The General Hohmann Transfer

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An analytical method is presented for tangent transfers (Hohman type transfers) between non-coaxial elliptical orbits. Since Hohmann transfers are thought not to apply to non-coaxial orbits, this method generalizes the Hohmann transfer, typically used only between circular orbits. Since tangent transfers are less complex, as they require no change in direction, they offer an alternative to other orbital transfer and rendezvous methods.

1 Introduction

A profound pioneering effect for space travel came in 1925, when Walter Hohmann published his book, "The Attainability of Heavenly Bodies" [1]. The book describes many aspects of space travel including liftoffs and landings, passenger considerations and destinations. More importantly, he describes the route taken to these destinations. The route taken from Earth to Venus, say, is an elliptical path, connecting the orbit of Earth around the sun, to the orbit of Venus around the sun, i.e. connecting circular orbits. The points of connection require, in the words of Walter Hohmann, "changes in velocity, but no changes in direction".

Since then, Hohmann transfers have been considered to apply only to circular orbits or coaxial elliptical orbits (see [2, chap. 6], for example). Here, we generalize Hohmann transfers between any two orbits in a plane, coaxial or not - offering an alternative procedure for orbital transfers and rendezvous. I am not an orbital mechanic, but I can speculate those versed in the skill, calculate transfers to minimize fuel - not position. In this procedure, a transfer or rendezvous is always tangent to the target. This means no maneuvering - so more reliable. And you never cross the orbit of the target - so safer. Just pull up to the target and apply the brakes, so to speak. As activities in space increase (space tourism, for instance), these advantages may be of interest to an orbital mechanic at a minimal increase in fuel (≈1.0%).

2 The Hohmann Transfer

A typical Hohmann transfer is shown in Fig. 1. An elliptical orbit is generated connecting the initial and final orbits through an impulse velocity at point A. Then, a circular orbit is generated through an impulse velocity at point B. At the apse points of the transfer ellipse, the velocities are tangent to the paths of the orbits. The procedure is listed in the appendix.

In contrast, two same plane non-coaxial orbits are shown in Fig. 2. The usual Hohmann transfer will not work. Any generated intermediate ellipse will not be tangent at the point on the target orbit that crosses the apse line of the initial orbit.

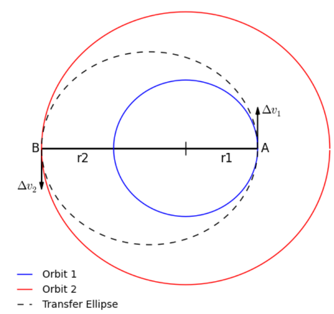

Fig. 1: Hohmann transfer example.

In Table 1 we define parameters for the initial and target orbits, designated by the subscripts 1 and 2 respectively, used to generate the plot in Fig. 2. The angle, θ_0, is the rotation of the target orbit apse line relative to the initial orbit apse line. The data will be used later to calculate the transfer.

3 General Transfer Method

There are at least two angles (or true anomalies) that designate points where the flight path angles of two orbits are equal, i.e. the orbits are tangent to each other along the radial line at these angles. If an (transfer) ellipse coincides with these points, then two orbital equations for the known distances and a third equation for the known flight angles would determine the three unknowns of the ellipse: the eccentricity (e), the angular momentum (h) and the degree of apse line rotation (ϕ) relative to the initial orbit.

3.1 The Tangent Points

Tangent points between any two orbits with a common focus can be found analytically by equating the flight path angles as follows:

$$\tan(\gamma_1) = \tan(\gamma_2) \tag{1}$$

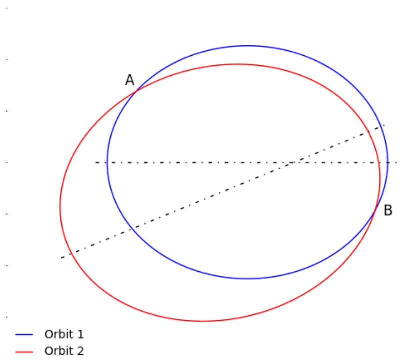

Fig. 2: Two non-coaxial orbits.

Table 1: Orbital definitions used in Fig. 2.

