On the Motion of Rotating Celestial Bodies and Polar Jets

Pantelis M. Pechlivanides
Atlantic IKE, Athens 11257, Greece
ppexl@teemail.gr

Abstract

The motion of rotating bodies is studied. The relativistic force between them, being non central, varies in orientation and magnitude depending on the rotational angular velocity of the bodies. Thus, the orbiting bodies experience acceleration or deceleration attraction or repulsion during their orbits leading to a rich variety of possible orbits. The conditions for existence of central force and in particular circular orbits that are stable is investigated. It is shown that stable circular orbits are possible only at particular distances from the origin. Application of these results to the Pluto – Charon dyad, allows us to determine the slippage constant λ for exponentially decreasing rotation of signals emanating from the rotating body. The creation of polar jets by large rotating bodies that attract and expel multitudes of minute bodies in their positive and negative directions of their axis of rotation is explained. The problem is then formulated as a relativistic flow problem for the far away observer. The ratio of the radial to axial velocity is calculated and the formation of jets in the axial direction is discussed for both microcosmos and macrocosmos.

1 Introduction

This paper is a continuation of [4], [5], and [6]. In [4] we studied the relativistic path of signals emitted at the origin of a rotating frame. Using the findings of [4] we determine in [5] the field (called **G**) created by a rotating point body at the origin and investigate its form. In [6] the interaction and the forces acting between two rotating bodies is investigated. In the present paper we apply the findings of previous papers to determine the orbits of rotating celestial bodies and specifically the orbit of a small rotating body that orbits around a large rotating body. The effect of rotation in accelerating or decelerating attraction or repulsion on the smaller body is shown and a variety of possible orbits are plotted. The total angle of deflection determines the direction in which the force acts, and when it is zero we have a central force. We study if it is possible to have stable circular orbits and at what distances. The answer for macrocosmos is that there is at least one and possibly two stable circulat orbits. For fast rotating orbiting bodies there is a multitude of distances where stable circular orbits are possible. The same is true for microcosmos. In the light of these findings we study the orbital motion of the Pluto and Charon dyad and we are able to calculate the slippage constant λ, for exponentially decreasing rotational angular velocity. The phenomenon of polar jets is also investigated based on the forces already studied in [6]. The problem is formulated as a special relativistic flow problem and the ratio of the radial to the axial velocity is studied showing the formation of jets for both the macrocosmos and microcosmos case. The paper is organized as follows: In section 2 we study the motion of one rotating body around another rotating body. In section 3 we focus on polar jets created by rotating bodies. In section 4 we conclude.

2 A small rotating body revolves around another big rotating body

For a body orbiting around the origin at radial distance ρ azimuthal angle θ that has velocity υ with radial component , $\upsilon_\rho = \frac{d}{dt}\rho$ and azimuthal component (perpendicular to the radial), $\upsilon_\theta = \rho\frac{d}{dt}\theta$ we may write the vector of velocity as ,

$$\boldsymbol{\upsilon} = \upsilon_\rho \mathbf{u}_\rho + \upsilon_\theta \mathbf{u}_\theta \tag{1}$$

Where $\mathbf{u}_\rho$ and $\mathbf{u}_\theta$ the unit vectors in the radial and azimuthal direction respectively.
From classical mechanics we know that taking the derivative of (1) with respect to time and noting that $\frac{d}{dt}\mathbf{u}_\rho = \mathbf{u}_\theta \frac{d}{dt}\theta$, $\frac{d}{dt}\mathbf{u}_\theta = -\mathbf{u}_\rho \frac{d}{dt}\theta$, we find the acceleration $\mathbf{a}$:

$$\mathbf{a} = \left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_\rho + \left[2\frac{d\rho}{dt}\frac{d\theta}{dt} + \rho\frac{d^2\theta}{dt^2}\right]\mathbf{u}_\theta \tag{2}$$

Then the equations of orbital motion of the body is in general (using Newton's dot notation for the time derivative)

$$\ddot{\rho} - \rho\dot{\theta}^2 = a_\rho \tag{3}$$

$$2\dot{\rho}\dot{\theta} + \rho\ddot{\theta} = a_\theta \tag{4}$$

Where a_ρ is the radial and a_θ the azimuthal acceleration respectively. If we want to allow motion in the z direction we add another equation,

$$\ddot{z} = a_z \tag{5}$$

Where a_z is the acceleration in the z direction. In what follows we will often simplify by assuming $z = 0$.

In our problem we have a body B that rotates with angular velocity w_{B0} and it orbits around a much more massive body A with mass m'_A and rotational angular velocity w_{A0}. Their rotation vectors point in the z direction. The signals of Force emitted by A at radial distance $\rho_{AB}(0.0)$ and distance z_{AB} in the z-direction from A, have angular velocity $w_A = w_{A0}e^{-(\lambda\rho_{AB}(0,0)+\mu z_{AB})}$. The same holds for w_B, where we measure the distance from body B . If $\lambda = \mu = 0$ then we say that there is no slippage or there is constant angular velocity of signals with respect to distance from the body.

In our problem the acceleration is provided by the force studied in [6] between two rotating bodies (see Figure 1)

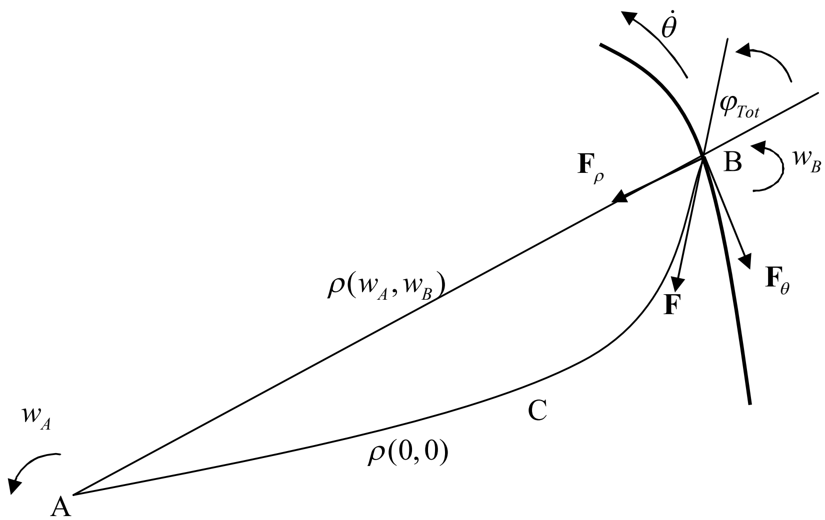


Figure 1 Body A with mass m_A rotates with angular velocity w_{A0}. Body B rotates with angular velocity w_{B0} and orbits A in the same direction as w_{A0} and on the plane of rotation of A with angular velocity $\dot{\theta}$. The signals from A travel the curve ACB whose length is $\rho(0,0)$ and hit body B at an angle of deflection φ_{Tot} from the radial AB whose length is $\rho(w_A, w_B)$.

The distance from A to B is $\rho(w_A, w_B)$, while the path of the signal of the force from A acting on B is $\rho(0,0)$. The $\mathbf{F}$ signals travel the curve ACB, and hit body B with an angle φ_{Tot} to the radial AB measured counterclockwise from AB and the force acts in the opposite direction to that of the signals. The total angle of deflection $\varphi_{Tot} = \varphi + \theta_B$ is the sum of the deflection φ due to the rotation of A and the deflection θ_B due to the rotation rotation of B as was studied in [4], [5], [6].The components of the acceleration along the radial and the azimuthal direction are $a_\rho = -\dfrac{1}{m'_B} F_\rho = -\dfrac{1}{m'_B} F \cos\varphi_{Tot}$ and $a_\theta = -F_\theta = -F \sin\varphi_{Tot}$. Where $F \triangleq |\mathbf{F}|$. Observers O' (the nearby non rotating observer) and O'' (the far away non rotating observer) will see the radial distance contract due to w_A and w_B and there will be a further contraction of the distance AB due to orbital motion around the center of mass of the bodies A an B with angular velocity $\dot{\theta}$.(Here we will assume that A has mass much greater than that of B so that the center of mass of the two bodies "coincides" with that of A) Hence, the magnitude of the field $\mathbf{F}$ will also depend on $\dot{\theta}$. Its direction will also be affected, because if for example $\dot{\theta}$ is close to w_A, then the signals from A to B will look to O' and O'' as if they fall on B radially. A similar argument holds for w_B, where the total deflection added by the rotation of B is diminished by the orbital angular velocity $\dot{\theta}$. Therefore, angles φ and θ_B will depend on the difference $w_A - \dot{\theta}$ and $w_B - \dot{\theta}$ instead of only w_A and w_B.

Below we will focus on the case of observer O'' and rotation with slippage.

