d\mathbf{x})/|\mathbf{x}|)^2 - H(|\mathbf{x}|)|d\mathbf{x}|^2. \tag{3.1a}$$

A term of the form $E(|\mathbf{x}|)(dx^0)((\mathbf{x}\cdot d\mathbf{x})/|\mathbf{x}|)$ is excluded because it is sensitive to time reversal $x^0 \to -x^0$.

It is of course very useful to write the Eq. (3.1a) spherically-symmetric invariant $(c\,d\tau)^2$ in terms of spherical polar coordinates, $\mathbf{x} = (r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta)$, which yields that, $|\mathbf{x}| = r$, $((\mathbf{x}\cdot d\mathbf{x})/|\mathbf{x}|)^2 = (dr)^2$ and $|d\mathbf{x}|^2 = (dr)^2 + r^2(d\theta)^2 + r^2(\sin\theta\,d\phi)^2$. Thereby Eq. (3.1a) becomes,

$$(c\,d\tau)^2 = D(r)(dx^0)^2 - F(r)(dr)^2 - H(r)\big(r^2(d\theta)^2 + r^2(\sin\theta\,d\phi)^2\big). \tag{3.1b}$$

where $F(r) = (G(r) + H(r))$.

The *determinant* of the Eq. (3.1b) metric form is of course $-D(r)F(r)(H(r))^2$, so for $r > 0$, the empty-space version of the Einstein equation should be solved using the Eq. (3.1b) metric form *subject to the additional stipulation that* $D(r)F(r)(H(r))^2 = 1$. That is precisely the mathematical problem which A. Einstein posed to K. Schwarzschild in 1915. Schwarzschild was a master of mathematical problem-solving techniques, so he did work out the metric solution Einstein requested; it was published in January 1916.

An extremely lamentable 1918 intercession by D. Hilbert resulted in Schwarzschild's impeccable 1916 solution metric *not being shown in gravitational-theory textbooks,* which *instead* show solution metrics for the static point source *which have an unphysical singularity that doesn't exist in Schwarzschild's 1916 solution.*

The *root cause of the unphysical singularity* is Hilbert's *application of a radial transformation which reduces the number of functions of the radial coordinate r in the* Eq. (3.1b) *metric form from three to two before it is inserted into the empty-space Einstein equation.* Radial transformations of solutions of the Einstein equation are still solutions of the Einstein equation because of its general covariance, *but they aren't necessarily physical solutions, whose metrics must, for example, have the signature* +, -, -, - *everywhere.* Radial transformations which *reduce* the number of functions of the radial coordinate present in the Eq. (3.1b) metric form *from three to two* will harm the physics *if the physics requires all three functions of the radial coordinate.* It in fact turns out that radial transformations *which successfully remove the singularity in the unphysical metric solutions for the static point source that have only two functions of the radial coordinate in their metric forms also increase to three the number of functions of the radial coordinate which are present in the transformed metric forms.*

A specific transformation $r'(r)$ of the radial coordinate r can be exhibited which changes Eq. (3.1b) to,

$$(c\,d\tau)^2 = C(r')(dx^0)^2 - G(r')\big((dr')^2 + (r')^2(d\theta)^2 + (r')^2(\sin\theta\,d\phi)^2\big), \tag{3.2a}$$

which has only two functions of r' in its Eq. (3.2a) *metric form;* it is dubbed the "isotropic" form of the static, spherically symmetric metric.

Another specific transformation $R(r)$ of the coordinate r can be exhibited which changes Eq. (3.1b) to,

$$(c\,d\tau)^2 = B(R)(dx^0)^2 - A(R)(dR)^2 - R^2(d\theta)^2 - R^2(\sin\theta\,d\phi)^2, \tag{3.2b}$$

which also has only two functions of R in its Eq. (3.2b) *metric form,* it is dubbed the "standard" form of the static, spherically symmetric metric. The above "isotropic" and "standard" *two-function truncations* of the Eq. (3.1b) *three-function general form* of the static, spherically symmetric metric *obviously aren't by any means guaranteed not to be unphysical.*

Notwithstanding *that it may well be unphysical,* almost *all* gravity theory textbooks follow D. Hilbert's recommendation to insert *the* Eq. (3.2b) *"standard" metric form* into the empty-space Einstein equation, which thereupon yields that $A(R)B(R) = K$, where K is a dimensionless constant, and that $B(R) =$

$1-(r_0/R)$, where r_0 is a constant with the dimension of length. *Both* of the constants K and r_0 *are determined by properties of the metric at sufficiently large values of* R.

