COMPLEX CIRCLES OF PARTITION AND THE ASYMPTOTIC LEMOINE CONJECTURE

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Abstract. Using the methods of the complex circles of partition (cCoPs), we study *interior* and *exterior* points of such structures in the complex plane. With simitarities to *quotient groups* inside of the group theory we define *quotient cCoPs*. With it we can prove an asymptotic version of the **Lemoine Conjecture**.

1. Introduction and Preliminaries

Lemoine's conjecture is the assertion that every odd number greater than 5 can be written as the sum of a prime number and a double of a prime number. More formally, the conjecture states

Conjecture 1.1. *The equation*

$$2n + 1 = p + 2q$$

always has a solution in the primes (not necessarily distinct) for all $n \geq 2$.

The conjecture was first posed by Émile Lemoine [1] in 1895 but was wrongly attributed to Hyman Levy [2], who had thought very deeply about it; hence, the name Lemoine or sometimes Levy conjecture. The conjecture is on par with other additive prime number problems like the binary Goldbach conjecture (see [3],[4],[5]) and the ternary Goldbach conjecture (see [6]). It is easy to see that the Lemoine conjecture is much stronger than and implies the ternary Goldbach conjecture.
We devised a method that we believe could be a useful tool and a recipe for analyzing issues pertaining to the partition of numbers in designated subsets of $\mathbb{N}$ in our work [7], which was partially inspired by the binary Goldbach conjecture and its variants. The technique is fairly simple, and it is similar to how the points on a geometric circle can be arranged. In [8], we have improved this strategy by switching from integer base sets to special complex number subsets. As a result, the *complex circle of partition* structure was defined (cCoP). The interior and exterior points of cCoPs as well as various applications are now introduced as we continue this work.

In an effort to make our work more self-contained, we have chosen to provide a little background of the method of circles of partition in the following sequel

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Definition 1.2. Let $n \in \mathbb{N}$ and $\mathbb{M} \subseteq \mathbb{N}$. We denote with

$$\mathcal{C}(n,\mathbb{M}) = \{[x] \mid x, y \in \mathbb{M}, n = x + y\}$$

the **Circle of Partition** generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n,\mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$. We denote with $\|[x]\| := x$ the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n,\mathbb{M})$ as $\|\mathcal{C}(n,\mathbb{M})\|$. Obviously holds

$$\|\mathcal{C}(n)\| = \{1, 2, \dots, n-1\}.$$

Definition 1.3. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n,\mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n,\mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then we call it as a **degenerated axis** $\mathbb{L}_{[x]}$ in comparison to the **real axes** $\mathbb{L}_{[x],[y]}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}), \text{ which means } [x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ with } x + y = n.$$

Important properties of CoPs are

- Each axis is uniquely determined by points $[x] \in \mathcal{C}(n,\mathbb{M})$.
- Each point of a CoP $\mathcal{C}(n,\mathbb{M})$ except its center has exactly one axis partner.

We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ resp. $\mathbb{L}_{[x]}$ to a CoP $\mathcal{C}(n,\mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}), \text{ which means } [x],[y] \in \mathcal{C}(n,\mathbb{M}) \text{ and } x + y = n \text{ resp.}$$
$$\mathbb{L}_{[x]} \hat{\in} \mathcal{C}(n,\mathbb{M}), \text{ which means } [x] \in \mathcal{C}(n,\mathbb{M}) \text{ and } 2x = n$$

and the number of real axes of a CoP as

$$\nu(n,\mathbb{M}) := \#\{\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n,\mathbb{M}) \mid x < y\}.$$

Obviously holds

$$\nu(n,\mathbb{M}) = \left\lfloor \frac{k}{2} \right\rfloor, \text{ if } |\mathcal{C}(n,\mathbb{M})| = k.$$

The *complex circle of partition* approach (see [8]) is an extension of the method of *circle of partition*, which is based on the following definition.