2.1 Observer O'' and Rotation with Slippage

In this case $w_A = w_{A0} e^{-(\lambda \rho''_{AB}(0,0)+\mu z_{AB})} = w_{A0} e^{-c\beta_{AB} t''_{AB}}$ and similarly for w_B ,with

$\beta''_{AB} = \lambda \sin \xi''_{AB}(0,0) + \mu \cos \xi''_{AB}(0,0)$, $t''_{AB} = t_{AB} = \dfrac{\rho''_{AB}(0,0)}{c}$

The $\mathbf{F}''_{AB}$ force that acts on B is due to A. Its magnitude, F''_{AB}, is given in [6],

$$F''_{AB} = \frac{k_G m'_A m'_B}{(\rho''_{AB}(0,0)^2 + z_{AB}^2)} \frac{1}{J_A} \frac{1}{J_B} \tag{6}$$

Where J_A and J_B are the Jacobians of the transformation due to the rotation of A and B respectively and are given by

$$\frac{1}{J_A} = \left[\frac{dV}{dV''}\right]_A = \frac{(c^2 + w_{A0}^2 (\rho''_{AB}(0,0) - r_B)^2 e^{-2(\lambda(\rho''_{AB}(0,0) - r_B \cos\zeta) + \mu(z_{AB} - r_B \sin\zeta))})^{\frac{3}{2}}}{c(c^2 + \lambda w_{A0}^2 (\rho''_{AB}(0,0) - r_B))^3 e^{-2(\lambda(\rho''_{AB}(0,0) - r_B \cos\zeta) + \mu(z_{AB} - r_B \sin\zeta))})} \tag{7}$$

$$\frac{1}{J_B} = \left[\frac{dV}{dV''}\right]_B = \frac{(c^2 + w_{B0}^2 r_B{}^2)^{\frac{3}{2}}}{c^3} \tag{8}$$

With $\sin\zeta = \dfrac{z_{AB}}{AB}$, r_B is the radius of body B, r_A is the radius of body A and $\rho(.,.)$ is the length of path from center of A to center of B. Note that body B is assumed rigid and therefore $\lambda = \mu = 0$ at distance r_B from the center of B. For simplicity, we will examine the case when $z_{AB} = 0$.

Still we have to take into account the orbital motion of body B with origin at A this will create a further relativistic contraction of the distance AB and the Jacobian of this extra transformation will affect the magnitude of the force F''_{AB}. Thus the force that appears at (6) must be amended to

$$F''_{AB} = \frac{k_G m'_A m'_B}{\rho''_{AB}(0,0)^2} \frac{1}{J_A J_B J_{orb}} \tag{9}$$

Where

$$\frac{1}{J_{orb}} = \frac{(c^2 + \dot{\theta}^2 \rho''_{AB}(0,0)^2)^{\frac{3}{2}}}{c^3} \tag{10}$$

Observe that $\rho''_{AB}(w_A, w_B)$ also depends on the orbital angular velocity $\dot{\theta}$ implying a further contraction due to the tangential orbital velocity. We will not deal with it assuming that it is small.

Further, because only the net rotation $w_{Anet} \triangleq w_A - \dot{\theta}$, $w_{Bnet} \triangleq w_B - \dot{\theta}$ will affect the direction of $\mathbf{F}''_{AB}$,. we may write (see [6])

$$\tan\varphi''_A = \frac{w_{Anet} t_{AB} (1 + w_{Anet}^2 t_{AB}^2 \sin^2 \xi''_{AB})}{1 + c\beta''_{AB} t_{AB}^3 w_{Anet}^2 \sin^2 \xi''_{AB}} = \frac{w_{Anet}^2 \sqrt{\rho''_{AB}(0,0)^2 + z_{AB}^2}}{c} \frac{1 + \dfrac{w_{Anet}^2 \rho''_{AB}(0,0)^2}{c^2}}{1 + \dfrac{w_{Anet}^2 \rho''_{AB}(0,0)^2 (\lambda \rho''_{AB}(0,0) + \mu z_{AB})}{c^2}} \tag{11}$$

and since $z_{AB} = 0$, $\sin \xi''_{AB} = 1$ hence

$$\tan\varphi''_A = \frac{w_{Anet}t_{AB}(1+w_{Anet}^2 t_{AB}^2)}{1+c\lambda t_{AB}^3 w_{Anet}^2} = \frac{w_{Anet}\rho''_{AB}(0,0)}{c}\frac{1+\dfrac{w_{Anet}^2\rho''_{AB}(0,0)^2}{c^2}}{1+\dfrac{\lambda w_{Anet}^2\rho''_{AB}(0,0)^3}{c^2}} \tag{12}$$

$$\theta_B = \int_0^{t_{AB}}(w_{B0}e^{-ct\beta''_{AB}}-\dot{\theta})dt = \frac{w_{B0}}{c\beta''_{AB}}(1-e^{-c\beta''_{AB}t_{AB}})+\theta(t_{AB})-\theta(0) \tag{13}$$

Where $\beta''_{AB} = \lambda\sin\xi''_{AB}(0,0)+\mu\cos\xi''_{AB}(0,0)$, $t''_{AB} = t_{AB} = \dfrac{\rho''_{AB}(0,0)}{c}$;

Then the total angle of deflection is defined as

$$\varphi''_{ATot} = \varphi''_A + \theta_B \tag{14}$$

The orbital equations for $z_{AB}=0$ are,

$$\ddot{\rho}''_{AB}(w_A,w_B)-\rho''_{AB}(w_A,w_B)\dot{\theta}''^2 = -\frac{k_G m'_A}{\rho''_{AB}(0,0)^2}\frac{1}{J_A J_B J_{orb}}\cos\varphi''_{ATot} \tag{15}$$

$$2\dot{\rho}''_{AB}(w_A,w_B)\dot{\theta}''+\rho''_{AB}(w_A,w_B)\ddot{\theta}'' = -\frac{k_G m'_A}{\rho''_{AB}(0,0)^2}\frac{1}{J_A J_B J_{orb}}\sin\varphi''_{ATot} \tag{16}$$

With

$$\frac{1}{J_A} = \left[\frac{dV}{dV''}\right]_A = \frac{(c^2+w_{A0}^2(\rho''_{AB}(0,0)-r_B)^2 e^{-2\lambda(\rho''_{AB}(0,0)-r_B)})^{\frac{3}{2}}}{c(c^2+\lambda w_{A0}^2(\rho''_{AB}(0,0)-r_B))^3 e^{-2\lambda(\rho''_{AB}(0,0)-r_B)})} \tag{17}$$

$$\frac{1}{J_B} = \left[\frac{dV}{dV''}\right]_B = \frac{(c^2+w_{B0}^2 r_B{}^2)^{\frac{3}{2}}}{c^3} \tag{18}$$

$$\frac{1}{J_{orb}} = \frac{(c^2+\dot{\theta}^2\rho''_{AB}(0,0)^2)^{\frac{3}{2}}}{c^3} \tag{19}$$

And where from [6] we know that,

$$\rho''_{AB}(w_A,0) = \rho''_{AB}(0,0)\frac{c}{\sqrt{c^2+w_{A0}^2 e^{-2\lambda\rho''_{AB}(0,0)}\rho''_{AB}(0,0)^2}} \tag{20}$$

$$\rho''_{AB}(w_A,w_B) = \rho''_{AB}(w_A,0)\frac{c}{\sqrt{c^2+w_{B0}^2 e^{-2\lambda\rho''_{AB}(0,0)}\rho''_{AB}(w_A,0)^2}} \tag{21}$$

Hence,

$$\rho''_{AB}(w_A,w_B) = \frac{c\rho''_{AB}(0,0)}{\sqrt{c^2+(w_{A0}^2+w_{B0}^2)e^{-2\lambda\rho''_{AB}(0,0)}\rho''_{AB}(0,0)^2}} \tag{22}$$

Using (12) we get $\cos\varphi''_A$ and $\sin\varphi''_A$ for $-90^\circ \le \varphi''_A \le 90^\circ$ used below.

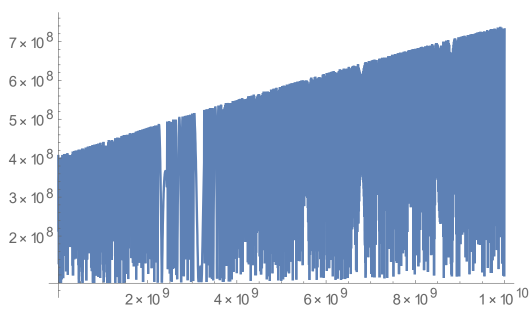
If $c >> w_{A0}e^{-\lambda\rho''(0,0)}\rho''(0,0)$ then $\rho''(w_A,0) \simeq \rho''(0,0)$ and if $c >> w_{A0}e^{-\lambda\rho''(w_A,0)}\rho''(w_A,0)$ then $\rho''(w_A,w_B) \simeq \rho''(w_A,0)$. These conditions are the *macrocosmos* conditions defined and used in [5] and they hold, when for example $\rho''(0,0)$ is big and $w_{A0}e^{-\lambda\rho''(0,0)}$ is small.

Hence, letting $\rho = \rho''(w_A,0) = \rho''(w_A,w_B) = \rho''(0,0)$, and for $z=0$ (15) and (16) can be simplified to,

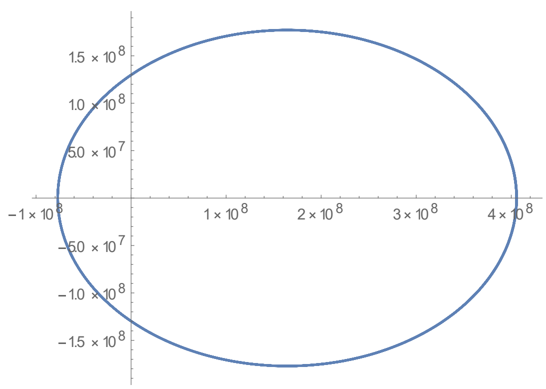
$$\ddot{\rho}-\rho\dot{\theta}^2 = -\frac{k_G m'_A}{\rho^2}\frac{1}{J_A J_B J_{orb}}\cos\varphi_{Tot} \tag{23}$$

$$2\dot{\rho}\dot{\theta}+\rho\ddot{\theta}=-\frac{k_G m'_A}{\rho^2}\frac{1}{J_A J_B J_{orb}}\sin\varphi_{Tot} \quad (24)$$