For the static point source located at $\mathbf{x} = \mathbf{0}$ we know from Eq. (2.6l) that at sufficiently large values of R, *where the static gravitational field becomes arbitrarily weak,* the metric matrix element g_{00} behaves as $1 + 2\phi/c^2$, where ϕ *is the corresponding Newtonian gravitational potential.* The Newtonian gravitational potential ϕ for a static point source located at $\mathbf{x} = \mathbf{0}$ is given by Eq. (2.6j) as $\phi = -GM/|\mathbf{x}| = -GM/R$, where M is the mass of that point source. (If the static point source had a static energy E, its effective mass M would of course be equal to E/c^2.) Thus for the static point source,

$$g_{00} \simeq 1 + 2\phi/c^2 \ \text{ with } \ \phi = -GM/R, \text{ as } R \to \infty, \ \text{ so } \ g_{00} \simeq 1 - (2GM/c^2)/R \text{ as } R \to \infty. \tag{3.2c}$$

Of course g_{00} is the same as the coefficient of $(dx^0)^2$ in Eq. (3.2b), which is $B(R)$. Therefore,

$$B(R) \simeq 1 - (2GM/c^2)/R \text{ as } R \to \infty. \tag{3.2d}$$

We noted below Eq. (3.2b) that the empty-space Einstein equation applied to the Eq. (3.2b) "standard" form of the static, spherically symmetric metric yields that $B(R) = 1-(r_0/R)$ where the constant r_0 is determined by the behavior of the metric at sufficiently large values of R. We therefore read off from Eq. (3.2d) that $r_0 = (2GM/c^2)$, so,

$$B(R) = 1 - (2GM/c^2)/R. \tag{3.2e}$$

We next obtain the value of the dimensionless constant K in the relation $A(R)B(R) = K$ which we noted below Eq. (3.2b) is a consequence of applying the empty-space Einstein equation to the Eq. (3.2b) "standard" form of the static, spherically symmetric metric. As $R \to \infty$, of course $g_{\mu\nu} \to \eta_{\mu\nu}$, which implies that as $R \to \infty$,

$$(c\,d\tau)^2 \to (dx^0)^2 - |d\mathbf{x}|^2 = (dx^0)^2 - (dR)^2 - R^2(d\theta)^2 - R^2(\sin\theta\, d\phi)^2, \tag{3.2f}$$

which, in conjunction with Eq. (3.2b), implies that as $R \to \infty$, $B(R) \to 1$ and $A(R) \to 1$. (Of course the fact that $B(R) \to 1$ as $R \to \infty$ follows as well from Eq. (3.2e).) Therefore, since the constant K satisfies $K = A(R)B(R)$, the value of the constant K must be consistent with what it is as $R \to \infty$, namely, $K = 1$. Since, therefore, $A(R)B(R) = 1$, it follows that $A(R) = 1/B(R)$, which, in conjunction with $B(R) = 1 - (2GM/c^2)/R$ from Eq. (3.2e) yields that $A(R) = [1/(1 - (2GM/c^2)/R)]$. Inserting these values of $B(R)$ and $A(R)$ into the Eq. (3.2b) "standard" form of the static, spherically symmetric metric yields *the "standard" form of the solution metric for a static point source of mass* M *located at* $R = 0$,

$$(c\,d\tau)^2 = (1 - (r_0/R))(dx^0)^2 - [1/(1 - (r_0/R))](dR)^2 - R^2(d\theta)^2 - R^2(\sin\theta\, d\phi)^2, \tag{3.2g}$$