Definition 1.4. Let $\mathbb{M} \subseteq \mathbb{N}$ and

$$\mathbb{C}_{\mathbb{M}} := \{z = x + iy \mid x \in \mathbb{M}, y \in \mathbb{R}\} \subset \mathbb{C}$$

be a subset of the complex numbers where the real part is from $\mathbb{M} \subseteq \mathbb{N}$. Then a CoP with a special requirement

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}}) = \{[z] \mid z, n - z \in \mathbb{C}_M, \Im(z)^2 = \Re(z)\,(n - \Re(z))\}$$

will be denoted as a **complex Circle of Partition**, abbreviated as **cCoP**. The special requirement will be called as the *circle condition*.

The components x and y we will call as *real weight* resp. *imaginary weight*. The CoP $\mathcal{C}(n,\mathbb{M})$ will be called as the *source CoP*.

In order to distinguish between points $[z]$ of cCoPs and points z in the complex plane $\mathbb{C}$ we denote the latter as *complex points*.

In the sequel we give a short outline about the basic properties of complex circles of partition.

The most important property is that all members of a cCoP are located on a circle in the complex plane $\mathbb{C}$ that has its center on the real axis at $\frac{n}{2}$ and has a diameter n. This is at once the length of each axis of $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$.

To each axis of a cCoP there exists a conjugate axis. For axis partners holds

$$\Im(z) = -\Im(n-z). \tag{1.1}$$

The circle in the complex plane with center on the real axis at $\frac{n}{2}$ and diameter n is called as the *embedding circle* $\mathfrak{C}_n$ of the cCop $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$. Any two, as well as all, embedding circles have only the origin as common point. Therefore any two cCoPs have no common point.

The length of a chord between any two points $[z_1] = [x_1+iy_1]$ and $[z_2] = [x_2+iy_2]$ of a cCoP $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ is given by

$$|\mathcal{L}_{[z_1],[z_2]}| = \Gamma([z_1],[z_2]) = |\sqrt{x_1(n-x_2)} \pm \sqrt{x_2(n-x_1)}|, \tag{1.2}$$

whereby $''-''$ will be taken if $\mathrm{sign}(y_1) = \mathrm{sign}(y_2)$ and $''+''$ else. The chord turns into an axis with length n if $[z_1],[z_2]$ are axis partners.

If $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ is a non-empty cCoP and $\mathfrak{I}_n$ resp. $\mathfrak{X}_n$ all complex points inside resp. outside of the embedding circle $\mathfrak{C}_n$, then for all complex points of $\mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}$ holds that their distances to each point of $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ is less than n

$$|z-w| < n \text{ for all } z \in ||\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})|| \text{ and all } w \in \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}. \tag{1.3}$$

And vice versa holds also

$$|z-w| > n \text{ for some } z \in ||\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})|| \text{ and all } w \in \mathfrak{X}_n \cap \mathbb{C}_{\mathbb{M}}. \tag{1.4}$$

2. Interior and Exterior Points of Complex Circles of Partition

In this section we introduce and develop the notion of **interior** and **exterior** points of complex circles of partition.

Definition 2.1. Since $\mathfrak{I}_n, \mathfrak{X}_n$ are defined in Definition 2.8 in [8] as all complex points **inside** resp. **outside** of the embedding circle $\mathfrak{C}_n$, we call the points $z \in \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}$ as **interior** points with respect to $\mathfrak{C}_n$ and denote the set of all such points as $\mathrm{Int}[\mathfrak{C}_n]$.

Correspondingly, we call the complex points $z \in \mathfrak{X}_n \cap \mathbb{C}_{\mathbb{M}}$ as **exterior** points with respect to $\mathfrak{C}_n$ and denote the set of all these points as $\mathrm{Ext}[\mathfrak{C}_n]$.

Obviously holds

$$\mathrm{Int}[\mathfrak{C}_n] = \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}} \text{ and } \mathrm{Ext}[\mathfrak{C}_n] = \mathfrak{X}_n \cap \mathbb{C}_{\mathbb{M}}.$$

Definition 2.2. Let $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be a non-empty cCoP and $\mathfrak{C}_n$ its embedding circle. Then we call the complex points $z \in \mathrm{Int}[\mathfrak{C}_n]$ as **interior** points with respect to the cCoP $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and denote the set of all these points as $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})]$ if and only if for **all** points $[w] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ holds $|z-w| < n$[1].