Orbit	Orbital Parameters	Values
1	e_1	1/3.0
	r_{1p} [km]	8000.0
	r_{1a} [km]	16000.0
	h_1 [km]	$\sqrt{r_{1p} u(1+e_1)}$
2	e_2	1/2.0
	r_{2p} [km]	7000.0
	r_{2a} [km]	21000.0
	h_2 [km]	$\sqrt{r_{2p} u(1+e_2)}$
	θ_0 [rad]	25.0 $(\pi/180)$

where the flight path angle, γ, is the angle between the orbiting body's velocity vector (the vector tangent to the instantaneous orbit) and the local horizontal, the line perpendicular to radial line passing through the object in orbit, as shown in Fig. 3.

The equations for the flight path angles can be written

$$e_1 \frac{\sin(\theta)}{(1+e_1\cos(\theta))} = e_2 \frac{\sin(\theta-\theta_0)}{(1+e_2\cos(\theta-\theta_0))} \tag{2}$$

resulting in

$$a\cos(\theta) + b\sin(\theta) = c \tag{3}$$

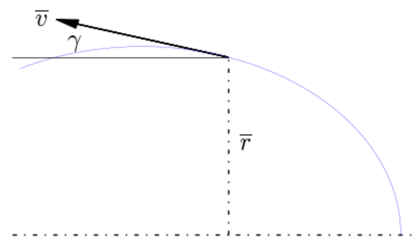

Fig. 3: Flight Path Angle (γ)

where

$$\begin{aligned} e &= \text{eccentricity} \\ \theta &= \text{true anomaly} \\ \theta_0 &= \text{rotation of target apse line} \\ a &= e_2\sin(\theta_0) \\ b &= (e_1 - e_2\cos(\theta_0)) \\ c &= e_1 e_2 \sin(\theta_0) \end{aligned}$$

and we finally solve for two angles where the two orbits are tangent (77° and 224° in Fig. 4 below).

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) \pm \cos^{-1}\left(c \quad \frac{\cos\left(\tan^{-1}\left(\frac{b}{a}\right)\right)}{a}\right). \tag{4}$$

3.2 Analytic Derivation

For convenience, we reference the initial, target and transfer orbits, orbits 1,2 and 3, respectively. We also reference the two angles (true anomalies) where the flight path angles are equal, θ_1 and θ_2. The distances of orbits 1 and orbit 2 are known at these angles. Since orbit 3 is tangent and touching both of these points, we have $r_3(\theta_1) = r_1(\theta_1)$ and $r_3(\theta_2) = r_2(\theta_2)$. This gives us two of the three equations needed to solve for orbit 3:

$$r_3(\theta_1) = \frac{h_3^2}{\mu(1+e_3\cos(\theta_1-\phi))} \tag{5}$$

and

$$r_3(\theta_2) = \frac{h_3^2}{\mu(1+e_3\cos(\theta_2-\phi))} \tag{6}$$

The third equation is obtained by equating flight path angles. First, the flight path angles, γ's, at θ_1 are equal and known, and the flight path angles at θ_2 are equal and known. Second, a simplification can be obtained by recognizing that the flight path angles at θ_1 are equal and opposite to the flight path angles at θ_2. For example, in Fig. 1, the flight path angles at $\theta = 0°$ are equal and opposite to the flight path angles at $\theta =$

180°. This is not obvious between non-coaxial orbits. However, it can be demonstrated as a general property by rotating orbit 2 relative to orbit 1 and evaluating the flight angles at various angular displacements. This was done at 45^o intervals and tabulated in Table 2 from the data in Table 1 .

Table 2: Flight angles (γ) at various rotations are equal and opposite. (in Deg.)