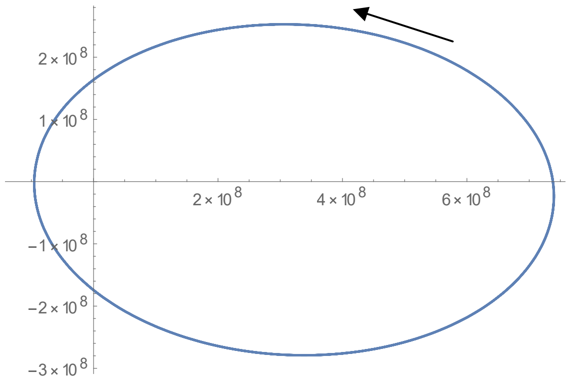
These equations give rise to a variety of orbits, shown in Figure 3:
In case A1 to A4 we show an orbit with $w_{A0}=7.27*10^{-5}$ rad/s $w_{B0}=2.7893*10^{-6}$ rad/s $m'_A=6.04586*10^{24}$ kg $\rho(0)=4.055*10^{8}$ m $\dot{\theta}(0)=1.392108*10^{-6}$ $\lambda=8.35*10^{-9}m^{-1}$ These parameters are close to earth (body A) and moon (body B) parameters but with much smaller orbital angular velocity than that of the real moon. Here body B spirals away in an orbit that becomes more and more eccentric which also precesses. A1 shows the distance from the origin ρ (where body B is located) as a function of time. A2 shows the orbital path at the beginning. A3 shows the orbit as it evolves to become very eccentric and shows precession as its major axis has rotated clockwise. A4 is the sine of the total angle of deflection which tends to become negative most of the time (Recall that the tangential force is accelerating in the counterclockwise direction when the sine is negative). In case B1 to B5 the angular velocity rotation of body A is increased so that $w_{A0}=7.27*10^{-4}$ rad/s This leads to a gradual deceleration of body B, which at first spirals down towards body A with increasing eccentricity and finally crashes on A. The total angle of deflection in B5, which when positive decelerates body B and when negative it accelerates it in the counterclockwise direction, remains almost always positive although it varies in magnitude with time. In C1 to C3 we have $w_{A0}=7.27*10^{-5}$ rad/s and we take negative rotation (=clockwise) for B so that $w_{B0}=-5.7893*10^{-3}$ rad/s. The sine of the total angle of deflection in C3, is negative during the orbit thus continuously accelerating body B, leading it to higher orbit until disengagement. In D1 to D3 we have an exotic case, where body B rotates very fast in the clockwise direction so that $w_{B0}=-2.7893*10^{2}$ rad/s. The body that starts with counterclockwise orbit changes and starts moving chaotically for a while, before the force becomes repulsive and the body moves away from A.



