Proving the Collatz Conjecture

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Abstract. Collatz sequences are formed by dividing an even number by two until it is odd. Then multiply by three and add one to get an even number. The Collatz conjecture states that if this process is repeated you always get back to one. Using geometric series summations we prove that a connected Collatz Structure exists, which contains all positive integers exactly once. The terms of the Collatz Structure are joined together via the Collatz algorithm. Every positive integer starts a Collatz sequence of unique terms ending in the number one.

History. The Collatz conjecture was made in 1937 by Lothar Collatz. Through 2017 the conjecture has been checked for all starting values up to $(87)(2^{60})$, but very little progress has been made toward proving the conjecture. Paul Erödos said about the Collatz conjecture: "Mathematics may not be ready for such problems." https://en.wikipedia.org/wiki/Collatz_conjecture

Introduction. Note! This paper is hard to follow the first time you read it. But your effort is rewarded.

The Collatz Structure (displayed in the diagram below) consists of horizontal branches and vertical towers. Vertical arrows ↓ represent descending Collatz towers, where each term is half the previous term. Horizontal arrows ← indicate the Collatz algorithm is applied to move from term to term in the branch. We show how different integer types fit in the Collatz Structure (Section 1) exactly once (Section 2). In section 3 we define the binary series of branches, which are used in section 4 to show by induction arguments that all positive integers are in the branches and towers. Section 4.0 shows why this proof covers the entire set of positive integers and not just a subset that is *100%* dense in the set of positive integers. We show there are no circular or unending Collatz sequences in Section 5. Appendix 1 proves there can be no more than two consecutive even integers in a branch. Appendix 2 gives vertical tower details described below. Appendix 3 provides details about the Collatz Structure. Appendix 4 gives details of **green towers** as described below. Appendix 5 shows that every possible binary series is realized. Appendix 6 shows the proportion calculation of branch/branch segments.

Section 1

Defining and populating the Collatz Structure

Collatz Structure Branches and Towers ↓ indicates a descending Collatz tower

		↓		↓				↓				↓				↓		↓							↓
↓		84		160	←	53	←	106	←	35	←	70	←	23	←	46	←	15							36
128		42		80					↓				↓				↓						↓		18
64	←	21		40	13	←	26	←	52	←	17	←	34	←	11	←	22	←	7	←	14	←	28	←	9
32				20	↓																				
16	←	5	←	10	3																				
8																									
4																									
2																									
1																									

The **Trunk Tower** is the left-most tower, where each term is a power of two 2^s, $s=0,1,2,3\ldots$. A Collatz sequence can begin anywhere within the Collatz Structure and eventually by applying the Collatz algorithm a 2^s term in the **Trunk Tower** will be reached. From there we repeatedly divide by two until the base term 1 is reached. Every Collatz sequence terminates at the **Trunk Tower** base term ***1.***

Notice that every **red tower** base term is of the form $24m+4$, $24m+10$, or $24m+22$. The rest of the **red tower** terms alternate between $12k+8$ terms $20, 80$ in **blue** and $24k+16$ terms $40, 160$ in **brown.**

We trace a **red tower** from its ***n-th*** term $24k_n+16 \rightarrow 12k_n+8 \rightarrow 6k_n+4 = 24k_{n-1}+16$ $(k_n = 4k_{n-1}+2)$…to its ***first*** (base) term. $24k_2+16 \rightarrow 12k_2+8 \rightarrow 6k_2+4 = 24k_1+16$ $(k_2 = 4k_1+2) \rightarrow 12k_1+8 \rightarrow 6k_1+4.$

If $k_1=4m$, $6k_1+4 = 24m+4$. **If** $k_1=4m+1$, $6k_1+4 = 24m+10$. **If** $k_1=4m+3$, $6k_1+4 = 24m+22$.

The Collatz Structure starts with the **Trunk Tower.** Each $(4^j)(4), j=1,2,3...$ Trunk Tower term is the last term in a branch. At every $a=24m+4, 24m+10,$ and $24m+22$ base term in the Trunk Tower branches is a $4^j a, j=1,2,3\ldots$ secondary **red tower**. Each of these $4^j a$ terms in the secondary **red towers** is the last term in a branch. At every $a=24m+4, 24m+10,$ and $24m+22$ base term in these secondary branches is a $4^j a$ secondary **red tower**. Each $4^j a$ is the last term in a branch. This process is repeated indefinitely.

Note that $24k+16$ terms, which are divisible by eight are the last term in a branch. All the other even terms that appear in the middle of a branch $24m+4 \rightarrow 12m+2$, $24m+10$, or $24m+22$, have even factors of at most four or two. In appendix 1 we show there can be no more than two consecutive even terms in a branch. Since they are divisible by eight, $24k+16$ terms must appear at the end of a branch. We show in section 4.0 that there are no unending branches.