where $r_0 \stackrel{\text{def}}{=} (2GM/c^2)$. This metric's four eigenvalues are $(1 - (r_0/R))$, $-[1/(1 - (r_0/R))]$, -1 and -1, so the physical requirement that a metric's signature must be +, -, -, - *clearly fails at* $R = r_0$. As well, at $R = r_0$ this metric's *determinant is undefined.* Very closely related, but even more striking, is that since $\sqrt{g_{00}(R)} = \sqrt{1 - (r_0/R)}$ for this metric, clock tick rates go to *zero* at $R = r_0$, so an object which is approaching $R = r_0$ never reaches that point because its approach speed is forced toward zero. This unphysical anomaly is dubbed an "event horizon". Due to D. Hilbert's influence, practically all textbooks refer to the unphysical Eq. (3.2g) metric as the solution which K. Schwarzschild found in 1916, whereas in actuality the solution metric in Schwarzschild's January 1916 paper *has no event horizon at* $R > 0$ *nor any other unphysical anomaly at* $R > 0$, and its mathematical form *is different from that of* Eq. (3.2g).

The "isotropic" form of the solution metric for a static point source *fares no better than the* Eq. (3.2g) *"standard" form* of the solution metric; it also has an unphysical anomaly at $r' > 0$ which is an event horizon. Some texts exhibit a harmonic form of the solution metric for a static point source *which is constructed directly from the two matrix elements* $B(R)$ *and* $A(R)$ *of the "standard" form of the solution metric,* and thus, unsurprisingly, *also* has an unphysical anomaly which is an event horizon. We have earlier pointed out that the harmonic coordinate condition $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$ *provides no inherent protection against the occurrence of unphysical anomalies in the metric tensor* $g_{\mu\nu}$, whereas the condition $\det(g_{\mu\nu}) = -1$ (which was applied by K. Schwarzschild at A. Einstein's request) *explicitly provides such protection to* $g_{\mu\nu}$, and it *as well* implies the four equations $\Gamma^{\mu}_{\mu\lambda} = 0$, which are similar to the harmonic coordinate condition $g^{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$.

Since D. Hilbert *induced* the unphysical anomaly located at $R = r_0 > 0$ in the Eq. (3.2g) "standard" form of the solution metric for the static point source *by making a damaging radial transformation to reduce* *from three to two the number of functions of the radial coordinate present in the* Eq. (3.1b) *metric form,* a radial transformation of the Eq. (3.2g) "standard" form of the solution metric *which removes its unphysical anomaly at $R = r_0 > 0$ is guaranteed to exist.* The "event horizon" located at $R = r_0 > 0$ *apparently belongs at the origin instead, where the clearly unphysical idealized point mass resides.* Moving $R = r_0$ to $\rho = 0$ is accomplished *by the simple radial transformation* $\rho(R) = R - r_0$,

$$\rho(R) = R - r_0 \;\Rightarrow\; R(\rho) = \rho + r_0 \;\Rightarrow\; (1 - (r_0/R(\rho))) = (\rho/(\rho + r_0)) = [1/(1 + (r_0/\rho))]. \tag{3.3a}$$

Insertion of $R(\rho) = \rho + r_0$ into the Eq. (3.2g) "standard" form of the solution metric, which is $(c\,d\tau)^2 = (1 - (r_0/R))(dx^0)^2 - [1/(1 - (r_0/R))](dR)^2 - R^2(d\theta)^2 - R^2(\sin\theta\, d\phi)^2$, produces,

$$(c\,d\tau)^2 = [1/(1 + (r_0/\rho))](dx^0)^2 - (1 + (r_0/\rho))(d\rho)^2 - (1 + (r_0/\rho))^2\big(\rho^2(d\theta)^2 + \rho^2(\sin\theta\, d\phi)^2\big), \tag{3.3b}$$