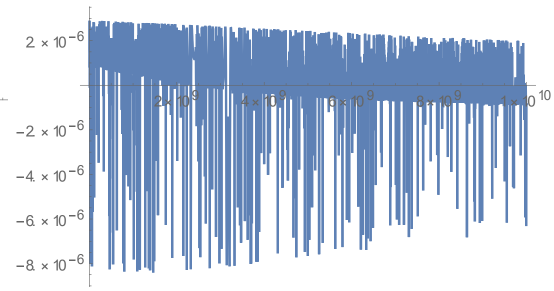
A1: ρ versus time



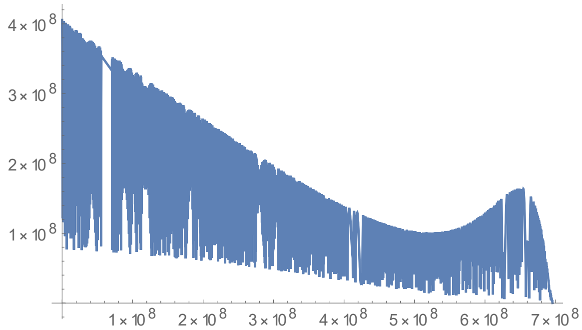
A2: Orbit path at start



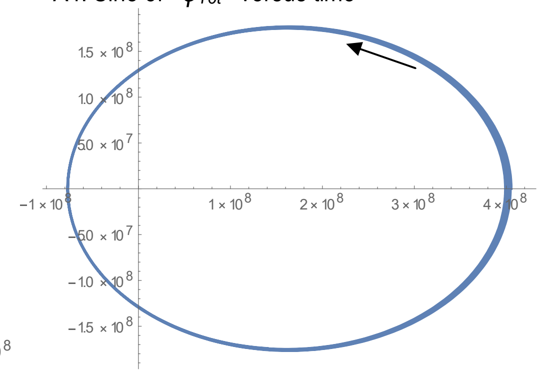
A3: Orbit curve at a late stage



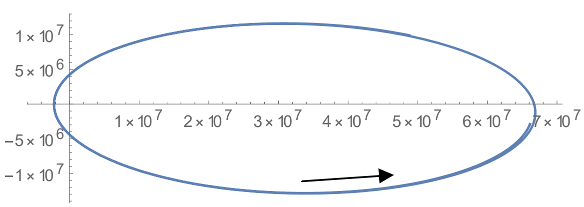
A4: Sine of φ_{Tot} versus time



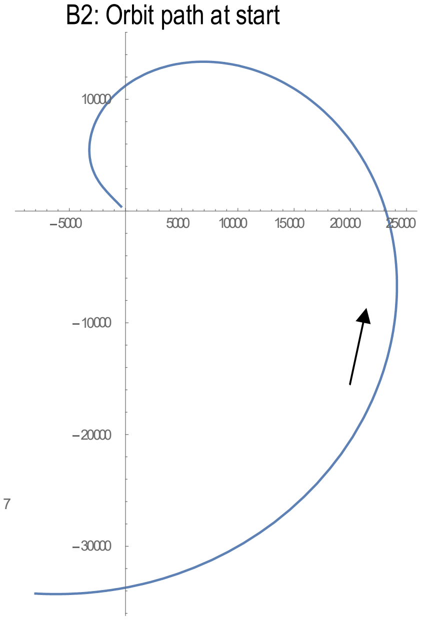
B1: ρ versus time



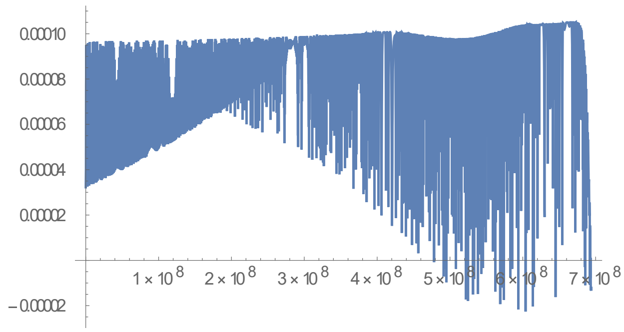
B2: Orbit path at start



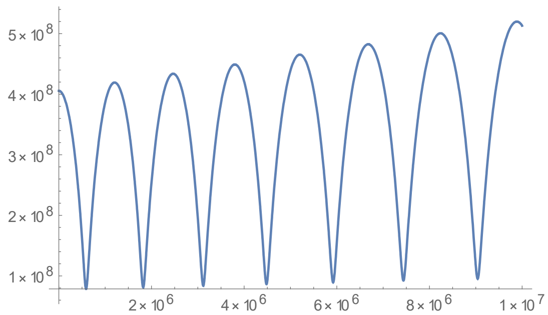
B3:: Orbit curve at a later stage



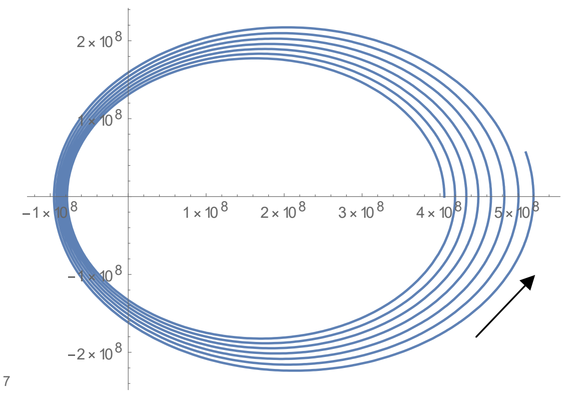
B4:: Orbit curve at final stage



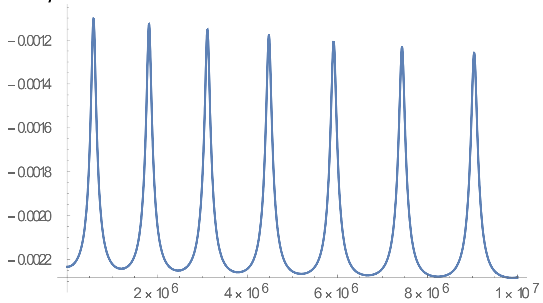
B5: Sine of φ_{Tot} versus time



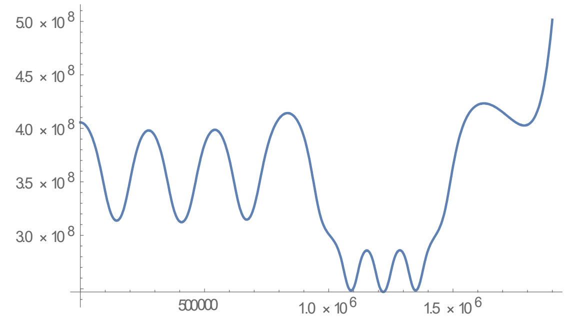
C1: ρ versus time



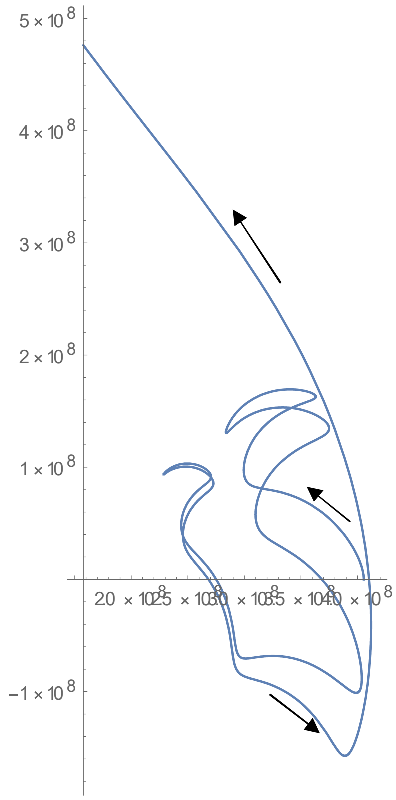
C2:Orbit path



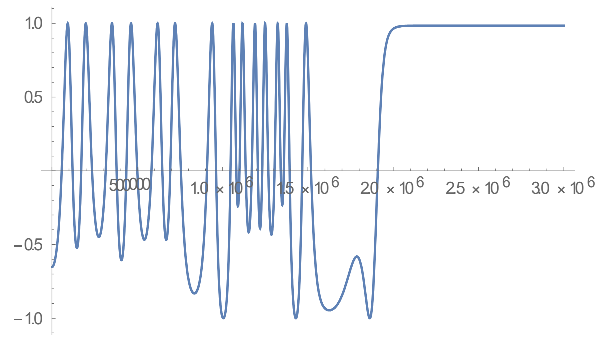
C3: Sine of φ_{Tot} versus time



D1: ρ versus time



D2:Orbit path



D3: Sine of φ_{Tot} versus time

Figure 3: Examples of orbits for observer O'' and rotation with slippage for a rotating body B around a big rotating body A located at the origin and for the macrocosmos. For A1 to A4 body B spirals away from A. The orbit becomes more eccentric, while the major axis changes orientation . In B1 to B5 we have deceleration of B leading to lower orbit that gradually turns more eccentric until Body B crashes on A. In C1 to C3 we have negative rotation of body B .This leads to acceleration of B until it disengages from A. In D1 to D3, body B rotates so rapidly that changes the direction of its orbit from positive to negative, moves chaotically for a while and then accelerates and escapes.

2.3 Central Force and circular orbits

For central force a necessary and sufficient condition is that the RHS of equation (16) is zero and this amounts to asking that $\varphi''_{Tot} = 0$. or, in general, that it is equal to an integer multiple of 2π . To demand that $\varphi''_{Tot} = 0$ along with (15), (16) and (22) makes the system over determined and infeasible. We will, therefore seek, solutions limiting ourselves to circular orbits. In this case $\ddot{\rho}(w_A, w_B) = 0 = \dot{\rho}(w_A, w_B) = \ddot{\theta}$ and then (16) becomes redundant (0=0). Thus we are left with (15) and $\varphi'' + \theta_B = 0$ plus equations (12), (13) and (22).
Substituting (22) into (15) we obtain

$$\dot{\theta}^2 = \frac{k_G m'_A}{c\rho_{00}^3}\sqrt{c^2 + (w_{A0}^2 + w_{B0}^2)e^{-2\lambda\rho_{00}}\rho_{00}^2}\,\frac{(c^2 + w_A^2(\rho_{00} - r_B)^2)^{\frac{3}{2}}}{(c^2 + \lambda w_A^2(\rho_{00} - r_B)^3)}\,\frac{(c^2 + w_{B0}^2 r_B^2))^{\frac{3}{2}}}{c^3}\,\frac{(c^2 + \dot{\theta}^2\rho_{00}^2)^{\frac{3}{2}}}{c^3} \qquad (25)$$

Where we denote for economy of space $\rho(0,0) = \rho_{00}$, $w_A = w_{A0}e^{-\lambda(\rho_{00} - r_B)}$, $w_B = w_{B0}e^{-\lambda r_B}$.

Using (12) and (13) on $\varphi''_{Tot} = \varphi'' + \theta_B = 2n\pi$ for $n = 0,1,2,3,...$ we have

$$\varphi''_{Tot} = \underbrace{\arctan\left(\frac{\frac{(w_{A0}e^{-\lambda\rho_{00}} - \dot{\theta})\rho_{00}}{c}(1 + \frac{(w_{A0}e^{-\lambda\rho_{00}} - \dot{\theta})^2 \rho_{00}^2}{c^2})}{1 + \lambda\frac{(w_{A0}e^{-\lambda\rho_{00}} - \dot{\theta})^2 \rho_{00}^3}{c^2}}\right)}_{\varphi''} + \underbrace{\frac{w_{B0}}{c\lambda}(1 - e^{-\lambda\rho_{00}}) + \theta(\frac{\rho_{00}}{c}) - \theta(0)}_{\theta_B} = 2n\pi$$

(26)

With the restriction that $-\pi/2 \le \varphi'' \le \pi/2$. For φ''_{Tot} to be an integer multiple of 2π we need $|w_{B0}|$ to be large enough.

Using (25) into (26) we see that the latter equation may have one or more solutions, which are the distances ρ_{00}, where circular orbits are possible. Then we may determine the observed distance (radius of the circular orbit) $\rho(w_A, w_B)$ using (22).

For the macrocosmos case where $(\rho_{00} - r_B)w_{A0}e^{-\lambda(\rho_{00} - r_B)} << c$, $r_B w_{B0} << c$, $(w_{A0}^2 + w_{B0}^2)e^{-2\lambda\rho_{00}}\rho_{00}^2 << c^2$ and $\rho_{00}\dot{\theta} << c$ equation (25) becomes,

$$\dot{\theta}^2 \simeq \frac{k_G m'_A}{\rho_{00}^3} \tag{27}$$

which is the classical equation for circular orbit. Substituting $\dot{\theta}$ in (26) we obtain,

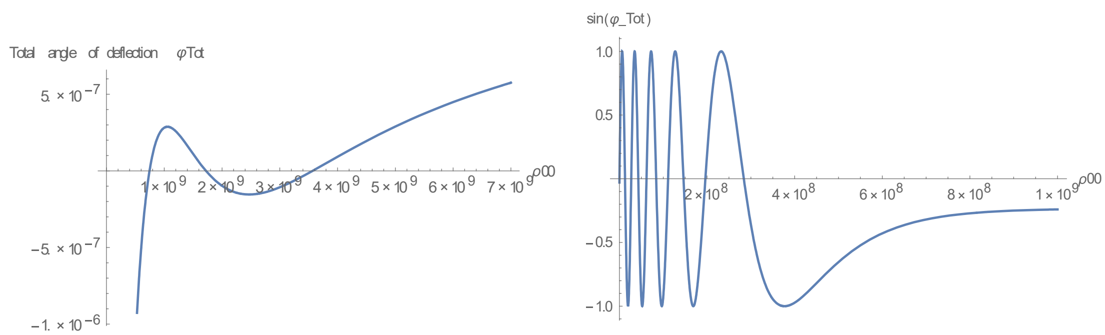
$$\varphi''_{Tot} = \arctan\left(\frac{\frac{(w_{A0}e^{-\lambda\rho_{00}} - \sqrt{\frac{k_G m'_A}{\rho_{00}^3}})\rho_{00}}{c}\left(1 + \frac{(w_{A0}e^{-\lambda\rho_{00}} - \sqrt{\frac{k_G m'_A}{\rho_{00}^3}})^2 \rho_{00}^2}{c^2}\right)}{1 + \lambda\frac{(w_{A0}e^{-\lambda\rho_{00}} - \sqrt{\frac{k_G m'_A}{\rho_{00}^3}})^2 \rho_{00}^3}{c^2}}\right) + \frac{w_{B0}(1 - e^{-\lambda\rho_{00}})}{\lambda c} - \sqrt{\frac{k_G m'_A}{\rho_{00}^3}}\frac{\rho_{00}}{c} = 2n\pi$$

(28)

Where we have approximated $\theta(\frac{\rho_{00}}{c})$ by $\dot{\theta}\frac{\rho_{00}}{c}$. The solutions to this equation, if they exist, give the distances, where circular orbits are possible in the macrocosmos case for $n = 0,1,2,3,...$.

Assume that $\dot{\theta} > 0$ and $w_{B0} > 0$. Then from (25) we see $\dot{\theta}\frac{\rho_{00}}{c} \to 0$ as $\rho_{00} \to \infty$, and therefore, using (26) $\varphi''_{Tot} \to \frac{w_{B0}}{\lambda c}$. This implies that at far away distances from body A only the self rotation of body B affects the direction of the force from A. Also, from (26) $\varphi''_{Tot} \to 0$ as $\rho_{00} \to 0$ since all terms go to 0. For the case of macrocosmos graphs of like that in Figure 4(a) show that there is a value of ρ_{00} where φ''_{Tot} changes from negative to positive values passing through zero. It is also possible to have another two crossings. One from positive to negative values and finally one from negative to positive values. Whenever we have

a crossing from negative to positive value, the zero attained by φ''_{Tot} is stable. This holds because φ''_{Tot} negative implies that body B is accelerating, therefore, ρ_{00} is increasing. If it exceeds the point where, φ''_{Tot} becomes zero it will change to positive decelerating the body and bringing it back to the ρ_{00} where φ''_{Tot} is zero. Thus the body will be orbiting, while φ''_{Tot} oscillates close to zero. The other point (if it exists) where φ''_{Tot} crosses from positive to negative values is not a stable point circular orbit because the forces act in the opposite way. In short, for the macrocosmos, there is a stable circular orbit and depending on the parameters there is another stable circular orbit further away with an unstable one in between. This argument holds symmetrically when $\dot{\theta} < 0$ and $w_{B0} < 0$; but it does not hold for $\dot{\theta} w_{B0} < 0$ since there is permanent acceleration that disengages Body B from A and no stability can be reached.
However, this pattern of stability does not hold when we have a fast rotating body B. For example, when $w_{B0} > 80$ rad/s $\sin(\varphi''_{Tot})$ oscillates with increasing period until it stabilizes for big enough ρ_{00}, thus allowing many distances, where a stable circular orbit can exist.(see Figure 4(b)) where $\sin(\varphi''_{Tot})$ using (28) is plotted.
In all the above cases a prerequisite for stable orbit is that the point, where $\sin(\varphi''_{Tot})$ becomes zero, $\cos(\varphi''_{Tot}) > 0$, meaning that the force is attractive



(a) (b)

Figure 4 The total angle of deflection for a circular orbit that crosses zero three times. **(a)** is a macrocosmos example :It is φ''_{Tot} versus ρ_{00}. The two crossings coming from negative to positive values are stable circular orbits. The one in between going from positive to negative values is unstable. In **(b)** we have $\sin\varphi''_{Tot}$ versus ρ_{00} .where body B is rotating fast in macrocosmos. There a multiplicity of distances where φ''_{Tot} becomes zero.