Correspondingly, we call the complex points $z \in \mathrm{Ext}[\mathfrak{C}_n]$ as **exterior** points with respect to $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ and denote the set of all these points as $\mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})]$ if and only if for **some** points $[w] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ holds $|z-w| > n$.

[1] $|.|$ means the usual distance between the points z and w in the complex plane $\mathbb{C}$.

Let $n_o \in \mathbb{N}$ be the least generator for all cCoPs. If $n > n_o$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ is an empty cCoP, then $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})]$ and $\mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})]$ are empty too.

Theorem 2.3. *If $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ is a non-empty cCoP then holds*

$$\begin{gathered}\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] = \mathrm{Int}[\mathfrak{C}_n] = \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}} \\ \textit{and} \\ \mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] = \mathrm{Ext}[\mathfrak{C}_n] = \mathfrak{X}_n \cap \mathbb{C}_{\mathbb{M}}.\end{gathered} \tag{2.1}$$

Proof. It suffices to prove that the distances of all complex points z of $\mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}$ to all points $[w] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ are less than n resp. of $\mathfrak{X}_n \cap \mathbb{C}_{\mathbb{M}}$ to some points $[w] \in \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ are greater than n. But this has already been proven in Theorem 3.3 in [8] for $\mathfrak{I}_n$ resp. $\mathfrak{X}_n$ instead of $\mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}$ resp. $\mathfrak{X}_n \cap \mathbb{C}_{\mathbb{M}}$. Hence the claim is proved. $\square$

Corollary 2.4. *If $\mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \neq \emptyset$ then $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ is non-empty too since there is at least an axis $\mathbb{L}_{[w],[n-w]} \hat{\in} \mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ such that the distances from both axis points to all complex points of $\mathrm{Int}[\mathfrak{C}_n]$ are less than n.*

Proposition 2.5. *Let $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be two non-empty cCoPs. If and only if $m < n$ holds*

$$\mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \textit{ and } \mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Ext}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})].$$

Proof. Let $m < n$, then since (2.1) holds

$$\begin{aligned}\mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})] &= \mathfrak{I}_m \cap \mathbb{C}_{\mathbb{M}} \text{ and since (2.4) in [8]} \\ &\subset \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}} = \mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})].\end{aligned}$$

Vice versa holds

$$\begin{aligned}\mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] &= \mathfrak{X}_n \cap \mathbb{C}_{\mathbb{M}} \text{ and since (2.4) in [8]} \\ &\subset \mathfrak{X}_m \cap \mathbb{C}_{\mathbb{M}} = \mathrm{Ext}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})].\end{aligned}$$

On the other hand from $\mathrm{Int}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})]$ follows $\mathfrak{I}_m \cap \mathbb{C}_{\mathbb{M}} \subset \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}$, which is only with $m < n$ solvable. Analogously follows from $\mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Ext}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ also $m < n$. $\square$

Proposition 2.6. *Let $\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})$ and $\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})$ be two non-empty cCoPs. If and only if $m < n$ holds*

$$||\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \textit{ and } ||\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Ext}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})].$$

Proof. Let $m < n$, then since (2.4) in [8] and $||\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})|| \subset \mathbb{C}_{\mathbb{M}}$ holds

$$\begin{aligned}||\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})|| &\subset \mathfrak{C}_m \cap \mathbb{C}_{\mathbb{M}} \\ &\subset (\mathfrak{C}_m \cap \mathbb{C}_{\mathbb{M}}) \cup \mathfrak{I}_m \\ &\subset (\mathfrak{C}_m \cup \mathfrak{I}_n) \cap \mathbb{C}_{\mathbb{M}} \text{ and since } \mathfrak{C}_m \subset \mathfrak{I}_n \\ &= \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}} \text{ and because of (2.1)} \\ &= \mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})].\end{aligned}$$

In a similar manner $||\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Ext}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ can be proved.