where $r_0 \stackrel{\text{def}}{=} (2GM/c^2)$. The Eq. (3.3b) solution metric is entirely free of singularities when $\rho > 0$; it does have an event horizon at $\rho = 0$ because of the presence there of the unphysical idealized point mass. Note that the "price" which is paid for this sensible behavior of the Eq. (3.3b) solution metric is that it has *three different powers of the entity* $(1 + (r_0/\rho))$, whereas the Eq. (3.2g) unphysically-behaved "standard" form of the solution metric has *only two different powers of the entity* $(1 - (r_0/R))$. It is now abundantly clear that D. Hilbert's *insistence on making a damaging radial transformation which reduces from three to two the number of functions of the radius variable present in the* Eq. (3.1b) *metric form is the root cause of the unphysical singularity which ensues in the solution metric.* It is extremely lamentable that D. Hilbert was able to *ensure* that his favored physically-defective solution metrics *are the only ones shown in textbooks on gravitational theory,* and that the physically sensible solution metric in K. Schwarzschild's January 1916 paper *is never shown in gravitational-theory textbooks.*

Returning to the physically well-behaved solution metric of Eq. (3.3b), we note that its determinant has the value $-(1 + (r_0/\rho))^4$, whereas we, like K. Schwarzschild in 1915, seek the solution metric whose determinant is equal to -1. We achieve that through a further radial transformation of the Eq. (3.3b) solution metric. Inserting a *general* radial transformation from ρ to r into Eq. (3.3b) produces,

$$(c\,d\tau)^2 = [1/(1 + (r_0/\rho(r)))](dx^0)^2 - (1 + (r_0/\rho(r)))(d\rho(r)/dr)^2(dr)^2 -$$
$$((\rho(r) + r_0)/r)^2\big(r^2(d\theta)^2 + r^2(\sin\theta\, d\phi)^2\big), \tag{3.3c}$$

whose determinant we equate to -1,

$$-(d\rho(r)/dr)^2((\rho(r) + r_0)/r)^4 = -1. \tag{3.3d}$$

We require $\rho(r)$ to *increase* monotonically with r, which implies that,

$$(d\rho(r)/dr)((\rho(r) + r_0)/r)^2 = 1. \tag{3.3e}$$

Separation of variables yields,

$$(\rho + r_0)^2\, d\rho = r^2\, dr, \tag{3.3f}$$

and integration yields,

$$(\rho(r) + r_0)^3 = r^3 + r_0^3 k, \tag{3.3g}$$

where k is a dimensionless integration constant. We require that $\rho(r = 0) = 0$, which implies that $k = 1$, so,

$$(\rho(r) + r_0) = (r^3 + r_0^3)^{\frac{1}{3}}, \tag{3.3h}$$

which implies that,

$$((\rho(r) + r_0)/r) = ((r^3 + r_0^3)^{\frac{1}{3}}/r), \tag{3.3i}$$

and also implies that,

$$(d\rho(r)/dr) = r^2/(r^3 + r_0^3)^{\frac{2}{3}} = (r/(r^3 + r_0^3)^{\frac{1}{3}})^2, \tag{3.3j}$$

and as well implies that,

$$(\rho(r)/r_0) = ((r^3 + r_0^3)^{\frac{1}{3}}/r_0) - 1. \tag{3.3k}$$

From Eq. (3.3k) we obtain,

$$(1+(r_0/\rho(r))) = \big(1+\big(1/(((r^3+r_0^3)^{\frac{1}{3}}/r_0)-1)\big)\big) = \big(((r^3+r_0^3)^{\frac{1}{3}}/r_0)/(((r^3+r_0^3)^{\frac{1}{3}}/r_0)-1)\big), \qquad (3.3l)$$

which yields that,

$$[1/(1+(r_0/\rho(r)))] = \big(1-\big(r_0/(r^3+r_0^3)^{\frac{1}{3}}\big)\big). \qquad (3.3m)$$