2.4 The Pluto-Charon case and an approximation of the slippage parameter λ

The dyad Pluto and Charon have the same rotation period of 6.4 earth days and they orbit around their center of mass in circular orbit with the same orbital period of 6.4 days. Their density is rather high implying a rocky interior. This limits the tidal effects giving more room to forces studied here. Further, the orbital plane is almost perpendicular to that of the ecliptic and thus forces due to rotation of the sun and other bodies of the type studied here have minimal effect on their orbit. This makes it a good candidate for finding an approximate value of the parameter λ.

We are given:
Mass of Pluton: $m_p=1.303*10^{22}$ kg
Mass of Charon: $m_c=0.1586*10^{22}$ kg
Radius of Pluto: $r_p=1.185*10^6$ m
Radius of Charon: $r_c=0.604*10^6$m
Distance between them:$d=1.964*10^7$m
The total distance from center to center is $\rho_{00}=21.429*10^6$m
Period of rotation and revolution:T=6.4 earth days which corresponds to angular velocity

$$\dot{\theta}=w_{P0}=w_{C0}=\frac{2\pi}{T}=1.13628*10^{-5} \text{ rad/s}$$

We let $M=m_P+m_C=1.461*10^{22}kg$

The Pluto Charon two-body problem requires that $\dot{\theta}^2=\frac{k_G M}{\rho_{00}^3}$. We substitute the observed values of the RHS and we find that $\dot{\theta}=1.134163*10^{-5}$ rad/s which is reasonably close to the observed value mentioned above.

Now we substitute $\dot{\theta}=\sqrt{\frac{k_G M}{\rho_{00}^3}}$ in (26) and we obtain

$$\arctan\left(\frac{\dfrac{(w_{P0}e^{-\lambda\rho_{00}}-\sqrt{\frac{k_G M}{\rho_{00}^3}})\rho_{00}^2}{c}\left(1+\dfrac{(w_{P0}e^{-\lambda\rho_{00}}-\sqrt{\frac{k_G M}{\rho_{00}^3}})^2\rho_{00}^2}{c^2}\right)}{1+\lambda\dfrac{(w_{P0}e^{-\lambda\rho_{00}}-\sqrt{\frac{k_G M}{\rho_{00}^3}})^2\rho_{00}^3}{c^2}}\right)+(1-e^{-\lambda\rho_{00}})\frac{w_{C0}}{\lambda c}-\sqrt{\frac{k_G M}{\rho_{00}^3}}\frac{\rho_{00}}{c}=0$$

(29)

This equation is symmetric in P and C since $w_{P0}=w_{C0}$. Therefore, $\varphi''_{P,Tot}=\varphi''_{C,Tot}=\varphi''_{Tot}=0$ and the force is central for both bodies.
Plugging into (29) the above values we solve numerically by trial and error for λ and we find that the solution is $\lambda=8.3581278*10^{-9}m^{-1}$
To check the solution we plot a graph of the left hand side of this equation (which is equal to φ''_{Tot}) versus ρ_{00}, with $\lambda=8.3581278*10^{-9}m^{-1}$ we see that φ''_{Tot} crosses from negative to positive values at $\rho_{00}=21.429*10^6 m$ as expected. This, as we explained above, is a stable circular orbit. See Figure 5(a).
The complete graph in (b) shows that there is no other crossing of the axis and therefore, this is the only stable circular orbit.

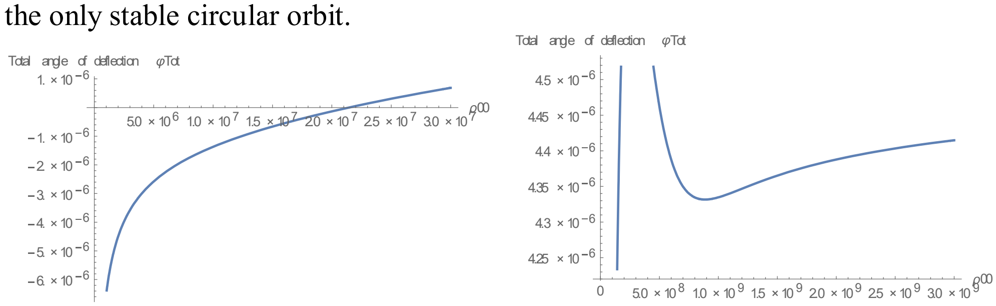


Figure 5 The plot of the total angle of deflection φ_{Tot} against distance ρ_{00} of the two bodies Pluton and Charon. In **(b)** is the total graph showing only one crossing. In **(a)** we see the detail of the crossing:

By choosing $\lambda = 8.35*10^{-9} m^{-1}$ we see that the total angle of deflection becomes zero at distance $\rho_{00} = 21.429*10^{6} m$ which is the distance between Pluton and Charon (center to center). At that distance the angle of deflection is zero and therefore, the force is central. Thus a circular orbit is possible and is stable.

3 Polar Jets

In this section we will study the problem of a multitude of small rotating bodies, B_ι , $\iota=1,2,3,...$ (also referred to as B type particles) that approach a much larger rotating body A, whose angular velocity is w_{A0} pointing in the z direction, the rotation is with slippage, from the point of view of the far away observer O'' .Then, as usual, the angular velocity of rotation of space around body A at distance (ρ, z) is $w_A = w_{A0}e^{-\lambda\rho-\mu z}$. A small body B_i with the z component of its rotational angular velocity is w_{B_i0}, has the z component of the angular velocity of space that rotates around it due to its rotation at distance (ρ, z), $w_{B_i} = w_{B_i0}e^{-\lambda\rho-\mu z}$. As it approaches body A, and depending on φ''_{ATot_i} ,

[[Recall (9). (11) and (13) : $\varphi''_{ATot_i} = \varphi''_{A_i} + \theta_{B_i}$,

$$\theta_{B_i} = \int_0^{t_{AB_i}} (w_{B_i0}e^{-ct_{AB_i}\beta_{AB_i}} - \dot{\theta}_i)dt = \frac{w_{B_i0}}{c\beta_{AB_i}}(1-e^{-ct_{AB_i}\beta_{AB_i}}) - \theta_i(t_{AB_i}) + \theta_i(0), \quad t''_{AB_i} = \frac{\sqrt{\rho''_{AB_i}(0,0)^2 + z_i^{\,2}}}{c},$$

$$\tan\varphi''_{Ai} = \frac{(w_A - \dot{\theta}_i)\sqrt{\rho''_{AB_i}(0,0)^2 + z_i^{\,2}}}{c} \frac{1 + \frac{(w_A - \dot{\theta}_i)^2 \rho''_{AB_i}(0,0)^2}{c^2}}{1 + \frac{(w_A - \dot{\theta}_i)^2 \rho''_{AB_i}(0,0)^2(\lambda\rho''_{AB_i}(0,0) + \mu z_i)}{c^2}} ,]]$$

it will feel a force that will or will not have a component towards A. If there is a component in the opposite direction from A, then there is repulsion from A and it will eventually be sent away from A unless its approaching initial velocity is big enough and the force changes as it approaches from repulsive to attractive. Those attracted will follow orbits that depend on φ''_{ATot_i} and the initial orbital velocity. If $\sin\varphi''_{ATot_i}$ is positive, the azimuthal component of the force points clockwise (in the opposite direction of the rotation of A, assuming $w_A > 0$ ie counterclockwise) decelerating body B_i when $\dot{\theta}_i > 0$ (ie counterclockwise) or vice versa. If the retarding force is not strong enough, when the initial orbital angular velocity is big enough, the body will eventually disengage from A. On the other hand, if it is very strong the body will spiral towards A and crash. Otherwise at some point equilibrium will be reached and φ''_{ATot_i} will vary from positive to negative and the opposite during the orbit. This equilibrium, as we saw in previous sections, can have $\varphi''_{ATot_i} = 0$ during the orbit thus the force is central which allows, as a particular case, the circular orbit. The inflow of B_i particles is bigger at $z = 0$, where the acceleration due to the force field is stronger and diminishes as $|z|$ increases (see (6)). The constant inflow of B type particles will create a congestion that can only be released if the particles flow away from the origin in the positive and negative z direction creating jets. The same congestion and unavoidable collisions between eccentric orbits will tend to make the particles move in circular orbits. We will, therefore, assume that the fluid element, which represents an average of the particles that constitute it, will have cyclic orbits with orbital angular velocity $\dot{\theta}$ so that $\varphi''_{ATot} = 0$. Further, each fluid element rotates around its own z axis with angular velocity $w_B = w_{B0} e^{-\lambda\rho-\mu z}$ where w_{B0} is the average of the w_{B0i}. We will approach the problem as a relativistic flow problem, from the point of view of observer O''.

