Schrödinger uncertainty relation which depends on the quantum transition

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Abstract

We review strictly the Schrödinger uncertainty relation. And we newly expand the formula of depending on the quantum transition. Based on the formula, the optimal upper limit of the Schrödinger uncertainty relation when we would measure simultaneously σ_x and σ_y in a two-level system (e.g., electron spin, photon polarizations, and so on) is reviewed. We show the optimal lower bound of the Schrödinger uncertainty relation is exactly zero if the two observables are commutative and a quantum state under study is a simultaneous eigenstate for the two observables. It turns out that the Schrödinger uncertainty relation is a very fundamental formula from the origin of the optimal upper limit (Bloch sphere) and the meaningful optimal lower limit (exactly zero).

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I. INTRODUCTION

Quantum mechanics (cf. [1–7]) gives accurate and at-times-remarkably accurate numerical predictions and much experimental data has fit to quantum predictions for long time.

As for foundations of quantum mechanics, Leggett-type non-local variables theory [8] is experimentally investigated [9–11]. The experiments report that quantum mechanics does not accept Leggett-type non-local variables interpretation.

As for applications of quantum mechanics [5–7], the implementation of a quantum algorithm to solve Deutsch's problem [12] on a nuclear magnetic resonance quantum computer is reported firstly [13]. An implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer is also reported [14]. There are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implement Deutsch's algorithm with polarization and transverse spatial modes of the electromagnetic field as qubits [15]. Single-photon Bell states are prepared and measured [16]. Also the decoherence-free implementation of Deutsch's algorithm is reported by using such single-photon and by using two logical qubits [17]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [18].

In quantum mechanics, the uncertainty principle is any of the variety of mathematical inequalities asserting a fundamental limit to the precision with which certain pairs of physical properties of a particle known as complementary variables, such as its position x and momentum p, can be known simultaneously. For instance, in 1927, Werner Heisenberg stated that the more precisely the position of some particle is determined, the less precisely its momentum can be known, and vice versa [19]. The formal inequality relating the standard deviation of position σ_x and the standard deviation of momentum σ_p was derived by Earle Hesse Kennard [20] later that year and by Hermann Weyl [21] in 1928.

Maccone and Pati discuss stronger uncertainty relations for all incompatible observables [22]. Quantum dynamics of simultaneously measured non-commuting observables is discussed [23]. Dynamics of a qubit while simultaneously monitoring its relaxation and dephasing are also discussed [24]

Recently, a universally valid reformulation of the Heisenberg uncertainty principle on noise and disturbance in measurement is discussed by Ozawa [25]. And an experimental demonstration of a universally valid error-disturbance uncertainty relation in spin measurements is discussed [26]. A violation of Heisenberg's error-disturbance uncertainty relation in neutron-spin measurements is also discussed [27].

The optimal upper limitation of the Schrödinger uncertainty relation in a two-level system (e.g., electron spin, photon polarizations, and so on) is proposed by Nagata and Nakamura [28]. The optimality is certified by the Bloch sphere when we would measure simultaneously σ_x and σ_y.

What is the motivation behind this work to be discussed in this paper? Concretely speaking, we want to know the most fundamental uncertainty relation. It turns out that the Schrödinger uncertainty relation is a very fundamental formula from the origin of the optimal upper limit (Bloch sphere) and the meaningful optimal lower limit (exactly zero).

In this paper, we review strictly the Schrödinger uncertainty relation. And we newly expand the formula of depending on the quantum transition. Based on the formula, the optimal upper limit of the Schrödinger uncertainty relation when we would measure simultaneously σ_x and σ_y in a two-level system (e.g., electron spin, photon polarizations, and so on) is reviewed. We show the optimal lower bound of the Schrödinger uncertainty relation is exactly zero if the two observables are commutative and a quantum state under study is a simultaneous eigenstate for the two observables. It turns out that the Schrödinger uncertainty relation is a very fundamental formula from the origin of the optimal upper limit (Bloch sphere) and the meaningful optimal lower limit (exactly zero).