On the other hand, the embedding $||\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})|| \subset \mathrm{Int}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})]$ implies $\mathfrak{I}_m \cap \mathbb{C}_{\mathbb{M}} \subset \mathfrak{I}_n \cap \mathbb{C}_{\mathbb{M}}$, which is only with $m < n$ solvable. Analogously follows from $\mathrm{Ext}[\mathcal{C}^o(n, \mathbb{C}_{\mathbb{M}})] \subset \mathrm{Ext}[\mathcal{C}^o(m, \mathbb{C}_{\mathbb{M}})]$ also $m < n$. $\square$

Definition 2.7. Let $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ be a non-empty cCoP and $[z_1],[z_2]\in \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$. Then we say the line $\mathcal{L}_{[z_1],[z_2]}\in \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$ joins the points $[z_1],[z_2]\in \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$.

Next we show that we can use information about the length of an axis of a cCoP and an interior point to determine an exterior point. We summarize this criterion in the following proposition.

Proposition 2.8. *Let $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})\neq\emptyset$. If $[z_1],[z_2]$ are axis partners of the cCoP $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ and $|\mathbb{L}_{[z_1],[z_2]}|=m>n$, then $z_2\in \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$.*

Proof. From the requirement $\mathbb{L}_{[z_1],[z_2]}\hat{\in}\,\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ with $m>n$ and Proposition 2.5, it follows that

$$||\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})||\subset \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})] \text{ and therefore}$$
$$z_2\in \mathrm{Ext}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})].$$

□

An important feature that governs the landscape of the complex circles of partition is the interplay between the points on the cCoP and their corresponding interior and exterior points. It is always plausible to find an interior with respect to a cCoP that is non-empty. In fact the interior with respect to a non-empty cCoP constitute the entire space bounded by the cCoP. On the other hand, if the interior (resp. exterior) is empty then the cCoP by itself is empty.

Proposition 2.9. *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})\neq\emptyset$. If $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]\subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$, then $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})\neq\emptyset$.*

Proof. The conditions above with Definition 2.1 implies that $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]\neq\emptyset$ and $\mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]\supset\emptyset$, and hence $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})\neq\emptyset$. □

We state a sort of converse of the above result in the following theorem.

Theorem 2.10. *Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}),\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})\neq\emptyset$. If $m<n$, then there exists a chord $\mathcal{L}_{[z_1],[z_2]}\hat{\in}\,\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$ such that the complex points $z_1,z_2\notin \mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]$.*

Proof. By virtue of Definition 2.8 in [8] holds $\mathfrak{C}_n\cap\mathfrak{I}_n=\emptyset$ and $||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})||\subset\mathfrak{C}_n$, it follows easily that $\mathfrak{I}_n\cap||\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})||=\emptyset$. Since $\mathcal{L}_{[z_1],[z_2]}\hat{\in}\,\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, we have $z_1,z_2\notin\mathfrak{I}_n$ and because of $m<n$ holds $\mathfrak{I}_m\subset\mathfrak{I}_n$ and hence

$$z_1,z_2\notin\mathfrak{I}_n\supset\mathfrak{I}_m\supset\mathfrak{I}_m\cap\mathbb{C}_{\mathbb{M}}=\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})].$$

□

3. Quotient Complex Circles of Partition

In this section we introduce and develop the notion of the **quotient** complex circles of partition. This notion is akin to and parallels the notion of quotient groups in group theory.

Definition 3.1. Let $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}),\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})\neq\emptyset$ with $\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})]\subset \mathrm{Int}[\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})]$. Then by the **quotient** cCoP $\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})/_z\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$ induced by $[z]\in\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})$, we mean the collection of all cCoPs

$$\mathcal{C}^o(n,\mathbb{C}_{\mathbb{M}})/_z\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}}):=\{\mathcal{C}^o(n_j,\mathbb{C}_{\mathbb{M}})\mid j=1,\ldots,k\}$$

determined by the generators

$$n_j = \Re(z) + u_j \mid u_j \in ||\mathcal{C}(m,\mathbb{M})||, j = 1, \dots, k$$

with $\mathcal{C}(m,\mathbb{M})$ as the source CoP of $\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ and $k = |\mathcal{C}(m,\mathbb{M})|$.