Collatz Structure Branches and Towers ↓ indicates a descending Collatz tower

		↓		↓				↓				↓				↓		↓							↓
↓		84		160	←	53	←	106	←	35	←	70	←	23	←	46	←	15							36
128		42		80					↓				↓				↓						↓		18
64	←	21		40	13	←	26	←	52	←	17	←	34	←	11	←	22	←	7	←	14	←	28	←	9
32				20	↓																				
16	←	5	←	10	3																				
8																									
4																									
2																									
1																									

The successor of any odd term is an even term $2j+1 \rightarrow 6j+4$ that leaves a remainder of one when divided by three. The **green** first terms in a branch are of the form $6j+3$. They are all divisible by three, as are all other terms in a **green tower.** They are of the form $(2^s)(6j+3)$ $s=1,2,3\ldots$ No odd term can precede a $6j+3$ term in a branch. An odd term can only precede an even term that leaves a remainder of one when divided by three $(5 \rightarrow 16)$. Each term above the $6j+3$ base term in a **green tower** is also of the form $24k, 24k+6, 24k+12,$ or $24k+18.$ The exact relation between the two form types is shown below. *** Note: every positive integer is of the form $(2j+1)(2^k)$. Take any positive integer and repeatedly divide by two until the remainder is odd. See appendix 4 for more details.

$$^{***}(2j+1)(24)(2^k) = (2^{k+3})(6j+3) \quad j=0,1,2,3\ldots \; k=0,1,2,3\ldots$$
$$24k+6 = (2)(6j+3), (j=2k) \quad j=0,2,4\ldots \; k=0,1,2,3\ldots$$
$$24k+12=(4)(6j+3), (j=k) \quad j=0,1,2,3\ldots \; k=0,1,2,3\ldots$$
$$24k+18=(2)(6j+3), (j=2k+1) \quad j=1,3,5\ldots \; k=0,1,2,3\ldots$$

Terms of the form $24k+2s, 0 \le s \le 11,$ and $6j+t, t=1,3,5$ fit within the Collatz Structure as follows:

24k **green tower**
24k+2 **successor of** $24j+4, j=2k$
24k+4 **red tower base middle of a branch**
24k+6 **green tower**
24k+8 **red tower successor of** $24j+16, j = 2k$
24k+10 **red tower base middle of a branch**
24k+12 **green tower**
24k+14 **successor of** $24j+4, j = 2k+1$
24k+16 **red tower end of a branch**
24k+18 **green tower**
24k+20 **red tower successor of** $24j+16, j = 2k+1$
24k+22 **red tower base middle of a branch**
6j + 1 **middle of a branch**
6j + 3 **green tower and beginning of a branch**
6j + 5 **middle of a branch**

Section 2

No individual term appears more than once in the Collatz structure. In Section 4.0 we show that all branches terminate with ***24h+16*** terms and are of finite length starting with a ***6j+3*** first term. There can be no duplicate terms in a branch. All the predecessors of a duplicate pair of terms would be duplicates. This would require first terms ***24h+3, 24h+9,*** or ***24h+15*** to be a duplicate term, and those terms only appear at the beginning of a branch. ***24h+21* have a *24(3h+2)+16*** term as an immediate successor without duplicates. No two distinct branches can contain the same term. Running the Collatz algorithm in reverse from the same term in two different branches would end in the same ***6j+3*** first term for two different branches. That is impossible. Both branches would be identical. All terms in the Trunk Tower are unique. That makes all terms in branches attached to the Trunk Tower unique. That makes all terms in secondary towers and all terms in branches that terminate in secondary towers unique. Thus, no individual term appears more than once in the Collatz structure.

Section 3

We define branch/branch segments and binary series, and provide examples. They will be used to prove all positive integers are in the Collatz Structure, and that there are no unending Collatz sequences.

Branches come in a group of four with odd first terms of the form ***24h+3, 24h+9, 24h+15,*** or ***24h+21*** and a ***24k+16*** last term ***h = 0,1,2,3,… k = 0,1,2,3,…***. **Branch segments start in the middle of branches.**

All **Branch segments** are odd. They come in two groups of four. The first group has first terms of the form ***24h+1, 24h+7, 24h+13,*** or ***24h+19.*** The second group has first terms of the form ***24h+5, 24h+11, 24h+17,*** or ***24h+23.*** Both groups have a ***24k+16*** last term. ***h = 0,1,2,3,… k = 0,1,2,3,…***

A branch/branch segment **binary series** counts the number of divisions by two on its **red tower base** terms: ***24m+4 (2), 24m+10 (1),*** and ***24m+22 (1)*** in a branch.

Branches/branch segments are characterized by their first term ***24h+2n-1,*** $1 \le n \le 12$ and a binary series of ***1*'s** and ***2*'s** (see examples below) and a last term ***24k+16.***

The **length *r*** of its binary series is the number of **red tower base** terms in a branch.

The sum of ***r 1's*** and ***2's*** in the binary series is ***s***.

First terms of branches *(24h+c, c=3, 9, 15, 21)* / branch segments *(24+c, c = 1,7,13,19* or *5,11,17,23)* have linear formulas $h = 2^s n+a$, ***n=0,1,2...*** that describe **branches / branch segments** sharing the same binary series. In $h = 2^s n+a$, ***n=0,1,2... a*** can be any value, even negative. Reducing or increasing ***a*** by 2^s will produce a formula for a branch with the same binary series as before. (See details below[1].) Each individual value of ***a***, $1 \le a \le 2^s$ is part of a different group of branches with the same binary series. Note: in ***24h+c, c*** can take on other values, as was done below, creating Collatz sequences of arbitrary length, but it's not necessary in defining the branch and branch segments.