First we need some notation.
r, θ, z are the cylindrical coordinates as observer O, who stands at the origin at body A and rotates with it, sees them. We use *r* instead of *ρ,* which we reserve for density. For observer O'', the corresponding coordinates are r'', θ'', z'' but as we know from previous theory, $\theta'' = \theta$,
$z'' = z$,

$$r'' = \frac{rc}{\sqrt{c^2 + (w_A^2 + w_B^2) r^2}}$$

Further, define for observer O
ρ : the static mass density of a fluid element which depends on the radial distance *r* and distance in the z direction *z*
υ_r : the velocity of flow of a fluid element in the r direction
υ_z : the velocity of flow of a fluid element in the z direction
υ_θ : the velocity of flow of a fluid element in the θ (tangential) direction
The total velocity of a fluid element for observer O is $\upsilon = \sqrt{\upsilon_r^2 + \upsilon_z^2 + (\upsilon_\theta - r w_A)^2}$
For observer O'' we have the same quantities with the double prime so that
$\upsilon'' = \sqrt{\upsilon_r''^2 + \upsilon_z''^2 + \upsilon_\theta''^2}$ where $\upsilon_z'' = \upsilon_z$

The relativistic flow problem for observer O'' expressed in cylindrical coordinates consists of (see [3])
-Energy conservation equation

$$\frac{\partial P''}{\partial t} - \frac{\partial}{\partial t} \frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}} - \nabla'' \cdot \frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}} \boldsymbol{\upsilon}'' = 0 \tag{30}$$

-Momentum conservation equations

$$\frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}} \left(\frac{\partial \upsilon_j''}{\partial t} + \boldsymbol{\upsilon}'' \cdot \nabla'' \upsilon_j'' \right) + \upsilon_j'' \frac{\partial P''}{\partial t} + \nabla_j'' P'' = a_j'' \rho'' \tag{31}$$

For $j \in \{x_1, x_2, x_3\}$ (Cartesian coordinates)
Where
P'' is the pressure, a_j'' is the jth component of acceleration, and $\rho'' c^2$ is the static (with respect to the fluid element with velocity $\boldsymbol{\upsilon}''$) energy density of a fluid element, since ρ'' is the static mass density of a fluid element for observer O''.
Fluid elements follow circular orbits with angular velocity $\dot{\theta}$, so that $\varphi''_{ATot} = 0$. Thus, Observer O'' will see fluid elements spiraling in towards A, especially close to $z = 0$, where the attraction force is biggest, as they orbit circularly around the z axis at the origin of which is A and then turn away from A in polar jets in the positive and negative direction of the z axis, while they continue to orbit around the z axis. As we are studying the problem at equilibrium, we will not be concerned about the dynamics of how the fluid reaches equilibrium.

At equilibrium (30) and (31) become,
-Energy equation at equilibrium

$$\nabla''\cdot\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}\boldsymbol{\upsilon}''=0 \quad (32)$$

Or in cylindrical coordinates,

$$\frac{\partial}{\partial r''}\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}\upsilon''_r+\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}\frac{\upsilon''_r}{r''}+\frac{1}{r''}\frac{\partial}{\partial\theta}\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}\upsilon''_\theta+\frac{\partial}{\partial z}\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}\upsilon_z=0 \quad (33)$$

And because $\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}$ does not depend on θ, it becomes

$$\frac{\partial}{\partial r''}\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}\upsilon''_r+\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}\frac{\upsilon''_r}{r''}+\frac{\partial}{\partial z}\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}\upsilon_z=0 \quad (34)$$

-Momentum equations at equilibrium.

$$\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}(\boldsymbol{\upsilon}''\cdot\nabla''\upsilon''_j)=-\nabla''_jP''+a''_j\rho'' \quad (35)$$

And changing to cylindrical coordinates $\{r,\theta,z\}$ and using the fact that $\mathbf{a}''$ is the acceleration acting on the fluid element, with magnitude $a''=|\mathbf{a}''|$, that has cylindrical components components $a''_r=a''\sin\xi''\cos\varphi''_{ATot}$, $a''_\theta=a''\sin\xi''\sin\varphi''_{ATot}$, $a''_z=a''\cos\xi''$, the momentum equations become,

$$\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}(\upsilon''_r\frac{\partial\upsilon''_r}{\partial r''}+\frac{\upsilon''_\theta}{r''}\frac{\partial\upsilon''_r}{\partial\theta}+\upsilon_z\frac{\partial\upsilon''_r}{\partial z})=-\frac{\partial P''}{\partial r''}+a''\rho''\sin\xi''\cos\varphi''_{ATot} \quad (36)$$

$$\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}(\upsilon''_r\frac{\partial\upsilon''_\theta}{\partial r''}+\frac{\upsilon''_\theta}{r''}\frac{\partial\upsilon''_\theta}{\partial\theta}+\upsilon_z\frac{\partial\upsilon''_\theta}{\partial z})=-\frac{\partial P''}{\partial\theta}+a''\rho''\sin\xi''\sin\varphi''_{ATot} \quad (37)$$

$$\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}(\upsilon''_r\frac{\partial\upsilon_z}{\partial r''}+\frac{\upsilon''_\theta}{r''}\frac{\partial\upsilon_z}{\partial\theta}+\upsilon_z\frac{\partial\upsilon_z}{\partial z})=-\frac{\partial P''}{\partial z}+a''\rho''\cos\xi'' \quad (38)$$

Taking into account the symmetry in θ, that $\frac{\partial P''}{\partial\theta}=0$ and $\sin\varphi''_{ATot}=0$ for circular motion, the above equations become

$$\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}(\upsilon''_r\frac{\partial\upsilon''_r}{\partial r''}+\upsilon_z\frac{\partial\upsilon''_r}{\partial z})=-\frac{\partial P''}{\partial r''}+a''\rho''\sin\xi'' \quad (39)$$

$$\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}(\upsilon''_r\frac{\partial\upsilon''_\theta}{\partial r''}+\upsilon_z\frac{\partial\upsilon''_\theta}{\partial z})=0 \quad (40)$$

$$\frac{\rho''c^2+P''}{1-\frac{\upsilon''^2}{c^2}}(\upsilon''_r\frac{\partial\upsilon_z}{\partial r''}+\upsilon_z\frac{\partial\upsilon_z}{\partial z})=-\frac{\partial P''}{\partial z}+a''\rho''\cos\xi'' \quad (41)$$

From calculus we know that $\frac{d\upsilon''_\theta}{dt} = \frac{\partial \upsilon''_\theta}{\partial r''}\frac{dr''}{dt} + \frac{\partial \upsilon''_\theta}{\partial z}\frac{dz}{dt} = \upsilon''_r \frac{\partial \upsilon''_\theta}{\partial r''} + \upsilon_z \frac{\partial \upsilon''_\theta}{\partial z}$

While from (40) we have

$$\frac{d\upsilon''_\theta}{dt} = \upsilon''_r \frac{\partial \upsilon''_\theta}{\partial r''} + \upsilon_z \frac{\partial \upsilon''_\theta}{\partial z} = 0 \tag{42}$$

Which means that υ''_θ is constant in time, which is expected as we assumed circular motion.

We define $Q \triangleq \frac{\upsilon''_r}{\upsilon_z}$ and using (42)

$$Q \triangleq \frac{\upsilon''_r}{\upsilon_z} = -\frac{\frac{\partial \upsilon''_\theta}{\partial z}}{\frac{\partial \upsilon''_\theta}{\partial r''}} \tag{43}$$

The quantity Q which will be used below is calculated in Appendix A as a function of the variables r and z

The auxiliary equations (see [1]) for solving the partial differential equations (39) and (41) are

$$\frac{dr''}{\upsilon''_r} = \frac{dz}{\upsilon_z} = -\left(\frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}}\right)\frac{d\upsilon''_r}{\frac{\partial P''}{\partial r''} - a''\rho''\sin\xi''} \tag{44}$$

And

$$\frac{dr''}{\upsilon''_r} = \frac{dz}{\upsilon_z} = -\left(\frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}}\right)\frac{d\upsilon_z}{\frac{\partial P''}{\partial z} - a''\rho''\cos\xi''} \tag{45}$$

respectively. Take (44). The first part $\frac{dr''}{\upsilon''_r} = \frac{dz}{\upsilon_z}$ is a tautology because it is equivalent to

$\frac{\frac{dr''}{dt}}{\frac{dz}{dt}} = \frac{\upsilon''_r}{\upsilon_z}$ which holds from the definition of υ''_r and υ_z. Therefore, no interrelation between r'' and z is demanded by the first part. Hence, only the second part remains,

$$\frac{dr''}{\upsilon''_r} = -\left(\frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}}\right)\frac{d\upsilon''_r}{\frac{\partial P''}{\partial r''} - a''\rho''\sin\xi''} \tag{46}$$

And similarly from (45),

$$\frac{dz}{\upsilon_z} = -\left(\frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}}\right)\frac{d\upsilon_z}{\frac{\partial P''}{\partial z} - a''\rho''\cos\xi''} \tag{47}$$

We may apply the method of auxiliary equations for solving the energy equation as well. To this end, define

$$U_r = \frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}} \upsilon''_r \tag{48}$$

And

$$U_z = \frac{\rho'' c^2 + P''}{1 - \frac{\upsilon''^2}{c^2}} \upsilon_z \tag{49}$$

Then the energy equation (34) becomes

$$\frac{\partial U_r}{\partial r''} + \frac{\partial U_z}{\partial z} + \frac{U_r}{r''} = 0 \tag{50}$$