We now insert Eqs. (3.3m), (3.3j) and (3.3i) into the Eq. (3.3c) solution metric to obtain its det $= -1$ form,

$$(c\,d\tau)^2 = \big(1-\big(r_0/(r^3+r_0^3)^{\frac{1}{3}}\big)\big)(dx^0)^2 - \big[1/\big(1-\big(r_0/(r^3+r_0^3)^{\frac{1}{3}}\big)\big)\big]\big(r/(r^3+r_0^3)^{\frac{1}{3}}\big)^4 (dr)^2 -$$
$$\big((r^3+r_0^3)^{\frac{1}{3}}/r\big)^2\big(r^2(d\theta)^2 + r^2(\sin\theta\, d\phi)^2\big), \qquad (3.3n)$$

where $r_0 \stackrel{\text{def}}{=} (2GM/c^2)$. It is easy to check that the determinant of the Eq. (3.3n) solution metric is equal to -1, and that it is free of singularities when $r > 0$. It has an event horizon at $r = 0$, the location of the unphysical idealized point mass. *This is the solution metric in K. Schwarzschild's January 1916 paper.* We obtained it here by transformation of the radial variable of the physically-defective "standard" form of the solution metric. Schwarzshild of course obtained it directly from the Eq. (3.1b) general form for a static, spherically-symmetric metric, the empty-space version of the Einstein equation and the stipulation that the determinant of the metric must be equal to -1.

It is extremely harmful to proper understanding of gravity theory that gravity-theory textbooks *show only physically-defective* "standard", "isotropic" and harmonic forms *of the solution metric for a static point source, and ignore the physically well-behaved* Eq. (3.3n) *solution metric for that case* published by K. Schwarzschild in January 1916. It is immensely regrettable that the misleading impact of the gravity-theory textbooks resulted in an enormous amount of wholly-wasted effort over the course of more than a century. An entire "discipline" of "geometrodynamics" arose based on the risibly unphysical "wormhole" present in the Eq. (3.2g) unphysical matrix element $g_{00} = (1-(r_0/R))$ in the region $0 < R < r_0$ beyond its "event horizon" located at $R = r_0$. The unphysical "event horizons" *themselves* which the physically-defective solution metrics exhibit resulted in a veritable avalanche of wasted effort on the thermodynamics and even particle physics of those physically nonexistent entities. D. Hilbert advocated the radial transformations which produce damaged two-function versions of the Eq. (3.1b) general three-function form of the static, spherically-symmetric metric, and was instrumental in persuading textbook authors *to show only physically-defective solution metrics for a static point source which are consequences of those damaging radial transformations.*

Fortunately we have in hand the physically well-behaved matrix element $g_{00}(r) = \big(1-\big(r_0/(r^3+r_0^3)^{\frac{1}{3}}\big)\big)$ of the Eq. (3.3n) solution metric for a static point source published by K. Schwarzschild in January 1916. At $r = r_0$ *there is no event horizon,* but even so,

$$\sqrt{g_{00}(r_0)} = \sqrt{1-\big(r_0/(r_0^3+r_0^3)^{\frac{1}{3}}\big)} = \sqrt{1-\big(1/2^{\frac{1}{3}}\big)} = .454. \qquad (3.4a)$$

Therefore the tick rate of a clock at $r = r_0$ *is less than half that of a clock at a very large value of* r *since,*

$$\sqrt{g_{00}(r)} = \sqrt{1-\big(1/((r/r_0)^3+1)^{\frac{1}{3}}\big)} \approx 1 - \tfrac{1}{2(r/r_0)} \quad \text{when} \quad (r/r_0) \gg 1, \qquad (3.4b)$$

but the tick rate of a clock at $r = r_0$ *is much greater than that of a clock much closer to* $r = 0$ *because,*

$$\sqrt{g_{00}(r)} = \sqrt{1-\big(1/(1+(r/r_0)^3)^{\frac{1}{3}}\big)} \approx \big((r/r_0)^{\frac{3}{2}}/\sqrt{3}\,\big) \quad \text{when} \quad (r/r_0) \ll 1. \qquad (3.4c)$$

Although there is no event horizon at $r = r_0$, *that point, where* $\sqrt{g_{00}(r)} \approx \frac{1}{2}$, *marks a transition in the behavior of* $\sqrt{g_{00}(r)}$ *from lingering near unity when* $r \gg r_0$ *to briskly proceeding toward zero as* $r \to 0$.