All branches end with ***24k+16***, $k=3^{r+1}n+b$, $r \ge 0$, ***n=0,1,2,...***

Starting with any constant ***a*** value, you can calculate the formula associated with it. Substituting ***5*** for ***h*** in ***24h+9*** gives

(24)(5)+9 = 129, 388(2), 97, 292(2), 73, 220(2), 55, 166(1), 83, 250(1), 125, 376 = (24)(15)+16

The binary series ***(2,2,2,1,1)*** has the linear formula $h = 2^8 n+5$. **Sub *1* for *n* or [1] adding** 2^8 to ***a*** gives

$(24)(2^8+5)+9 = 6273, 18820(2), 4705, 14116(2), 3529, 10588(2), 2647, 7942(1), 3971, 11914(1), 5957, 17872 = (24)(3^6+15)+16$

We have 3 branches for ***h = 24h+9*** with the binary series ***(2,2,1).***

(1) ***h = 1 24+9 = 33, 100(2), 25, 76(2), 19, 58(1), 29, 88 = (24)(3)+16. k = 3*** **If *h = a, k = b***

(2) $h = ((2^5)(1)+1)+9$ ***(24)(33)+9=801, 2404(2), 601, 1804(2), 451, 1354(1), 677, 2032=(24)(84)+16.*** $k=3^4+3$

(3) $h = ((2^5)(2)+1)+9$ ***(24)(65)+9 = 1569, 4708(2), 1177, 3532(2), 883, 2650(1), 1325, 3976 = (24)(165)+16.*** $k=(3^4)(2)+3$

Creating branch (2) from branch (1)

$(24)(2^5)+33, (3)(24)(2^5)+100, (3)(24)(2^3)+25, (3^2)(24)(2^3)+76, (3^2)(24)(2)+19, (3^3)(24)(2)+58, (3^3)(24)+29, (3^4)(24)+88.$

By using repeating Collatz sequences you can build Collatz sequences of arbitrary length.

$1 \to 4(2) \to 1 \to 4(2) \to 1 \to 4(2) \to 1$ $24h - 23$, $h = 2^6+1$ $(24)(65) - 23$ $1537, 4612(2), 1153, 3460(2), 965, 2596(2), 649$

$-3 \to -8(3) \to -1 \to -2(1) \to -1 \to -2(1) \to -1$ $24h - 27$, $h = 2^5+1$ $(24)(33) - 27$ $765, 2296(3), 287, 862(1), 431, 1294(1), 647$

$24h - 29, -5 \to -14(1) \to -7 \to -20(2) \to -5 \to -14(1) \to -7 \to -20(2) \to -5$

$h = 2^6+1$ $(24)(65) - 29$ $1531, 4594(1), 2297, 6892(2), 1723, 5170(1), 2585, 7756(2), 1938$

Section 4

All positive integers appear in branches or towers. Note! The proofs in sections 4.2 and 4.3 are exactly the same as section 4.1. You could go directly to the summary after section 4.3.

Section 4.0 All branch binary series are of finite length with *24k+16* last terms.

Every possible binary series is realized **(see Appendix 5 for details)**. The proportion of ***24k+16*** terms in branches with a binary series **length *r*** is $2^r/3^{r+1}$. Proof by induction. ***24h+21→ 24(3h+2)+16*** is the formula for a branch with **length *r = 0*** binary series. The proportion of ***24k+16*** terms in branch with an empty binary series is $1/3^{[1]}$. For $r = 0$. $1/3 = 2^0/3^{0+1} = 2^r/3^{r+1}$.

Assume the proportion of ***24k+16*** terms in branches with a binary series of **length** $r \geq 0$ is $2^r/3^{r+1}$.
2^r is the number of different branch binary series of **length *r***. There are ***r+1*** applications of ***2j+1→6j+4*** .
2^{r+1} is the number of branch binary series of **length *r+1***. There are ***r+2*** applications of ***2j+1→6j+4***.

Thus, the proportion of ***24k+16*** terms in branches with a binary series of **length *r+1*** is $2/3^{[2]}$ the proportion of ***24k+16*** terms in branches with a binary series of **length *r***. $(2/3)(2^r/3^{r+1}) = 2^{r+1}/3^{r+2}$. The total proportion is $(1/3^{[1]})/(1 - 2/3^{[2]}) = 1$. **All branch binary series are of finite length with binary series of every combination of *1's* and *2's* for every value of *r* with *24k+16* last terms.**

Section 4.1 *24h+3, 24h+9,* and *24h+15* for all values of *h* are the first terms of the Collatz structure branches with binary series of every combination of *1's* and *2's* for every value of *r*.

Theorem 4.1.1: All ***24h+9*** are first terms in branches of the Collatz structure. (Proof by induction.)

Lemma 4.1.1.: The first ***24h+9*** term binary series is ***(2)*** for ***h=3,7,11,… 1/4*** of all the terms.
All other binary series begin with ***(2,...)***.

For ***h=4n+3, 24h+9 = 96n+81→288n+244(2)→72n+61→(9n+7)(24)+16.***

24h+9→72h+28(2)→18h+7→…

By Lemma 4.1.1 The binary series for ***r =1*** is ***(2)*** $= 1/2^{2[1]} = 1/4^{[2]} = 3^{r-1}/2^{2r}$. $s=2^{[1]}$ **binary series sum.**

Assume the proportion of ***24h+9*** terms in branches with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

The ***r+1*** position in the branch binary series can be ***(1)*** or ***(2).*** The proportion of ***24h+9*** terms of binary series **length** $r+1 = (1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$. **See Appendix 6 for more details.**

The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r}) / (3^r/2^{2r+2}) = 3/4^{[3]}$

The total proportion of ***24h+9*** terms in the Collatz structure is $(1/4^{[2]})/(1 - 3/4^{[3]}) = 1$.

All ***24h+9*** terms are in branches of the Collatz structure.

Theorem 4.1.2: All ***24h+3*** are first terms in branches of the Collatz structure. (Proof by induction.)

Lemma 4.1.2: The first two ***24h+3*** term binary series are ***(1)*** for ***h=2,4,6,…*** $1/2^{[1]}$, ***(1,2)*** for ***h=3, 11, 19,...*** and ***(1,2,...)*** for all other binary series with ***h*** an odd number.