We call the total number of all distinct cCoPs belonging to the **quotient** cCoP $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})/_z\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ induced by the point $[z] \in \mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ the **index** of the $\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ in $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$ induced by $[z]$

$$\mathrm{Ind}_z[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M}) : \mathcal{C}^o(m,\mathbb{C}_\mathbb{M})].$$

We call the union

$$\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})/\mathcal{C}^o(m,\mathbb{C}_\mathbb{M}) := \bigcup_{[\Re(z)]\in\mathcal{C}(n,\mathbb{M})} \mathcal{C}^o(n,\mathbb{C}_\mathbb{M})/_z\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$$

a **complete quotient** cCoP. We call the total number of all *distinct* cCoPs in $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})/\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ the **index** of the cCoP $\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$ in $\mathcal{C}^o(n,\mathbb{C}_\mathbb{M})$

$$\mathrm{Ind}[\mathcal{C}^o(n,\mathbb{C}_\mathbb{M}) : \mathcal{C}^o(m,\mathbb{C}_\mathbb{M})].$$

Obviously each member of the collection $\{\mathcal{C}^o(n_j,\mathbb{C}_\mathbb{M}) \mid j = 1, \dots, k\}$ has an axis

$$\mathbb{L}_{[z],[w_j]} \hat{\in} \mathcal{C}^o(n_j,\mathbb{C}_\mathbb{M}) \text{ with } w_j = u_j + i\Im(w_j) \in ||\mathcal{C}^o(n_j,\mathbb{C}_\mathbb{M})||.$$

Lemma 3.2 (The squeeze principle). *Let* $\mathbb{B} \subset \mathbb{M} \subseteq \mathbb{N}$ *and* $\mathcal{C}^o(m,\mathbb{C}_\mathbb{B}), \mathcal{C}^o(m+t,\mathbb{C}_\mathbb{B}) \neq \emptyset$ *with*

$$\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})] \subset \mathrm{Int}[\mathcal{C}^o(m+t,\mathbb{C}_\mathbb{M})]$$

for $t \geq 4$. *If* $m < s < m+t$ *such that* s, m, t *are of the same parity with*

$$\{u \in ||\mathcal{C}(m,\mathbb{M})|| \mid u \in \mathbb{B}\} \subseteq \{u \in ||\mathcal{C}(m+t,\mathbb{M})|| \mid u \in \mathbb{B}\}$$

and

$$||\mathcal{C}(m,\mathbb{M})|| \subset ||\mathcal{C}(m+t,\mathbb{M})||$$

and there exists $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{M})$ *with* $x \in \mathbb{B}$ *and* $x < y$ *such that*

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{M})|| \mid u \in \mathbb{B}\} \tag{3.1}$$

and $x > m - w$, *then there exists*

$$\mathcal{C}^o(s,\mathbb{C}_\mathbb{M}) \in \mathcal{C}^o(m+t,\mathbb{C}_\mathbb{M})/\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})$$

such that

$$\mathrm{Int}[\mathcal{C}^o(m,\mathbb{C}_\mathbb{M})] \subset \mathrm{Int}[\mathcal{C}^o(s,\mathbb{C}_\mathbb{M})] \subset \mathrm{Int}[\mathcal{C}^o(m+t,\mathbb{C}_\mathbb{M})].$$

Proof. In virtue of (3.1) holds $w \in \mathbb{B}$. As required the axis $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{M})$ exists with $x \in \mathbb{B}$ such that $m - w < x < y$. Then under the requirement

$$\{u \in ||\mathcal{C}(m,\mathbb{M})|| \mid u \in \mathbb{B}\} \subseteq \{u \in ||\mathcal{C}(m+t,\mathbb{M})|| \mid u \in \mathbb{B}\}$$

and

$$||\mathcal{C}(m,\mathbb{M})|| \subset ||\mathcal{C}(m+t,\mathbb{M})||$$

we have the inequality

$$\begin{aligned} m = w + (m-w) < w + x = w + (m+t-y) = m + t + (w-y) \\ < m+t, \text{ since } y > w \end{aligned} \tag{3.2}$$