And because $U_z \frac{\upsilon''_r}{\upsilon_z} = U_r$ we obtain,

$$\frac{\partial}{\partial r''}\left(\frac{\upsilon''_r}{\upsilon_z} U_z\right) + \frac{\partial U_z}{\partial z} = -\frac{\upsilon''_r}{\upsilon_z r''} U_z \tag{51}$$

Or

$$\frac{\upsilon''_r}{\upsilon_z}\frac{\partial U_z}{\partial r''} + \frac{\partial U_z}{\partial z} = -\frac{\upsilon''_r}{\upsilon_z r''} U_z - U_z \frac{\partial}{\partial z}\left(\frac{\upsilon_z}{\upsilon''_r}\right) \tag{52}$$

The auxiliary equations for this partial differential equation are

$$\frac{dr''}{\frac{\upsilon''_r}{\upsilon_z}} = \frac{dz}{1} = \frac{dU_z}{-U_z\left(\frac{1}{r''}\frac{\upsilon''_r}{\upsilon_z} + \frac{\partial}{\partial r''}\left(\frac{\upsilon''_r}{\upsilon_z}\right)\right)} \tag{53}$$

Rearranging terms we find,

$$\frac{dU_z}{U_z} = -\frac{\upsilon_z}{\upsilon''_r}\left(\frac{1}{r''}\frac{\upsilon''_r}{\upsilon_z} + \frac{\partial}{\partial r''}\left(\frac{\upsilon''_r}{\upsilon_z}\right)\right) dr'' \tag{54}$$

Simplifying and integrating

$$\ln U_z = -\ln r'' - \ln \frac{\upsilon''_r}{\upsilon_z} + const \tag{55}$$

Thus

$$U_z = C_1 \frac{\upsilon_z}{r'' \upsilon''_r} \tag{56}$$

Or

$$U_r = \frac{C_1}{r''} \tag{57}$$

Where C_1 is a constant depending on boundary conditions. This is also the general solution to (50) since no restriction is imposed by the first part of the auxiliary equations (53).
Applying (57) on (48) we find,

$$\rho'' c^2 + P'' = \left(1 - \frac{\upsilon''^2}{c^2}\right)\frac{C_1}{\upsilon''_r r''} \tag{58}$$

We apply this to the momentum equations (46) and (47) to obtain,

$$\frac{d\upsilon''_r}{dr''} = \frac{r''}{C_1}\left(\frac{\partial P''}{\partial r''} - a'' \rho'' \sin \xi''\right) \tag{59}$$

$$\frac{d\upsilon_z}{dz} = \frac{\upsilon''_r}{\upsilon_z}\frac{r''}{C_1}\left(\frac{\partial P''}{\partial z} - a'' \rho'' \cos \xi''\right) \tag{60}$$

We can transform the last equation noting that $\frac{d}{dz}=\frac{\partial r''}{\partial z}\frac{d}{dr''}=\frac{\upsilon_r''}{\upsilon_z}\frac{d}{dr''}=Q\frac{d}{dr''}$,

$$Q\frac{d\upsilon_z}{dr''}=Q\frac{r''}{C_1}(Q\frac{\partial P''}{\partial r''}-a''\rho''\cos\xi'') \tag{61}$$

Dividing this by Q^2 and subtracting from (59) we get rid of P'' and we find

$$\frac{d\upsilon_r''}{dr''}-\frac{1}{Q}\frac{d\upsilon_z}{dr''}=\frac{r''}{C_1}a''\rho''(\frac{\cos\xi''}{Q}-\sin\xi'') \tag{62}$$

using the fact that $\upsilon_z=\frac{1}{Q}\upsilon_r''$,which implies that

$\frac{d\upsilon_z}{dr''}=\frac{d}{dr''}(\frac{1}{Q}\upsilon_r'')=\upsilon_r''\frac{d}{dr''}(\frac{1}{Q})+\frac{1}{Q}\frac{d\upsilon_r''}{dr''}$, (62) becomes,

$$(Q-\frac{1}{Q})\frac{d\upsilon_r''}{dr''}+\frac{\upsilon_r''}{Q^2}\frac{dQ}{dr''}=\frac{r''}{C_1}a''\rho''(\cos\xi''-Q\sin\xi'') \tag{63}$$

we can solve this to find

$$\rho''=\frac{C_1}{r''a''(\cos\xi''-Q\sin\xi'')}\left((Q-\frac{1}{Q})\frac{d\upsilon_r''}{dr''}+\frac{\upsilon_r''}{Q^2}\frac{dQ}{dr''}\right) \tag{64}$$

Observe that $\frac{d}{dr''}=\frac{\partial r}{\partial r''}\frac{d}{dr}$ where $\frac{\partial r''}{\partial r}=\frac{c(c^2+\lambda r^3(w_A^2+w_B^2))}{(c^2+(w_A^2+w_B^2)r^2)^{\frac{3}{2}}}$ since $r''=\frac{rc}{\sqrt{c^2+(w_A^2+w_B^2)r^2}}$

Hence, the above differential equations, from (61) onward, can all be brought to a form having derivatives with respect to the variables r and z only.

In order to proceed further, will make an assumption about the equation of state. Namely, that

$$P''=C_2\rho'' \tag{65}$$

Where C_2 is a constant. We will use the equation of state to substitute P'' in the equations of momentum (59) and (60) which thus take the form,

$$\frac{d\upsilon_r''}{dr''}=\frac{r''}{C_1}(C_2\frac{\partial\rho''}{\partial r''}-a''\rho''\sin\xi'') \tag{66}$$

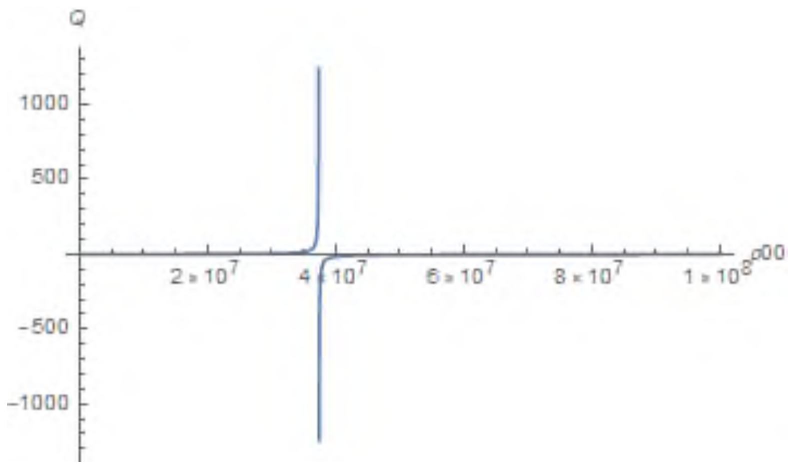
$$\frac{d\upsilon_z}{dz}=Q\frac{r''}{C_1}(C_2\frac{\partial\rho''}{\partial z}-a''\rho''\cos\xi'') \tag{67}$$

Substituting (64) into (66) we obtain an ordinary linear second order homogeneous differential equation in υ_r'' with non constant coefficients. This, hopefully, may be numerically solved with the use of math packages.

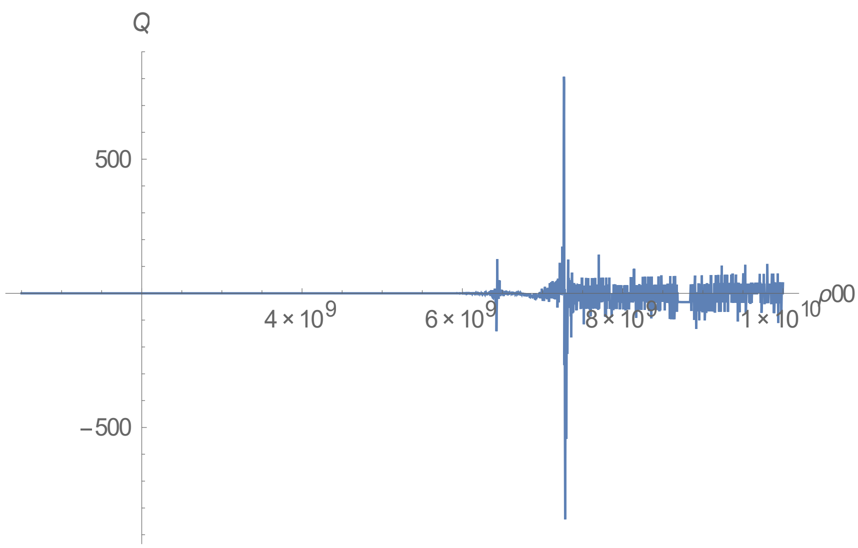
Once υ_r'' is determined we can find ρ'' from (64) and υ_z from $Q=\frac{\upsilon_r''}{\upsilon_z}$.

But solving this equation analytically to find υ_r'', is difficult so we will explore the behavior of $Q=\frac{\upsilon_r''}{\upsilon_z}$, which is known and calculated in Appendix A. In Figure 6 we present the graph of Q versus r with macrocosmos parameters in (a) while in (b) we have a graph for the microcosmos case. For the macrocosmos case, (a), the graph is drawn for $z\geq 0$, where $\upsilon_z\geq 0$ because as we explained above fluid elements approach radially towards the origin as they orbit the body at the origin and flow away in two jets , one in the positive z direction and one in the negative direction. The graph 6(a) shows that $\upsilon_r\leq 0$ .ie. the radial velocity is negative (pointing towards the z axis) and increases in magnitude relative to the velocity in the z direction from far away till some distance from the z axis. Then after that point (point of reversal) the radial velocity becomes positive, while the velocity in the z direction increases much more than the radial as we approach the z axis. The reversal implies a higher concentration of particles around the point of reversal. Further, as z increases the reversal point moves away from the z axis. For $z \le 0$, $\upsilon_z \le 0$ and the graph is the mirror image of the Figure 6(a) wrt the horizontal. In this case $\upsilon_r \le 0$ from far away until the same point of reversal and in short it follows the same pattern as for $z \ge 0$

For microcosmos in (b) it appears that there is a multitude of reversal points that appear as "noise" in the graph. There are two prominent reversals or spikes. The outer one behaves like in the macrocosmos case, while the inner does the opposite, so that in between them υ_r is mainly positive and is bigger closer to the spikes. Just outside the two spikes υ_r is mainly negative. So the region between the two spikes acts like a containment region that moves closer to the origin with increasing λ . The two prominent spikes appear to remain at constant radial distance ρ_{00} as z varies.