For ***h=2n, 24h+3 = 48n+3→144n+10(1)→72n+5→(24n)(9)+16.***
For ***h=8n+3, 24h+3 = 192n+75→576n+226(1)→288n+113→864n+340(2)→216n+85→(27n+10)(24)+16***

For ***h=2n+1, 24h+3 = 48n+27→144n+82(1)→72n+41→216n+124(2)→54n+41→…***

By Lemma 4.1.2 The binary series proportion for ***r=2*** is $(1,2) = 1/2^{3[2]} = 1/8^{[3]} = 3^{r-2}/2^{2r-1}$ $s=3^{[2]}$ **binary series sum.**

Assume the proportion of ***24h+3*** terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

The proportion of ***24h+3*** terms of binary series **length *r+1*** is: $(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.

The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1}) / (3^{r-1}/2^{r+1}) = 3/4^{[3]}$

The total proportion of ***24h+3*** terms in the Collatz structure is $1/2^{[1]}+(1/8)^{[3]}/(1 - 3/4^{[3]}) = 1$.

All ***24h+3*** terms are in branches of the Collatz structure.

Theorem 4.1.3: All $24h+15$ are first terms in branches of the Collatz structure. (Proof by induction.)

Lemma 4.1.3: The first $24h+15$ term binary series is *(1,1)* for $h=3,7,11,\ldots$ $1/4$ of all the terms.
All other binary series with an odd number of terms begin with *(1,1,...)*.

For $h=4n+3$, $24h+15 = 96n+87 \to 288n+262(1) \to 144n+131 \to 432n+394(1) \to 216n+197 \to (24)(27n+24)+16$
$24h+15 \to 72h+46(1) \to 36h+23 \to 108h+70(1) \to 54h+35 \to \ldots$

By Lemma 4.1.3.1 The binary series for $r=2$ is $(1,1) = 1/2^2 = 1/4$[1]$= 3^{r-2}/2^{2r-2}$.

Assume the proportion of $24h+15$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The proportion of $24h+15$ terms of binary series **length** $r+1$ is
$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}$.
The ratio between successive terms is $(3^{r-2}/2^{2r-2}) / (3^{r-1}/2^{2r}) = 3/4$[2].
The total proportion of $24h+15$ terms in the Collatz structure is $1 = (1/4$[1]$)/(1 - 3/4$[2]$)$.
All $24h+15$ terms are in branches of the Collatz structure.

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Collectively all $24h+3$, $24h+9$, $24h+15$, are first terms in finite branches with binary series of all 2^r combinations of *1's* and *2's* for every value of r. As shown in section 4.0, all $24h+16$ are last terms in finite branches with binary series of all 2^r **combinations of** *1's* and *2's* for all r. **There are no unending** $24h+3$, $24h+9$, $24h+15$ **branches.**

Section 4.2 $24h+1$, $24h+7$, and $24h+19$ are the first terms of branch segments with binary series of every combination of *1's* and *2's* for every value of r.

Theorem 4.2.1: All $24h+19$ terms are in branches of the Collatz structure. (Proof by induction.)

Lemma 4.2.1: The first two $24h+19$ term binary series are *(1)* for $h=2,4,6,\ldots$ $1/2$[1] **of all terms** *(1,2)* for $h = 5,13,21\ldots$ and *(1,2,…)* for all other binary series with h an odd number.

For $h=2n$, $24h+19 = 48n+19 \to 144n+58(1) \to 72n+29 \to (24)(9n+3)+16$.

For $h=8n+5$, $192n+139 \to 576n+418(1) \to 288n+209 \to 864n+628(2) \to 216n+157 \to (27n+24)(24)+16$

For $h=2n+1$, $48n+43 \to 144n+130(1) \to 72n+65 \to 216n+196(2) \to 54n+49 \to \ldots$

By Lemma 4.2.1 The binary series for $r=2$ is $(1,2)=1/2^3 = 1/8$[2]$= 3^{r-2}/2^{2r-1}$

Assume the proportion of $24h+19$ terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

The proportion of $24h+19$ terms of binary series **length** $r+1$ is: $(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.
The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1}) / (3^{r-1}/2^{r+1}) = 3/4$[3]
The total proportion of $24h+19$ terms in the Collatz structure is $1/2$[1]$+(1/8)$[2]$/(1 - 3/4$[3]$) = 1$.

All $24h+19$ terms are in branches of the Collatz structure.

Theorem 4.2.2: All $24h+1$ terms are in branches of the Collatz structure. (Proof by induction.)

Lemma 4.2.2.: The first $24h+1$ term binary series is *(2)* for $h=2, 6, 10,\ldots$ $1/4$ of all the terms.
All other binary series begin with *(2,...)*.

For $h=4n+2$ $24h+1 = 96n+49 \to 288n+148(2) \to 72n+37 \to (24)(9n+4)+16$
$24h+1 \to 72h+4(2) \to 18h+1 \to \ldots$

By Lemma 4.2.2 The binary series for $r=1$ is $(2) = 1/2^2$[1]$= 3^{r-1}/2^{2r}$.

Assume the proportion of $24h+1$ terms in branches with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

The proportion of $24h+1$ terms of binary series **length** $r+1 = (1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$.

The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r}) / (3^r/2^{2r+2}) = 3/4$[2]

The total proportion of $24h+1$ terms in the Collatz structure is $(1/4$[1]$)/(1 - 3/4$[2]$) = 1$.

All $24h+1$ terms are in branches of the Collatz structure.

