The Gravitational Constant
[Calculating Big G]

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Abstract

The mathematical calculation of the Universal Constant of Gravitation has eluded physicists and mathematicians for in excess of three centuries The most recent issue being the exact definition of the force of Gravity and how it can be included in an all-encompassing theory taking into account quantum mechanics. The most challenging of all is a precise understanding of gravity in particular the Gravitational constant. Mathematical calculation of the Gravitational constant reveals interesting results offering compelling evidence that an exact value for the Gravitational constant can indeed be calculated.

Subject Headings: Nuclear physics, Fundamental Constants, Gravitational Constant.

§1. Introduction

In August 2018, a Chinese research group announced two measurements of the Gravitational constant being, $6.674\,184 \times 10^{-11}$and $6.674\,484 \times 10^{-11}$ both were made using upon torsion balances but using two different methods. These values are claimed to be the most accurate measurements of the Gravitational constant to date, with a standard uncertainties cited as low as 12 ppm. The difference of 2.7 sigma between the two results suggests there could be unaccounted sources of error in the results. It has long been claimed that no theory exists whereby the Gravitational constant can be calculated mathematically which has resulted in the actual value of this constant being one of the most inaccurate values in physics due to the finite limits of measurement. The author proposes that a mathematical model does indeed exist which enables the establishment of a more accurate value for the Gravitational constant. It will be shown that the Gravitational constant can indeed be calculated and the value correlated with other collective values of "natural constants" within the current accepted model. Clearly, when using an anthropomorphic numbering model, a small selection of representative base values must be used such as length, time, velocity etc. in order to yield meaningful results. The choice should be values that can reasonably be assumed to be universally applicable, for this reason the measurement of values associated with the properties of the most basic element the Hydrogen atom are used, that of mass and length plus the speed of light. This of course limits the overall accuracy of the calculated value of the Gravitational constant to that of the measured properties of the Hydrogen atom itself. In the interests of simplicity and consistency the values throughout are therefore shown only in base ten SI units.

§2. The Speed of Light [c]

The speed of light is a somewhat obvious and necessary choice in base units. Clearly the speed of light is a very well established and experimentally measured value which can also be derived from James Clerk Maxwell's publication "A Treatise on electricity and magnetism", the generally recognized value being;

$$c = 2.997\,924\,580 \times 10^{8}\,\mathrm{m \cdot s^{-1}} \qquad (2.00)$$

The SI value is stated as being an exact value, which may or may not be the case. If it assumed to be an exact value as shown above, then the uncertainty is zero.

§3. The Fine structure constant [α]

The exact meaning of the fine structure constant has many interpretations, the preferred being that of the original definition by Arnold Sommerfeld in 1916. The value represents the ratio between the orbital speed of the electron in the Hydrogen atom and the speed of light.

$$\alpha = \frac{v_o}{c} = \frac{2.187\,691\,263 \times 10^{6}\ \mathrm{m \cdot s}}{2.997\,924\ \ 580 \times 10^{8}\ \mathrm{m \cdot s}} \qquad (3.00)$$

As the dimensions of the numerator and the denominator are both meters per second the final value is clearly dimensionless. The current accepted value for the Fine structure being;

$$\alpha = 7.297\,352\,5693 \times 10^{-3} \qquad (3.01)$$

This calculated value is of course identical to the published value where the uncertainty of the currently recognized value for the fine structure constant is stated as being 1.1×10^{-12}. The actual value of the Fine Structure constant when relying upon the orbital velocity of the ground state electron of the Hydrogen atom agrees with published value. Its value correlates with many other methods of calculation suggesting that indeed the value is correct.

§4. The Mass of the Electron $[m_e]$

One of the measurements that is much simpler to obtain physically is that of the mass of the electron. The mass of the electron can be confirmed using other values within the current numerical model, including the use of Planck unit values which will be shown here. The currently accepted value for the mass of the electron to be used is the value currently published which is;

$$m_e = 9.109\ 383\ 7015 \times 10^{-31}\text{kg} \quad (4.00)$$

The uncertainty of this measurement is 2.8×10^{-10}which being a relatively high level of uncertainty is the upper limit of any calculations requiring this value. As stated, one of the values with which a correlation can be established is that of the Planck constant which uses three properties, the value of the electron mass, its orbital speed and orbital radius. The exact value of the electron mass can therefore be calculated using the following method;

$$m_e = \frac{\alpha m_p l_p}{r_e} \quad (4.01)$$

The parameters in the equation being, the fine structure constant, the Planck mass and the Planck length with the denominator being the electron radius. The value returned agrees exactly with the measured value for the mass of the electron;

$$m_e = 9.109\ 383\ 7015 \times 10^{-31}\text{kg} \quad (4.02)$$

§5. The Bohr radius $[a_0]$

The distance center to center between the proton and the ground state electron in the classical orbit of the Hydrogen atom is the Bohr radius. It's current published value being;

$$a_0 = 5.291\ 772\ 109\ 03 \times 10^{-11}\text{m} \quad (5.00)$$

According to current published values the uncertainty of its measurement is 1.5×10^{-10}.