II. SCHRÖDINGER UNCERTAINTY RELATION WHICH DEPENDS ON THE QUANTUM TRANSITION

In this section, we derive the rigorous Schrödinger uncertainty relation. And we newly expand it of depending on the quantum transition. Here, we introduce the notation t which relates the quantum transition probability. The quantum transition probability between a quantum state $|\Psi(t)\rangle$ and a quantum state $|\Psi(t')\rangle$ is given by $|\langle\Psi(t)|\Psi(t')\rangle|^2$.

Parts of this derivation shown here incorporate and build off of those shown in Robertson [29], Schrödinger [30], and standard textbooks such as Griffiths [31]. As for the derivation of the Schrödinger uncertainty relation, the main point is the Cauchy-Schwarz inequality [32] as we show below.

For any Hermitian operator $\hat{A}$, based upon the definition of variance, we have

$$\sigma_A^2(t) = \langle(\hat{A} - \langle\hat{A}\rangle(t))\Psi(t)|(\hat{A} - \langle\hat{A}\rangle(t))\Psi(t)\rangle, \tag{1}$$

where $\langle\hat{A}\rangle(t)=\langle\Psi(t)|\hat{A}|\Psi(t)\rangle$. We let $|f(t)\rangle=|(\hat{A}-\langle\hat{A}\rangle(t))\Psi(t)\rangle$ and thus

$$\sigma_A^2(t)=\langle f(t)|f(t)\rangle. \tag{2}$$

Similarly, for any other Hermitian operator $\hat{B}$ in the state $|\Psi(t')\rangle$

$$\sigma_B^2(t')=\langle(\hat{B}-\langle\hat{B}\rangle(t'))\Psi(t')|(\hat{B}-\langle\hat{B}\rangle(t'))\Psi(t')\rangle=\langle g(t')|g(t')\rangle, \tag{3}$$

for $|g(t')\rangle=|(\hat{B}-\langle\hat{B}\rangle(t'))\Psi(t')\rangle$ and $\langle\hat{B}\rangle(t')=\langle\Psi(t')|\hat{B}|\Psi(t')\rangle$. Thus, the product of the two variances can be expressed as

$$\sigma_A^2(t)\sigma_B^2(t')=\langle f(t)|f(t)\rangle\langle g(t')|g(t')\rangle. \tag{4}$$

In order to relate the two vectors $|f(t)\rangle$ and $|g(t')\rangle$ with each other, we use the Cauchy-Schwarz inequality [32] which is defined as

$$\langle f(t)|f(t)\rangle\langle g(t')|g(t')\rangle\geq|\langle f(t)|g(t')\rangle|^2, \tag{5}$$

and thus Eq. (4) can be written as

$$\sigma_A^2(t)\sigma_B^2(t')\geq|\langle f(t)|g(t')\rangle|^2. \tag{6}$$

Since $\langle f(t)|g(t')\rangle$ is generally a complex number, we use the fact that the modulus squared of any complex number z is defined as $|z|^2=zz^*$, where z^* is the complex conjugate of z. The modulus squared can also be expressed as

$$|z|^2=(\mathrm{Re}(z))^2+(\mathrm{Im}(z))^2=\left(\frac{z+z^*}{2}\right)^2+\left(\frac{z-z^*}{2i}\right)^2. \tag{7}$$

We let $z=\langle f(t)|g(t')\rangle$ and $z^*=\langle g(t')|f(t)\rangle$ and substitute these into the equation above in giving

$$|\langle f(t)|g(t')\rangle|^2=\left(\frac{\langle f(t)|g(t')\rangle+\langle g(t')|f(t)\rangle}{2}\right)^2+\left(\frac{\langle f(t)|g(t')\rangle-\langle g(t')|f(t)\rangle}{2i}\right)^2. \tag{8}$$