Theorem 4.2.3: All ***24h+7*** terms are in branches of the Collatz structure. (Proof by induction.)

Lemma 4.2.3: The first ***24h+7*** term binary series is ***(1,1)*** for ***h=2,6,10,… 1/4*** of all ***24h+7*** terms.
All other binary series with an odd number of terms begin with ***(1,1,...)***.

For ***h=4n+2, 24h+7 = 96n+55→288n+166(1)→144n+83→432n+250(1)→216n+125→(24)(27n+15)+16***
24h+7→72h+22(1)→36h+11→108h+34(1)→54h+17→...

By Lemma 4.2.3.1 The binary series for ***r=2*** is ***(1,1)*** $= 1/2^2 = 1/4$[1] $= 3^{r-2}/2^{2r-2}$.

Assume the proportion of ***24h+7*** terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The proportion of ***24h+7*** terms of binary series **length** ***r+1*** is

$$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}.$$

The ratio between successive terms is $(3^{r-2}/2^{2r-2}) / (3^{r-1}/2^{2r}) = 3/4$[2].

The total proportion of ***24h+7*** terms in the Collatz structure is $1 = (1/4$[1]$)/(1 - 3/4$[2]$)$.

All ***24h+7*** terms are in branches of the Collatz structure.

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Collectively all ***24h+1, 24h+7,*** and ***24h+19*** are first terms in finite branch segments with binary series of all 2^r combinations of ***1's*** and ***2's*** for every value of ***r***. **There are no unending *24h+1, 24h+7,* or *24h+19* branch segments.**

Section 4.3 *24h+11, 24h+17,* and *24h+23* are the first terms of branch segments with binary series of every combination of *1's* and *2's* for every value of *r*.

Theorem 4.3.1: All ***24h+11*** terms are in branches of the Collatz structure. (Proof by induction.)

Lemma 4.3.1: The first two ***24h+11*** term binary series are ***(1)*** for ***h=1,3,5,… 1/2***[1] **of all *24h+11* terms** ***(1,2)*** for ***h = 8,16,24…*** and ***(1,2,…)*** for all other value of ***h***.

For ***h=2n+1, 24h+11 = 48n+35→144n+106(1)→72n+53→(24)(9n+6)+16.***

For ***h=8n+8, 24h+11=192n+203→576n+610(1)→288n+305→864n+916(2)→216n+229→(27n+28)(24)+16***

24h+11→72h+34(1)→36h+17→108h+52(2)→27h+13→ …

By Lemma 4.3.1 The binary series for ***r=2*** is ***(1,2)***$=1/2^3 = 1/8$[2]$= 3^{r-2}/2^{2r-1}$

Assume the proportion of ***24h+11*** terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-1}$.

The proportion of ***24h+11*** terms of binary series **length** ***r+1*** is: $(1/2)(3^{r-2}/2^{2r-1})+(1/2^2)(3^{r-2}/2^{2r-1}) = 3^{r-1}/2^{2(r+1)-1}$.

The ratio between terms in the geometric series formed by the binary series is $(3^{r-2}/2^{2r-1}) / (3^{r-1}/2^{r+1}) = 3/4$[3]

The total proportion of ***24h+11*** terms in the Collatz structure is $1/2$[1]$+(1/8)$[2]$/(1 - 3/4$[3]$) = 1$.

All ***24h+11*** terms are in branches of the Collatz structure.

Theorem 4.3.2: All ***24h+17*** terms are in branches of the Collatz structure. (Proof by induction.)

Lemma 4.3.2.: The first ***24h+17*** term binary series is ***(2)*** for ***h=4, 8, 12,… 1/4*** of all the terms.
All other binary series begin with ***(2,...)***.

For ***h=4n+4 24h+17 = 96n+113→288n+340(2)→72n+85→(24)(9n+10)+16***
24h+17→72h+52(2)→18h+13→…

By Lemma 4.3.2 The binary series for ***r =1*** is ***(2)*** $= 1/2^2$[1] $= 3^{r-1}/2^{2r}$.

Assume the proportion of ***24h+17*** terms in branches with a binary series of **length** $r \geq 1$ is $3^{r-1}/2^{2r}$.

The proportion of ***24h+17*** terms of binary series **length** ***r+1*** $= (1/2)(3^{r-1}/2^{2r})+(1/2^2)(3^{r-1}/2^{2r}) = 3^r/2^{2(r+1)}$.

The ratio between terms in the geometric series formed by the binary series is $(3^{r-1}/2^{2r}) / (3^r/2^{2r+2}) = 3/4$[2]

The total proportion of ***24h+17*** terms in the Collatz structure is $(1/4$[1]$)/(1 - 3/4$[2]$) = 1$.

All ***24h+17*** terms are in branches of the Collatz structure.

Theorem 4.3.3: All ***24h+23*** terms are in branches of the Collatz structure. (Proof by induction.)

Lemma 4.3.3: The first ***24h+23*** term binary series is ***(1,1)*** for ***h=4,8,12,… 1/4*** of all the terms. All other binary series with an odd number of terms begin with ***(1,1,...)***.

For ***h=4n+4, 24h+23=96n+119→288n+358(1)→144n+179→432n+538(1)→216n+269→(24)(27n+33)+16***
24h+23→72h+70(1)→36h+35→108h+106(1)→54h+53→...

By Lemma 4.3.3.1 The binary series for ***r=2*** is ***(1,1)*** $= 1/2^2 = 1/4$[1] $= 3^{r-2}/2^{2r-2}$.

Assume the proportion of ***24h+23*** terms in branches with a binary series of **length** $r \geq 2$ is $3^{r-2}/2^{2r-2}$.