Besides $\sqrt{g_{00}(r)}$, another entity of interest obtained from $g_{00}(r)$ is $\phi(r) \stackrel{\text{def}}{=} \frac{1}{2}c^2(g_{00}(r)-1)$, the generalization of the Newtonian gravitational potential. Since $r_0 \stackrel{\text{def}}{=} (2GM/c^2)$,

$$\phi(r) \stackrel{\text{def}}{=} \tfrac{1}{2}c^2(g_{00}(r)-1) = -\tfrac{1}{2}c^2 r_0/(r^3+r_0^3)^{\frac{1}{3}} = -GM/(r^3+r_0^3)^{\frac{1}{3}} =$$
$$(-GM/r)\big(1/(1+(r_0/r)^3)^{\frac{1}{3}}\big) = -\tfrac{1}{2}c^2\big(1/(1+(r/r_0)^3)^{\frac{1}{3}}\big), \qquad (3.4d)$$

which implies that,

$$\phi(r_0) = \big(-\tfrac{1}{2}c^2\big)/2^{\frac{1}{3}} = (-GM/r_0)/2^{\frac{1}{3}}. \qquad (3.4e)$$

Eq. (3.4d) also implies that,

$$\phi(r) \approx -GM/r \quad \text{when} \quad (r/r_0) \gg 1, \tag{3.4f}$$

and Eq. (3.4d) as well implies that,

$$\phi(r) \approx -\tfrac{1}{2}c^2\big(1 - \tfrac{1}{3}(r/r_0)^3\big) \quad \text{when} \quad (r/r_0) \ll 1. \tag{3.4g}$$

Unlike the Newtonian gravitational potential $-GM/r$, which is unbounded below as $r \to 0$, its Eq. (3.4d) generalization $\phi(r)$ *cannot become more negative than* $-\frac{1}{2}c^2$, just as the speed of a relativistic particle cannot exceed c. Again, the point $r = r_0$ *marks a transition in the behavior of $\phi(r)$ from lingering near $-GM/r$ when $r \gg r_0$ to briskly leveling off toward its maximally negative value $-\frac{1}{2}c^2$ as $r \to 0$.*

In the Newtonian case, with gravitational potential $\phi(r) = -GM/r$, a test particle placed at rest at time zero a distance d from the static point-mass M accelerates toward the static point mass, and as the test particle closes in on the static point mass its acceleration $-d\phi(r)/dr = -GM/r^2$ gets stronger without bound. It is readily shown that at time $t = (\pi/2)\sqrt{d^3/(2GM)}$ *the test particle reaches the location of the static point mass, and there the test particle's speed is infinite.* If the test particle initially *has nonzero velocity directed away from the point mass,* it will *still* reach the location of the point mass *in a finite time,* and there its speed *will still be infinite.* This simple picture *is a time-reversed microcosm of the ostensible Big Bang.* Details to be added include many particles instead of one, which permits the static point mass M at $r = 0$ to be removed in favor of the mutual gravitational attraction of those particles, and also that when those particles crash into each other at infinite speed *the consequence is an infinite temperature.*

In the case of the potential $\phi(r) = -GM/(r^3+r_0^3)^{\frac{1}{3}}$ of Eq. (3.4d), however, the test particle's acceleration $-d\phi(r)/dr = -GMr^2/(r^3 + r_0^3)^{\frac{4}{3}}$, *actually goes to zero as* $r \to 0$, instead of, as happens in the Newtonian case, going to $-\infty$ as $r \to 0$. At the same time, the clock tick-rate factor $\sqrt{g_{00}(r)} = (1 - (r_0/(r^3 + r_0^3)^{\frac{1}{3}}))^{\frac{1}{2}}$ goes to zero as $r \to 0$, which forces the test particle's speed toward zero as it approaches $r = 0$; this of course is aided by the fact that the test particle's acceleration goes to zero as $r \to 0$. Thus the situation in relativistic gravity is the *opposite* of that in Newtonian gravity: instead of the test particle reaching $r = 0$ in a finite time with infinite speed, the clock tick rate factor *forces its speed toward zero as it approaches $r = 0$, so it takes forever to reach $r = 0$.* At sufficiently large values of r, however, *the relativistic test particle's behavior obviously must be almost the same as the Newtonian test particle's behavior.* As we have twice noted above, *the transition in behavior occurs around $r = r_0$.*

With these simple lessons in relativistic gravitational physics which flow from the Eq. (3.3n) solution metric for the static point source published by K. Schwarzschild in January 1916, *we are obliged to rule out Big Bang cosmology,* but can present a very rough sketch of the universe's likely early evolution.