Apse Line Rotation	θ_1	θ_2	γ_1	γ_2
45	106.16	247.28	20.21	-20.21
90	139.79	287.58	16.53	-16.53
135	161.53	323.98	8.84	-8.84
180	180	0	0	0
225	36.01	198.46	8.84	-8.84
270	72.41	220.20	16.53	-16.53
315	112.71	253.83	20.21	-20.21

The flight angle equation (see equation 2) for orbit 3, at the appropriate true anomalies, θ_1 and θ_2, can then be placed in the following relation:

$$e_3 \frac{\sin(\theta_1 - \phi)}{(1 + e_3 \cos(\theta_1 - \phi))} = (-)e_3 \frac{\sin(\theta_2 - \phi)}{(1 + e_3 \cos(\theta_2 - \phi))} \tag{7}$$

Using equations 5 and 6 gives us:

$$e_3 \frac{r_3(\theta_1)\mu \sin(\theta_1 - \phi)}{h_3^2} = (-)e_3 \frac{r_3(\theta_2)\mu \sin(\theta_2 - \phi)}{h_3^2} \tag{8}$$

To solve for ϕ, rearrange Eq. 8 using a trig identity and A $= (-)r_3(\theta_2)/r_3(\theta_1)$:

$$\tan(\phi) = \frac{A \sin(\theta_2) - \sin(\theta_1)}{A \cos(\theta_2) - \cos(\theta_1)} \tag{9}$$

Using equations 5 and 6 and the definition of A above:

$$e_3 = \frac{(-)(A + 1)}{\cos(\theta_1 - \phi) + A \cos(\theta_2 - \phi)} \tag{10}$$

$$h_3 = \sqrt{r_3(\theta_1)\mu(1 + e_3 \cos(\theta_1 - \phi))} \tag{11}$$

The transfer ellipse, plotted in Fig. 4, is now fully defined and represented by:

$$r_3(\theta) = \frac{h_3^2}{\mu(1 + e_3 \cos(\theta - \phi))} \tag{12}$$

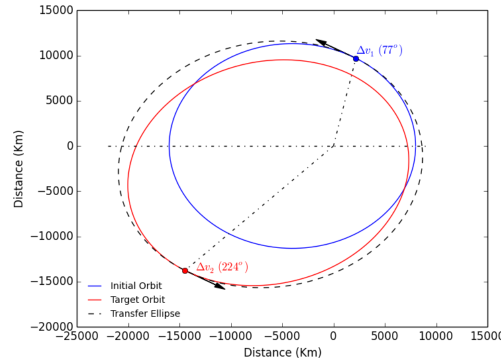

Fig. 4: Analytically generated transfer ellipse

4 Conclusion

This procedure generalizes the Hohmann transfer. Similar tangent transfers can be generated numerically at most points on the initial orbit where they exist (for example, a transfer would not exist at points of intersection). Numerically we can demonstrate the analytic method shown here is the lowest energy transfer of these types.

5 Appendix

Calculation of Hohmann transfers for circular orbits, as shown in Fig. 1, is straightforward. In a circular orbit, the centrifugal force is equal to the gravitational force, and the velocity can be obtained from the relation

$$\frac{v^2}{r} = \frac{GM}{r^2} \quad \text{or} \quad v = \sqrt{\frac{GM}{r}} \tag{13}$$

where

r is the radius of the orbit

v is the velocity of the object in orbit

G is the gravitational constant

M is the mass of the primary

In an elliptical orbit, motion is governed by the equation

$$r(\theta) = \frac{h^2}{GM(1 + e\cos(\theta))} \tag{14}$$

where

θ is the angle of the orbit (true anomaly)

e is the eccentricity, $(r_2 - r_1)/(r_2 + r_1)$

h is the specific angular momentum

At periapsis, the angle is zero, and the velocity, with no radial component, can be determined by

$$v_{ellipse@A} = \sqrt{\frac{GM(1+e)}{r_1}} \tag{15}$$

At apoapsis, since the angular momentum of an orbit around a central force is a constant, the velocity can be determined by

$$v_{ellipse@B} = r_1 \frac{v_{ellipse@A}}{r_2} \tag{16}$$

The impulsive thrusts are then calculated as follows:

$$\Delta v_1 = v_{ellipse@A} - v_1 \tag{17}$$

$$\Delta v_2 = v_2 - v_{ellipse@B} \tag{18}$$

$$\Delta v_{Total} = \Delta v_1 + \Delta v_2 \tag{19}$$

References

[1] Walter Hohmann, *Die Erreichbarheit der Himmelskorper*, Oldenbourg, Munich 1925

[2] Howard D Curtis, *Orbital Mechanics for Engineering Students*, Elsevier, 2005.

[3] Battin, R., *An Introduction to the Mathematics and Methods of Astrodynamics*, AIAA Education Series, AIAA, New York, 1987.