The proportion of ***24h+23*** terms of binary series **length** ***r+1*** is

$$(1/2)(3^{r-2}/2^{2r-2})+(1/2^2)(3^{r-2}/2^{2r-2}) = 3^{r-1}/2^{2(r+1)-2}.$$

The ratio between successive terms is $(3^{r-2}/2^{2r-2}) / (3^{r-1}/2^{2r}) = 3/4$[2].

The total proportion of ***24h+23*** terms in the Collatz structure is $1 = (1/4$[1]$)/(1 - 3/4$[2]$)$.

All ***24h+23*** terms are in branches of the Collatz structure.

***

Collectively all ***24h+11, 24h+17,*** and ***24h+23*** are first terms in finite branch segments with binary series of all 2^r combinations of ***1's*** and ***2's*** for every value of ***r***. **There are no unending *24h+11, 24h+17,* or *24h+23* branch segments.**

Section 4 Summary. All positive integers are in branches or towers of the Collatz structure.

The terms of the form ***24h+3***[3]***, 24h+11***[4]***,*** and ***24h+19***[4] have proportion formulas $3^{r-2}/2^{2r-1}$ and are the first terms in **branch**[3] or **branch segments**[4] with binary series of ***(1), (1,2)*** and ***(1,2,…)***.

The terms of the form ***24h+7***[4]***, 24h+15***[3]***,*** and ***24h+23***[4] have proportion formulas $3^{r-2}/2^{2r-2}$ and are the first terms in **branch**[3] or **branch segments**[4] with binary series of ***(1,1)*** and ***(1,1,…)***.

The terms of the form ***24h+1***[4]***, 24h+9***[3]***,*** and ***24h+17***[4] have proportion formulas $3^{r-1}/2^{2r}$ and are the first terms in **branch**[3] or **branch segments**[4] with binary series of ***(2)*** and ***(2,…)***.

The proportion formulas create geometric series that all sum to ***1 (100%)***. All odd terms with **length** $r > 0$ are in branches. The terms of the form ***24h+5***[4]***, 24h+13***[4]***,*** and ***24h+21***[3] are the first terms in **branch**[3] or **branch segments**[4] with an empty **length** $r = 0$ binary series. ***24h+5→24(3h)+16. 24h+13→24(3h+1)+16. 24h+21→ 24(3h+2)+16.***

All even terms are also in branches and/or towers.

(***2n+1→6n+4) 24m+4 (n = 4m)→12m+2 (24j+2, m=2j, 24j+14, m=2j+1), 24m+10 (n = 4m+1), 24m+16 (n = 4m+2), and 24m+22 (n = 4m+3)*** are in branches. **See table bottom of Section 1.**

All ***(2^s)(6j+3) 24k, 24k+6, 24k+12,*** and ***24k+18*** terms are in **green towers. See Section 1 and Appendix 4.**
All ***24k+16→12k+8 (24j+8, k=2j, 24j+20, k=2j+1)*** terms are in **red towers. See table bottom of Section 1.**

All terms ***24k+2s, k=0,1,2,…*** $0 \leq s \leq 11$, are in the branches or towers.

Section 5

There are no circular or unending Collatz sequences.

A circular Collatz sequence could not contain any ***6j+3*** terms. The only predecessors of ***6j+3*** terms are of the form ***(2^s)(6j+3)*** and they cannot be in a circular sequence. They have no predecessors but themselves. No ***6j+1*** or ***6j+5*** terms can be in a circular Collatz sequence. They are all in branches, which contain ***6j+3*** terms.
All even terms are in branches or towers. Therefore, there are no circular Collatz sequences.

Collatz sequences start anywhere in the Collatz structure and join the Trunk Tower at 4^{j+1} ***j=1,2,3…*** terminating at ***1*** its base. There are no unending Collatz sequences (details in Theorem 5.1 proof).

Theorem 5.1 The usage factor for **red tower base** terms in Collatz sequences with a binary series of **length *r*** **is** $3^r/4^r$. We will prove theorem 5.1 by induction.

There are two binary series of length one ***(1), (2)*** for a Collatz sequence with one **red tower base** term.
The usage factor is $1/2^1+1/2^2= 3/4$[1] verifying the formula for $r = 1$.

Assume usage factors for all binary series of **length *r*** is $3^r/4^r$.

The $r+1$ **red tower base** term of a Collatz sequence of that length contains ***(1)*** one or ***(2)*** two divisions by two. The **length** $r+1$ usage factor is $(1/2)(3^r/4^r)+(1/4)(3^r/4^r) = 3^{r+1}/4^{r+1}$.

The usage factor proportion ratio of Collatz sequence binary series is $(3^r/4^r)/(3^{r+1}/4^{r+1}) = 3/4^{[2]}$. The sum of the geometric series of the binary series usage factors is $(3/4^{[1]})/(1 - 3/4^{[2]}) = 3$. This equals the combined total proportion of ***24h+3, 24h+9,*** and ***24h+15*** terms in all branches. This indicates that ***100%*** of all three **red tower base** terms appear in Collatz sequences. Each Collatz sequence binary series is of finite length, but there is no longest binary series and no longest Collatz sequence.

Appendices

Appendix 1. A branch cannot have more than two consecutive even terms.

6n+1 → 18n+4

If $n = 4j$, $18n+4 = 72j+4$ $(24m+4,\ m=3j) \rightarrow 36j+2 \rightarrow 18j+1$.
If $n = 4j+1$, $18n+4 = 72j+22$ $(24m+22,\ m=3j) \rightarrow 36j+11$.
If $n = 4j+2$, $18n+4 = 72j+40$ $(24m+16,\ m=3j+1)$ Last term in the branch.
If $n = 4j+3$, $18n+4 = 72j+58$ $(24m+10,\ m=3j+2) \rightarrow 36j+29$

6n+3 → 18n+10.