The inner product $\langle f(t)|g(t')\rangle$ is written out explicitly as

$$\langle f(t)|g(t')\rangle=\langle(\hat{A}-\langle\hat{A}\rangle(t))\Psi(t)|(\hat{B}-\langle\hat{B}\rangle(t'))\Psi(t')\rangle, \tag{9}$$

and using the fact that $\hat{A}$ and $\hat{B}$ are Hermitian operators, we find

$$\begin{aligned}
\langle f(t)|g(t')\rangle &= \langle\Psi(t)|(\hat{A}-\langle\hat{A}\rangle(t))(\hat{B}-\langle\hat{B}\rangle(t'))\Psi(t')\rangle \\
&= \langle\Psi(t)|(\hat{A}\hat{B}-\hat{A}\langle\hat{B}\rangle(t')-\hat{B}\langle\hat{A}\rangle(t)+\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t'))\Psi(t')\rangle \\
&= \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle-\langle\Psi(t)|\hat{A}\langle\hat{B}\rangle(t')\Psi(t')\rangle-\langle\Psi(t)|\hat{B}\langle\hat{A}\rangle(t)\Psi(t')\rangle+\langle\Psi(t)|\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\Psi(t')\rangle \\
&= \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle-\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle-\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle+\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle \\
&= \langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle-\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle.
\end{aligned} \tag{10}$$

Similarly, it can be shown that $\langle g(t')|f(t)\rangle=\langle\Psi(t)|\hat{B}\hat{A}\Psi(t')\rangle-\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle$. For a pair of operators $\hat{A}$ and $\hat{B}$, we may define their commutator as $[\hat{A},\hat{B}]=\hat{A}\hat{B}-\hat{B}\hat{A}$. Thus we have

$$\begin{aligned}
&\langle f(t)|g(t')\rangle-\langle g(t')|f(t)\rangle=\langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle-\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle \\
&-\langle\Psi(t)|\hat{B}\hat{A}\Psi(t')\rangle+\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle \\
&=\langle\Psi(t)|[\hat{A},\hat{B}]|\Psi(t')\rangle
\end{aligned} \tag{11}$$

and

$$\begin{aligned}
&\langle f(t)|g(t')\rangle+\langle g(t')|f(t)\rangle=\langle\Psi(t)|\hat{A}\hat{B}\Psi(t')\rangle-\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle \\
&+\langle\Psi(t)|\hat{B}\hat{A}\Psi(t')\rangle-\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle \\
&=\langle\Psi(t)|\{\hat{A},\hat{B}\}|\Psi(t')\rangle-2\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle,
\end{aligned} \tag{12}$$

where we have introduced the anticommutator $\{\hat{A},\hat{B}\}=\hat{A}\hat{B}+\hat{B}\hat{A}$. We now substitute the above two equations into Eq. (8) in giving

$$|\langle f(t)|g(t')\rangle|^2=\left(\frac{1}{2}\langle\Psi(t)|\{\hat{A},\hat{B}\}|\Psi(t')\rangle-\langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle\right)^2+\left(\frac{1}{2i}\langle\Psi(t)|[\hat{A},\hat{B}]|\Psi(t')\rangle\right)^2. \tag{13}$$

Substituting the above into Eq. (6), we get the Schrödinger uncertainty relation which depends on the quantum transition,

$$\sigma_A(t)\sigma_B(t') \geq \sqrt{\left(\frac{1}{2}\langle\Psi(t)|\{\hat{A},\hat{B}\}|\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi(t)|[\hat{A},\hat{B}]|\Psi(t')\rangle\right)^2}. \tag{14}$$

III. REVIEW OF THE OPTIMAL UPPER LIMIT OF THE SCHRÖDINGER UNCERTAINTY RELATION

In this section, we review [28] an example that the Schrödinger uncertainty relation is optimal in the case where $t = t'$. The optimality is certified by the Bloch sphere. In fact, a violation of the Schrödinger uncertainty relation is equivalent to a violation of the Bloch sphere in a specific case. We re-derive the Schrödinger uncertainty relation by using the Bloch sphere in the specific case. Let $V(X)$ be the variance of X, i.e., $\langle X^2\rangle - \langle X\rangle^2$.