and $m-w<x=m+t-y$ holds $y-w<t$. With $w+x=s$ there is an axis $\mathbb{L}_{[x],[w]} \hat{\in} \mathcal{C}(s,\mathbb{B})$ and it follows that $\mathcal{C}(s,\mathbb{B}) \neq \emptyset$ and hence $\mathcal{C}^o(s,\mathbb{C}_{\mathbb{B}}) \neq \emptyset$ with

$$\mathcal{C}^o(s,\mathbb{C}_{\mathbb{M}}) \in \mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{M}})/\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})$$

by virtue of our construction and

$$\operatorname{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{M}})] \subset \operatorname{Int}[\mathcal{C}^o(s,\mathbb{C}_{\mathbb{M}})] \subset \operatorname{Int}[\mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{M}})]$$

since $\mathcal{C}^o(s,\mathbb{C}_{\mathbb{B}}) \subset \mathcal{C}^o(s,\mathbb{C}_{\mathbb{M}})$ and Proposition 2.5. This completes the proof. □

Lemma 3.2 can be viewed as a basic tool-box for studying the possibility of partitioning numbers of a particular parity with components belonging to a special subset of the integers. It works by choosing two non-empty cCoPs with the same base set and finding further non-empty cCoPs with generators trapped in between these two generators. This principle can be used in an ingenious manner to study the broader question concerning the feasibility of partitioning numbers with each summand belonging to the same subset of the positive integers. We launch the following proposition as an outgrowth of Lemma 3.2.

Proposition 3.3 (The first interval Lemoine partition detector)**.** *Let $\mathbb{P}$ and $2\mathbb{P}$ be the set of all prime numbers and their doubles, respectively, and $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{P}\cup 2\mathbb{P}}), \mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{P}\cup 2\mathbb{P}}) \neq \emptyset$ by $t \geq 4$. If $m<s<m+t$ such that $s,m \equiv 1 \pmod 2$ with $t \equiv 0 \pmod 2$ and there exists $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{N})$ with $x \in \mathbb{P}$ and $x<y$ such that*

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \mid u \in \mathbb{P}\cup 2\mathbb{P}\} \in 2\mathbb{P} \tag{3.3}$$

and $x>m-w$ then there must exist $m<s<m+t$ such that $\mathcal{C}^o(s,\mathbb{C}_{\mathbb{P}\cup 2\mathbb{P}}) \neq \emptyset$.

Proof. This is a consequence of Lemma 3.2 by taking $\mathbb{M}=\mathbb{N}$ and $\mathbb{B}=\mathbb{P}\cup 2\mathbb{P}$ since its requirements are satisfied with

$$\begin{aligned}\{u \in ||\mathcal{C}(m,\mathbb{N})|| \mid u \in \mathbb{P}\cup 2\mathbb{P}\} &= \{u \in \mathbb{P}\cup 2\mathbb{P} \mid 3 \leq u \leq m-1\}\\ &\subseteq \{u \in \mathbb{P}\cup 2\mathbb{P} \mid 3 \leq u \leq m+t-1\}\\ &= \{u \in ||\mathcal{C}(m+t,\mathbb{N})|| \mid u \in \mathbb{P}\cup 2\mathbb{P}\}\end{aligned}$$

and

$$||\mathcal{C}(m,\mathbb{N})|| = \{1,2,\ldots,m-1\} \subset ||\mathcal{C}(m+t,\mathbb{N})|| = \{1,2,\ldots,m-1+t\}.$$

And in virtue of Proposition 2.6 due to $m<m+t$ holds also

$$\operatorname{Int}[\mathcal{C}^o(m,\mathbb{C}_{\mathbb{P}\cup 2\mathbb{P}})] \subset \operatorname{Int}[\mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{P}\cup 2\mathbb{P}})].$$