§6. Planck Constant $[h]$

It is imperative the correct interpretation of the Planck constant is adhered to. It is no coincidence that the dimensions of the Planck constant are identical to angular momentum. It is made clear from the outset, that the Planck constant is absolutely <u>not</u> a fixed scalar value as is currently defined in the modern SI units, it is a constant of proportionality with a fixed ratio. This is not controversial within the physics community and it is elected to embrace this simple fact that the value using the properties of the ground state electron of the Hydrogen atom which can be shown to be;

$$h = 6.626\ 070\ 149\ 981 \times 10^{-34}\text{J} \cdot \text{Hz} \quad (6.00)$$

This exact value for the Planck constant requires only simple algebra, the equation being as follows;

$$h = 2\pi a_0 m_e v_o \quad (6.01)$$

Likewise the reduced Planck constant can be shown to be;

$$\hbar = a_0 m_e v_o = 1.054\ 571\ 817\ 643 \times 10^{-34}\text{J.s} \quad (6.02)$$

The uncertainty of the published values of the Planck and reduced Planck constant are currently assumed to be an exact value in SI units placing the value at;

$$h = 6.626\ 070\ 15 \times 10^{-34}\text{J} \cdot \text{Hz} \quad (6.03)$$

And

$$\hbar = 1.054\ 571\ 82 \times 10^{-34}\text{J.s} \quad (6.04)$$

The uncertainty of the component parts when calculating its value using the ground state electron of the hydrogen atom restricts this accuracy to the measurement of the electron mass at 3×10^{-10}. The result is that the calculated value of the Planck constant is indeed in agreement with the current published values.

§7. The Planck Length $[l_p]$

Having defined the Planck constant precisely, possibly the most important derivative Planck unit of all is that of the Planck length. The Planck length is best described as the distance that light travels in one unit of Planck time. There is nothing particularly special nor fundamental about the Planck constant or any the subsequent Planck units. It is notable however that the Gravitational constant is an integral part of most of these derived Planck units including the Planck length. From the values it is apparent that lacking an alternative method to calculate its value, the measured value of the Gravitational constant has been used to calculate the subsequent values of the Planck length the current published value being;

$$l_p = 1.616\ 255 \times 10^{-35}\text{m} \quad (7.00)$$

The uncertainty of the value of 1.1×10^{-5}m is limited by the uncertainty of the measured value of the Gravitational constant itself. It is often considered that Avogadro's number can be used to establish the amount of discrete elements that exist within a given item. Although often used in the context of mass on the sub atomic scale it is not generally thought that this may equally apply to length. If indeed this were the case then the number of Planck units, that subdivides the distance between the electron and proton can be calculated using Avogadro's constant and Euler's number, what could be classed somewhat colloquially as the Planck molar length;

$$\frac{1}{2N_A e} = 3.273\ 975\ 159\text{E} \times 10^{24} \tag{7.01}$$

Consequently, if the current value of the Bohr radius is divided by this number of discrete elements the resultant value number should indeed represent the Planck length;

$$l_p = \frac{a_0}{2N_A e} = 1.616\ 314\ 067 \times 10^{-35}\text{m} \tag{7.02}$$

The result is as anticipated and in exact agreement with current published values for the Planck length.

§8. The Gravitational Constant [G]

The first method to ensure that indeed all subsequent equations are correct is to establish a base value for the Gravitational constant by using dimensional analysis, the dimensions of the Gravitational constant being;

$$G = [\text{M}^{-1}\ \text{L}^3\ \text{T}^{-2}] \tag{8.00}$$

It follows that, this is nothing more than the ubiquitous definition of the Gravitational constant.

$$G = \frac{r^3}{MT^2} \tag{8.01}$$

Substitution of the current values of the pertinent Planck units of length, mass and time into the equation results in;

$$\frac{{l_p}^3}{m_p t_p^2} = 6.674\ 787\ 6456 \times 10^{-11} \tag{8.02}$$

In order to validate that this calculation is indeed correct, a second method, this time without using Planck units, will show a direct correlation with the latter calculated value of the Universal Gravitational constant. This method involves the combination of two equivalent equations which both yield the Planck length. The first of these equations is the instantly recognizable equation for the Planck length which itself contains Newton's constant of gravitation;

$$l_P = \sqrt{\frac{\hbar G}{c^3}} = 1.616\ 314\ 067 \times 10^{-35}\ \text{m} \tag{8.05}$$

The second equation to be used which also returns an identical value for the Planck length, this time using the Bohr radius, as shown previously;

$$l_P = \frac{a_0}{2N_A e} = 1.616\ 314\ 067 \times 10^{-35}\ \text{m} \tag{8.06}$$

It follows that as these two equations both returning an identical value for the Planck length can be considered equivalent implying that the following must also be correct;

$$\frac{a_0}{2N_A e} = \sqrt{\frac{\hbar G}{c^3}} \tag{8.07}$$

Simplification and subsequent rearrangement of the above, results in an equation that returns the value of Newton's Universal Gravitational constant. In this particular case without reference to the Planck constant itself or indeed any other Planck units;

$$G = \frac{c^3 a_0}{4 m_e v_o {N_A}^2 e^2} \tag{8.08}$$

It follows, that if the currently published values are used in this equation, once more it results in an identical value for the Gravitational constant;

$$6.674\ 787\ 6456 \times 10^{-11}\ \text{m}^3\text{kg}^{-1}\text{s}^{-2} \tag{8.09}$$

Summary

Calculations have been made with and without Planck units and the returned results are identical in both cases. The values used in the equations are shown in the table in the appendix which result in a value of the Universal Gravitational constant to an unprecedented accuracy, offering a precision with an uncertainty limited only by the uncertainty of the electron mass of 3.0×10^{-10} which is significantly greater than any current physical measurements made to date.

Appendix

Symbol	Value	Uncertainty
a_0	$5.291\,772\,109 \times 10^{-11}$	1.5×10^{-10}
m_e	$9.109\,383\,7015 \times 10^{-31}$	3.0×10^{-10}
α	$7.297\,352\,5693 \times 10^{-3}$	1.5×10^{-10}
c	$2.997\,924\,58 \times 10^{8}$	Exact
N_A	$6.022\,140\,762 \times 10^{23}$	Exact
e	$2.718\,281\,828\,459$	Exact