Statement 1

$$\sqrt{V(\sigma_x)V(\sigma_y)} \geq \sqrt{\left(\frac{1}{2i}\langle[\sigma_x,\sigma_y]\rangle\right)^2 + \langle\sigma_x\rangle^2\langle\sigma_y\rangle^2}. \tag{15}$$

Proof. By using $1 - \langle\sigma_x\rangle^2 - \langle\sigma_y\rangle^2 \geq \langle\sigma_z\rangle^2$, we have

$$\begin{aligned}&V(\sigma_x)V(\sigma_y) = (1-\langle\sigma_x\rangle^2)(1-\langle\sigma_y\rangle^2) = 1-\langle\sigma_x\rangle^2-\langle\sigma_y\rangle^2+\langle\sigma_x\rangle^2\langle\sigma_y\rangle^2 \geq \langle\sigma_z\rangle^2+\langle\sigma_x\rangle^2\langle\sigma_y\rangle^2\\ &= \left(\frac{1}{2i}\langle[\sigma_x,\sigma_y]\rangle\right)^2 + \langle\sigma_x\rangle^2\langle\sigma_y\rangle^2.\end{aligned} \tag{16}$$

Thus,

$$\sqrt{V(\sigma_x)V(\sigma_y)} \geq \sqrt{\left(\frac{1}{2i}\langle[\sigma_x,\sigma_y]\rangle\right)^2 + \langle\sigma_x\rangle^2\langle\sigma_y\rangle^2}. \tag{17}$$

QED

We define B as follows:

$$B := \left(\frac{1}{2i}\langle[\sigma_x,\sigma_y]\rangle\right)^2 + \langle\sigma_x\rangle^2\langle\sigma_y\rangle^2. \tag{18}$$

We define S as follows:

$$S := \overbrace{\left(\frac{1}{2}\langle\{\sigma_x,\sigma_y\}\rangle - \langle\sigma_x\rangle\langle\sigma_y\rangle\right)^2 + \left(\frac{1}{2i}\langle[\sigma_x,\sigma_y]\rangle\right)^2}^{\text{Schrodinger term}}. \tag{19}$$

We show the relation between B and S in the following statement:

Statement 2

$$B = S. \tag{20}$$

Proof. We have the following relation:

$$\langle\{\sigma_x,\sigma_y\}\rangle = 0. \tag{21}$$

QED

Thus the Schrödinger uncertainty relation is optimal in the specific case. The optimality is certified by the Bloch sphere. A violation of the Schrödinger uncertainty relation is equivalent to a violation of the Bloch sphere in the specific physical situation. It turns out that the Schrödinger uncertainty relation is a very fundamental formula from the origin of the optimal upper limit (Bloch sphere).

IV. SIMULTANEOUS MEASUREMENTS ON COMMUTING OBSERVABLES

We suppose that $\hat{A}, \hat{B}$ are two Hermitian operators on an N-dimensional unitary space. Let us consider a simultaneous pure eigenstate $|\Psi_i\rangle, (i = 1, 2, ..., N)$, that is, $\langle\Psi_i|\Psi_j\rangle = \delta_{ij}$, for the two Hermitian operators $\hat{A}, \hat{B}$ such that $\langle\Psi_i|\hat{A}|\Psi_i\rangle = a_i, \langle\Psi_i|\hat{B}|\Psi_i\rangle = b_i$.