If $n = 4j$, $18n+10 = 72j+10$ $(24m+10,\ m=3j) \rightarrow 36j+5$.
If $n = 4j+1$, $18n+10 = 72j+28$ $(24m+4,\ m=3j+1) \rightarrow 36j+14 \rightarrow 18j+7$
If $n = 4j+2$, $18n+10 = 72j+46$ $(24m+22,\ m=3j+1) \rightarrow 36j+23$.
If $n = 4j+3$, $18n+10 = 72j+64$ $(24m+16,\ m=3j+2)$ Last term in the branch.

6n+5 → 18n+16.

If $n = 4j$, $18n+16 = 72j+16$ $(24m+16,\ m=3j)$ Last term in the branch.
If $n = 4j+1$, $18n+16 = 72j+34$ $(24m+10,\ m=3j+1) \rightarrow 36j+17$.
If $n = 4j+2$, $18n+16 = 72j+52$ $(24m+4,\ m=3j+2) \rightarrow 36j+26 \rightarrow 18j+13$.
If $n = 4j+3$, $18n+16 = 72j+70$ $(24m+22,\ m=3j+2) \rightarrow 36j+35$.

Appendix 2. The repeating binary series structure of towers.

Within a tower if the sum of ***r 1's*** and ***2's*** in the binary series of a branch is ***s,*** there are three groups of branches having the same binary series ***24h+3, 24h+9,*** and ***24h+15***.

The first begins with $24h+3+(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$, $h,k,s=0,1,2,3,\dots$, $p=1,2,3\dots$, $x=3^{r+1}$ and ends with $(24k+16)+(24k+16)(4^{(x)(p-1)})$, $p=1,2,3\dots$, $x=3^{r+1}$, where ***24h+3*** becomes ***24k+16*** after ***r+1*** applications of $2j+1 \rightarrow 6j+4$ applied to ***24h+3*** and its odd successors and ***s*** divisions by two applied to ***72h+10*** and its even successors.

Applied to $(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$ it becomes $(24k+16)(4^{(x)(p-1)} - 1)$.

This gives $(24k+16)+(24k+16)(4^{(x)(p-1)} - 1) = (24k+16)(4^{(x)(p-1)})$.

The other two groups that begin with ***24h+9…*** and ***24h+15…*** have the same form as ***24h+3***...

Link between the formulas for branch and tower first terms.

For some t, $24h+3+(t-1)(24)(2^s) = 24h+3+(2^s)(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$.

For $x=3^{r+1}$ every power of three in $4^{(x)(p-1)} - 1 = (3+1)^{(x)(p-1)} - 1$ has a coefficient divisible by 3^{r+1} and is a multiple of ***3***. $(24k+16)(4^{(x)(p-1)} - 1) / 3^{r+1}$ is a multiple ***24.*** The same is true for the forms beginning with ***24h+9…*** and ***24h+15…***

Each tower's branch binary series structure is a microcosm of the total branch binary series structure.

$x=3^{r+1}$, $(24k+16)(4^{(x)(p-1)})$ vs $24k+16+(p-1)(24)(3^{r+1})$ $p=1,2,3\dots$ **Compare with Section 3 bottom.**

Now we apply the formulas to the Trunk Tower to calculate some branches

$((24)(0)+16)(4^{(9)(1-1)})$, $16 \leftarrow 5 \leftarrow 10(1) \leftarrow 3$, $(24)(0)+3+(2^1)((24)(0)+16)(4^{(9)(1-1)} - 1)/3^2$, $h=0, k=0, p=1, s=1$

$((24)(2)+16)(4^{(3)(1-1)})$, $64 \leftarrow 21$, $24h+21+(2^0)(24(2)+16)(4^{(3)(1-1)} - 1)/3$, $h=0, k=2, p=1, s=0$

$((24)(2)+16)(4^{(3)(2-1)})$, $4096 \leftarrow 1365$, $24h+21+((2^0)(24)(3h+2)+16)(4^{(3)(2-1)} - 1)/3$, $h=0, k=2, p=2, s=0$

$((24)(0)+16)(4^{(9)(2-1)})$, $(174762)(24)+16 \leftarrow (58254)(24)+5 \leftarrow (116508)(24)+10(1) \leftarrow (38836)(24)+3$,
$(24)(0)+3+(2^1)((24)(0)+16)(4^{(9)(2-1)} - 1)/3^2$, $h=0, k=0, p=2, s=1$

Appendix 3. Collatz Structure Details.

Groups of similar Collatz sequence segments. If a Collatz sequence segment has a first term a and a last term b with r, $2j+1 \rightarrow 6j+4$ and s divisions by two, there is a series of Collatz sequence segments containing the same number of terms of the same size and structure with a first term $a+(p-1)(24)(2^s)$ and last term $b+(p-1)(24)(3^r)$, $p=1,2,3\ldots$ **(Compare with the bottom of Section 3.)**

The average branch binary series length: $3r=(1)(3/4)+(2)(9/16)+(3)(27/64)+\ldots$ $3r-(3)(3/4)r=3$, $r=4$. The binary series usage factor is three. Three lengths $3r$ are being calculated. $3/4$ is the proportion of length one. $9/16$ of length two…Multiply the equation by $3/4$ and subtract. $3r-(3)(3/4)r = 3/4 + 9/16 + \ldots = 3$.