4. The universe's likely early evolution, a very rough sketch based on relativistic gravity

As we have said previously, a key fact about the universe is that it apparently is expanding, and therefore it must have been arbitrarily compact and dense at a sufficiently remote time in the past. We assume that in the sufficiently remote past the universe was compact enough to be well inside the radius $r_0 = (2GM/c^2)$, where M is the universe's mass. At such times the universe's behavior *would be dominated by gravitational time dilation* (see Eq. (3.4c)); *even gravity's acceleration $-d\phi(r)/dr$ would be diminished.*

In a universe dominated by gravitational time dilation all physical processes would have greatly slowed and radiation frequencies would have been greatly reduced; it would have been a cold, dark universe with almost no discernible physical processes. Even the expansion speed of such a universe would have been reduced. *Going still further back in time accentuates these features of the very early universe.* Going *forward* in time *eventually* brings us to a universe whose radius is approximately r_0. The accompanying *decrease in gravitational time dilation* allows the universe's *expansion speed to increase,* which *still further reduces gravitational time dilation,* causing the universe's expansion speed *to increase still further,* etc.

Thus a universe whose radius is approximately r_0 is one on the cusp *of rapid acceleration of its expansion,* which is termed inflation. Physical process rates in such a universe would greatly increase as the dead hand of extreme gravitational time dilation rapidly lifts. *Notwithstanding its inflationary expansion,* such a universe *would still be vastly more compact and dense than today's universe is,* with its billions of years *of additional expansion.* So *dense* a universe, *which was liberated from extreme gravitational time dilation,* would have been able to give birth to every conceivable kind of young star at an utterly enormous rate, with particular emphasis on immensely massive, extremely short-lived giants. *Considering how much even denser than that the universe was when it reached the liberating radius* r_0, only a small fraction of its matter would have participated *directly* in those fireworks; *by far the bulk of that matter would have taken the form of primordial black holes* (but do remember that black holes *don't have event horizons*). But those primordial black holes would have *profoundly modulated* the spectacular events underway by, for example, becoming the active nuclei of galaxies and quasars. Primordial black holes of lower mass *were crucial to galaxy formation* by providing the necessary cold, dark gravitational "glue".

As the universe continued its expansion, and its density decreased, the enormous rate of star and galaxy formation of its early post-inflationary era would of course have decreased as well. It appears that the James Webb Space Telescope may, at this time, possibly be accumulating evidence of unusually rapid galaxy formation in the early post-inflationary era.

The inflationary expansion of the universe is a consequence of gravitational time dilation: the universe's expansion decreases the intensity of its gravitation, which diminishes gravitational time dilation, causing the universe's expansion to accelerate, etc. As the universe continued expanding for billions of years after it reached the liberating radius r_0, the intensity of its gravitation would have greatly diminished, but it still may be enough that the gravitational time-dilation acceleration of the universe's expansion overcomes the natural deceleration of the universe's expansion which is caused by its gravitational force. It has been found observationally that the universe's expansion is still very, very slightly accelerating; the existence of a cosmological term in the Einstein gravitational field equation is currently postulated as the explanation. Such a term *conflicts* with the idea that the Einstein gravitational field equation must reproduce Newtonian gravity for static, weak gravitational fields, so *the not so ad hoc explanation* that the universe's gravitational intensity is still sufficient for gravitational time-dilation acceleration of the universe's expansion to overcome the natural deceleration of its expansion by its gravitational force is worthy of serious consideration.