We have the Schrödinger uncertainty relation which depends on the quantum transition

$$\sigma_A(t)\sigma_B(t') \geq \sqrt{\left(\frac{1}{2}\langle\Psi(t)|\{\hat{A}, \hat{B}\}|\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\langle\Psi(t)|\Psi(t')\rangle\right)^2 + \left(\frac{1}{2i}\langle\Psi(t)|[\hat{A}, \hat{B}]|\Psi(t')\rangle\right)^2}. \tag{22}$$

The Schrödinger uncertainty relation becomes as follows in the case where $\langle\Psi(t)|\Psi(t')\rangle = \delta_{tt'}$:

$$\sigma_A(t)\sigma_B(t') \geq \sqrt{\left(\frac{1}{2}\langle\Psi(t)|\{\hat{A}, \hat{B}\}|\Psi(t')\rangle - \langle\hat{A}\rangle(t)\langle\hat{B}\rangle(t')\delta_{tt'}\right)^2 + \left(\frac{1}{2i}\langle\Psi(t)|[\hat{A}, \hat{B}]|\Psi(t')\rangle\right)^2}. \tag{23}$$

Statement 3

When $[\hat{A}, \hat{B}] = 0$, the Schrödinger uncertainty relation becomes

$$\sigma_A\sigma_B \geq \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle, \tag{24}$$

and the optimal lower bound is zero.

Proof. We consider the Schrödinger uncertainty relation in the case where $[\hat{A}, \hat{B}] = 0$

$$\sigma_A\sigma_B \geq \sqrt{\left(\frac{1}{2}\langle\{\hat{A}, \hat{B}\}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle\right)^2}. \tag{25}$$

Thus, we have

$$\sigma_A\sigma_B \geq \langle\hat{A}\hat{B}\rangle - \langle\hat{A}\rangle\langle\hat{B}\rangle. \tag{26}$$

On the other hand, we have

$$\begin{aligned} &\langle\Psi_i|\hat{A}\hat{B}|\Psi_i\rangle = a_i b_i, \\ &\langle\Psi_i|\hat{A}|\Psi_i\rangle\langle\Psi_i|\hat{B}|\Psi_i\rangle = a_i b_i, \end{aligned} \tag{27}$$

where $[\hat{A}, \hat{B}] = 0$ and a_i, b_i are respectively the real numbers of the diagonal elements of the two Hermitian operators $\hat{A}, \hat{B}$.

QED

We show that the optimal lower bound of the Schrödinger uncertainty relation is exactly zero if the two observables are commutative. It turns out that the Schrödinger uncertainty relation says a precise measurement on commuting observables is possible. Let us consider one observable case as follows: If $A = B$ and **Statement 3**, then

$$\sigma_A^2 \geq \langle\hat{A}^2\rangle - \langle\hat{A}\rangle^2 = 0, \tag{28}$$

where we use a pure eigenstate $|\Psi_i\rangle$ of $\hat{A}$. Thus, it turns out that the Schrödinger uncertainty relation says a precise measurement on one observable is possible. It turns out that the Schrödinger uncertainty relation is a very fundamental formula from the meaningful optimal lower limit (exactly zero).

V. CONCLUSIONS

In conclusions, we have reviewed strictly the Schrödinger uncertainty relation. We have newly expanded the formula of depending on the quantum transition. Based on the formula, the optimal upper limit of the Schrödinger uncertainty relation when we would measure simultaneously σ_x and σ_y in a two-level system (e.g., electron spin, photon polarizations, and so on) has been reviewed. We have shown the optimal lower bound of the Schrödinger uncertainty relation is exactly zero if the two observables are commutative and a quantum state under study is a simultaneous eigenstate for the two observables. It has turned out that the Schrödinger uncertainty relation is a very fundamental formula from the origin of the optimal upper limit (Bloch sphere) and the meaningful optimal lower limit (exactly zero).

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DECLARATIONS

Ethical Approval

The authors are in an applicable thought to Ethical Approval.

Competing Interests

The authors state that there is no conflict of interest.

Authors' Contributions

Koji Nagata, Do Ngoc Diep, and Tadao Nakamura wrote and read the manuscript.

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