The average branch binary series sum: $((2,1,1,1)+(2,2,1,1)+(2,1,1,1))/3 = (5+6+5)/3 = 4.333\ldots$ There are twice as many binary series components with one division by two $24j+10$ (1), $24j+22$ (1) than there are components with two divisions by two $24j+4$ (2). Three binary series of length four with twice as many ***1's*** as ***2's*** make up the computation.

Appendix 4. The green towers consist of $(2^s)(6j+3)$ $s=0,1,2,\ldots$ $j=0,1,2,\ldots$

The second term in each tower is an odd multiple of six. $(1)(6), (3)(6), (5)(6), \ldots$ Each subsequent tower term has twice the multiples of six as the previous term. The total number of multiples of six is $(2n-1)(2^s)$ $n=1,2,3\ldots$ $s=0,1,2,3\ldots$ This accounts for all multiples of six.

Appendix 5. Every possible binary series is realized. The terms of the form $24h+7$[1], $24h+15$[2], and $24h+23$[1] are the first terms in branch[2] or branch segments[1] with binary series of $(1,1)$ and $(1,1,\ldots)$.

The terms of the form $24h+1$[1], $24h+9$[2], and $24h+17$[1] are the first terms in branch[2] or branch segments[1] with binary series of (2) and $(2,\ldots)$.

The proportion of the branch/branch segments of each of these six term types with binary series sums of $s=2,3,4,\ldots$ is $1/2^2 + 1/2^3 + 2/2^4 + 3/2^5 + 5/2^6 + \ldots = 1$ (see explanation of series sum $=1$ below).

The denominators are 2^s $s=2,3,4,\ldots$ for the binary series sums. The numerators are the number of different binary series with a binary series sum of $s=2,3,4,\ldots$ Each binary series is associated with a linear formula of the form $h=2^s n+a$. The proportion of each formula in the above sum is $1/2^s$. Since for each value s there is an $s+1$ and an $s+2$, the numerators form a Fibonacci sequence.

$[2], [2,1], [(2,1,1)\ (2,2)], [(2,1,1,1), (2,2,1), (2,1,2)], \ldots$ For a binary series of length s and b of length $s+1$ there will be $a+b$ binary series of length $s+2$.

The proportional amount remaining after the corresponding proportional amounts of the above binary series are subtracted $(2/2^1-1/2^2=3/2^2-1/2^3=5/2^3-2/2^4=8/2^4-3/2^5=13/2^5\ldots)$ also forms a partial Fibonacci sequence. $2/2^1, 3/2^2, 5/2^3, 8/2^4, 13/2^5, \ldots$ This occurs because we are doubling the numerator and denominator of the previous term in this sequence. We then subtract the term with the corresponding denominator from the above Fibonacci sequence sum. This is the same as subtracting the numerator of the third previous term in this sequence; another way of generating a Fibonacci sequence. $(a, b, a+b, a+2b)$ $(a, b, a+b, 2a+2b-a)$.

The denominators are increasing exponentially, while the numerators are increasing linearly. This sequence has a limit of zero as s approaches infinity. Proving the original sum whose numerators are a Fibonacci sequence has a limit of one as s approaches infinity. **Every possible binary series is realized, because their total proportion is *1 (100%)*.**

The terms of the form $24h+3$[2], $24h+11$[1], and $24h+19$[1] are the first terms in branch[2] or branch segments[1] with binary series of (1), $(1,2)$ and $(1,2,\ldots)$.

The proportion of the branch/branch segments of each of these three term types with binary series sums of $s=3,4,5,\ldots$ is $1/2^3 + 1/2^4 + 2/2^5 + 3/2^6 + 5/2^7 + \ldots = 1/2$ plus $1/2$ for $s=1$. **Every possible binary series is realized, because their total proportion is *1 (100%)*.**

Appendix 6. The proportion calculation of branch/branch segments

The formula for $24h+9$ is $3^{r-1}/2^{2r}$. $r=1,2,3,\ldots$ $1/4$[1]$+3/16+9/64+\ldots$ $(1/4$[1]$)/(1-3/4)=1$; the total proportion of $24h+9$ terms. For $r=1$ binary series (2) accounts for $1/4$[1] of the positive integers generated from $h=2^2 n+3$. The $r=2$ binary series $2, 1$ is generated from $h=2^3 n+6$. Generating the binary series $2, 2$ is $h=2^4 n+13$. Together they account for $3/16$ of all the positive integers. Similar linear formulas exist that sum to all $3^{r-1}/2^{2r}$. $r=1,2,3,\ldots$ r value proportions. If the total number of $3^{r-1}/2^{2r}$ formulas are $h=2^{sm} n+a_m$ $m=1,2,3\ldots$

where m is the total number of different binary series of length r, $3^{r-1}/2^{2r} = 1/2^{s1}+ 1/2^{s2}+1/2^{s3}+\ldots+1/2^{sm}$
For length $r+1$ $3^{r}/2^{2(r+1)}$ each of the existing binary series sums will be increased by ***1*** and by ***2.***
$3^{r}/2^{2(r+1)} = 1/2^{s1+1}+ 1/2^{s2+1}+1/2^{s3+1}+\ldots+1/2^{sm+1}+ 1/2^{s1+2}+ 1/2^{s2+2}+1/2^{s3+2}+\ldots+1/2^{sm+2}= (1/2)(3^{r-1}/2^{2r})+(1/2^{2})(3^{r-1}/2^{2r})$

Thanks for your interest in this paper. If you wish to make comments send them to Jim Rock at collatz3106@gmail.com**.**