(a)



(b)

Figure 6 Plot of $Q = \dfrac{\upsilon_r''}{\upsilon_z}$ versus *r*(denoted as ρ_{00} on the graph axis). **(a)** Example from macrocosmos. The parameters for this example are $w_{A0} = 8.2*10^{-5}$ rad/s, $w_{B0} = 8.3*10^{-5}$ rad/s, $\lambda = 8.98*10^{-9} m^{-1}$,

$\mu = 8.94 * 10^{-9} m^{-1}$, $z = 2.64 * 10^{7} m$ **(b)** is a microcosmos example with $w_{A0} = 8.4 * 10^{22}$ rad/s, $w_{B0} = 8.82 * 10^{23}$ rad/s, $\lambda = 8.35 * 10^{-9} m^{-1}$, $\mu = 4.02 * 10^{-9} m^{-1}$, $z = 4.64 * 10^{-8} m$. The noise at the right side of the graph is due to the big value of w_{B0} that makes φ''_{Tot} rotate rapidly as *r*, changes. In fact this noise exists on all the r-axis but on the left the amplitude of Q is small. The spikes at the reversal points are similar to the macrocosmos case and but they are stable as z changes.

4 Conclusion

In studying the motion of two rotating bodies as one orbits around the other, we exploited the characteristics of the non central force acting between them. This force as we have shown in a previous paper [6] can be attractive or repulsive and accelerating or decelerating since its direction is affected by the angular velocity of rotation and distance of the bodies involved. As a result, depending on the rotational angular momentums a small body can orbit in a variety of orbits around a bigger rotating body since changing the rotation of either the big or the small body we may change the direction of the force acting on the orbiting body. It is shown that, when a small body orbits a large one, there is an equilibrium distance, where the force acting on the orbiting body is central and the orbit is circular. In macrocosmos, when the orbital angular velocity and the angular velocity of rotation are in the same direction, there is at least one such point, where circular orbit is stable. Depending on the parameters and for non rapidly rotating celestial bodies, there may be another such point with stable orbit further away with an unstable one in between. If the bodies are of comparable size the circular orbit is valid for both bodies. As an application, the Pluto - Charon case and the parameters measured by NASA exploration, are used to determine the slippage constant λ that is used for exponentially decreasing rotational angular velocity of signals around a rotating body. However, if the orbiting body has a very fast rotation, then there will be a multitude of points, where circular orbits are stable in macrocosmos and the same is true for microcosmos.

We finally applied the theory on forces between rotating bodies to explain how the rotation of a large body may attract multitudes of minute bodies of the right rotational characteristics. This continuous attraction of minute bodies creates a congestion that can only be released by the particles forming jets in the positive and negative z direction of the large rotating body. The problem was formulated as a relativistic flow problem for the far away observer O'' and the behavior of the ratio of the radial to the z-axial velocity was found and explained showing similar behavior of polar jet formation for both the microcosmos and macrocosmos cases.

5 References

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APPENDIX A

Calculation of Q

The orbital equations for given z are (15) and (16):

$$\ddot{\rho}''_{AB}(w_A, w_B) - \rho''_{AB}(w_A, w_B)\dot{\theta}''^2 = -\frac{k_G m'_A}{(\rho''_{AB}(0,0)^2 + z^2)} a'' \cos\varphi''_{ATot} \sin\xi''_{AB}(w_A, w_B) \quad (1)$$

$$2\dot{\rho}''_{AB}(w_A, w_B)\dot{\theta}'' + \rho''_{AB}(w_A, w_B)\ddot{\theta}'' = -\frac{k_G m'_A}{(\rho''_{AB}(0,0)^2 + z^2)} a'' \sin\varphi''_{ATot} \sin\xi''_{AB}(w_A, w_B) \quad (2)$$

We assumed that we have cyclical orbits with constant in time orbital angular velocity. This leaves us with one orbital equation (1) with $\varphi''_{ATot} = 0$, where we know from [5] (see section on "Forces for rotation with slippage") that

$$\tan\xi''_{AB}(w_A, w_B) = \tan\xi''_{AB}(0,0)\frac{\sqrt{w_A^2 t_{AB}^2 (1 + (w_A^2 + w_B^2) t_{AB}^2 \sin\xi''^2_{AB}(0,0))^2 + (1 + c\beta_{BA} t_{BA}^3 (w_A^2 + w_B^2)\sin^2\xi''_{AB}(0,0))^2}}{(1 + (w_A^2 + w_B^2) t_{AB}^2 \sin\xi''^2_{AB}(0,0))^{\frac{3}{2}}} \quad (3)$$

With

$$t_{AB} = \frac{\sqrt{\rho_{AB}(0,0)^2 + z^2}}{c}, \ \rho''_{AB}(0,0) = \rho''_{BA}(0,0) = \rho \quad \beta_{AB} = \beta_{BA} = \lambda\sin\xi''_{AB}(0,0) + \mu\cos\xi''_{AB}(0,0),$$

$$\sin\xi = \sin\xi(0,0) = \sin\xi''(0,0) = \sin\xi''_{AB}(0,0) = \sin\xi''_{BA}(0,0) = \frac{\rho}{\sqrt{\rho^2 + z^2}},$$

Note that, (3) may also be written as

$$\tan\xi''_{AB}(w_A, w_B) = \tan\xi''_{AB}(0,0) c \frac{\sqrt{w_A^2 \frac{\rho^2 + z^2}{c^2}(c^2 + (w_A^2 + w_B^2)\rho^2)^2 + (c^2 + (\lambda\rho + \mu z)(w_A^2 + w_B^2)\rho^2)^2}}{(c^2 + (w_A^2 + w_B^2)\rho^2)^{\frac{3}{2}}} \quad (4)$$

The acceleration is given by

$$a'' = \frac{k_G m'_A}{(\rho''_{AB}(0,0)^2 + z^2)} \frac{1}{J_A} \frac{1}{J_B} \frac{1}{J_{orb}} \quad (5)$$

Where

$$\frac{1}{J_A} = \frac{(c^2 + w_A^2 (\rho''_{AB}(0,0) - r_B)^2)^{\frac{3}{2}}}{c(c^2 + \lambda w_A^2 (\rho''_{AB}(0,0) - r_B)^3)} \quad (6)$$

$$\frac{1}{J_B} = \frac{(c^2 + w_{B0}^2 r_B^2))^{\frac{3}{2}}}{c^3} \quad (7)$$

Where r_B is the radius of body B

$$\frac{1}{J_{orb}} = \frac{(c^2 + \dot{\theta}^2 \rho''^2_{AB}(0,0))^{\frac{3}{2}}}{c^3} \quad (8)$$

From (1) and letting, $\rho''_{AB}(w_A,w_B)=r''_{AB}(w_A,w_B)=r''$ and $\rho''_{AB}(0,0)=\rho=r$ it takes the form

$$r''\dot{\theta}^2=\frac{k_G m'_A}{(r^2+z^2)}\frac{1}{J_A}\frac{1}{J_B}\frac{1}{J_{orb}}\sin\xi''_{AB}(w_A,w_B) \tag{9}$$

Where $\sin\xi''_{AB}(w_A,w_B)$ is calculated from $\tan\xi''_{AB}(w_A,w_B)$ above.

Using $r''=\dfrac{rc}{\sqrt{c^2+(w_A^2+w_B^2)r^2}}$ on (9) we solve for $\dot{\theta}$ and substitute it in

$$\upsilon''_\theta=r''\dot{\theta} \tag{10}$$

To obtain

$$\upsilon''_\theta=\sqrt{\frac{k_G m'_A}{(r^2+z^2)}}\frac{\sqrt{cr}}{(c^2+(w_A^2+w_B^2)r^2)^{\frac{1}{4}}}\sqrt{\frac{1}{J_A J_B J_{orb}}}\sqrt{\sin\xi''_{AB}(w_A,w_B)} \tag{11}$$

J_B is constant and if r_B is small enough compared to r and the $\dot{\theta}$ is small enough, then $J_B\simeq 1$ and $J_{orb}\simeq 1$

Then (11) simplifies to

$$\upsilon''_\theta\simeq\sqrt{\frac{k_G m'_A}{(r^2+z^2)}}\frac{\sqrt{r}}{(c^2+(w_A^2+w_B^2)r^2)^{\frac{1}{4}}}\frac{(c^2+(w_{A0}e^{-\lambda r-\mu z})^2r^2)^{\frac{3}{4}}}{(c^2+\lambda(w_{A0}e^{-\lambda r-\mu z})^2r^3)^{\frac{1}{2}}}\sqrt{\sin\xi''_{AB}(w_A,w_B)} \tag{12}$$

We may use this to find,

$$Q=\frac{\upsilon''_r}{\upsilon_z}=-\frac{\dfrac{\partial\upsilon''_\theta}{\partial z}}{\dfrac{\partial\upsilon''_\theta}{\partial r''}} \tag{13}$$