□

Proposition 3.4 (The second interval Lemoine partition detector)**.** *Let $\mathbb{P}$ and $2\mathbb{P}$ be the set of all prime numbers and their doubles, respectively, and $\mathcal{C}^o(m,\mathbb{C}_{\mathbb{P}\cup 2\mathbb{P}}), \mathcal{C}^o(m+t,\mathbb{C}_{\mathbb{P}\cup 2\mathbb{P}}) \neq \emptyset$ by $t \geq 4$. If $m<s<m+t$ such that $s,m \equiv 1 \pmod 2$ with $t \equiv 0 \pmod 2$ and there exists $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{N})$ with $x \in 2\mathbb{P}$ and $x<y$ such that*

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \mid u \in \mathbb{P}\cup 2\mathbb{P}\} \in \mathbb{P} \tag{3.4}$$

and $x>m-w$ then there must exist $m<s<m+t$ such that $\mathcal{C}^o(s,\mathbb{C}_{\mathbb{P}\cup 2\mathbb{P}}) \neq \emptyset$.

Proof. The proof is the same as in Proposition 3.3. □

Theorem 3.5 (Conditional Lemoine)**.** *Let $\mathbb{P}$ and $2\mathbb{P}$ be the set of all prime numbers and their doubles, respectively, and $m \in 2\mathbb{N}+1$ such that $\mathcal{C}(m,\mathbb{P}) \neq \emptyset$ for m* ***sufficiently*** *large. If for* ***all*** *$t \geq 4$ there exists $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{N})$ with $x \in \mathbb{P}$ and $x < y$ such that*

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \mid u \in \mathbb{P} \cup 2\mathbb{P}\} \in 2\mathbb{P}$$

and $m - w < x$, or there exists $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t,\mathbb{N})$ with $x \in 2\mathbb{P}$ and $x < y$ such that

$$y > w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \mid u \in \mathbb{P} \cup 2\mathbb{P}\} \in \mathbb{P}$$

and $m - w < x$ then there are CoPs $\mathcal{C}(s,\mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$ for all (sufficiently large) $s \in 2\mathbb{N}+1 \mid s > m$.

Proof. It is known that there are infinitely many odd numbers that can be written as the sum of a prime and a double of a prime, so that for $m \in 2\mathbb{N}+1$ **sufficiently** large with $\mathcal{C}(m,\mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$ then $t \geq 4$ can be chosen **arbitrarily** large such that $\mathcal{C}(m+t,\mathbb{P}) \neq \emptyset$. Under the requirements and appealing to Proposition 3.3 and 3.4 there must exist some $s \equiv 1 \pmod 2$ with $m < s < m+t$ such that $\mathcal{C}(s,\mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$. Now we continue our arguments on the intervals of generators $[m,s]$ and $[s,s+r]$. If there exist some $u,v \in 2\mathbb{N}+1$ such that $m < u < s$ and $s < v < s+r$, then we repeat the argument under the requirements (for arbitrary t) to deduce that $\mathcal{C}(u,\mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$ and $\mathcal{C}(v,\mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$. We can iterate the process repeatedly so long as there exists some odd generators trapped in the following sub-intervals of generators $[m,u],[u,s],[s,v],[v,v+r]$ where $v+r = m+t$ for $t \geq 4$. Since t can be chosen arbitrarily so that $\mathcal{C}(m+t,\mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$, the assertion follows immediately. □

4. Application to the Lemoine Conjecture

In this section we apply the notion of the quotient complex circles of partition and the squeeze principle to study Lemoine's conjecture in the very large. We begin with the following preparatory elementary results.

Lemma 4.1 (The prime number theorem)**.** *Let $\pi(m)$ denotes the number of prime numbers less than or equal to m and $p_{\pi(m)}$ denotes the $\pi(m)^{th}$ prime number. Then we have the asymptotic relations*

$$p_{\pi(m)} \sim m\left(1 - \frac{\log\log m}{\log m}\right) \sim 2p_{\pi(\frac{m}{2})}.$$

Proof. This is an easy consequence by combining the two versions of the prime number theorem

$$\pi(m) \sim \frac{m}{\log m} \quad \textbf{and} \quad p_k \sim k\log k$$

where p_k denotes the k^{th} prime number. Since with $k = \pi(m)$ we get

$$\begin{aligned} p_k = p_{\pi(m)} &\sim \frac{m}{\log m}\log\left(\frac{m}{\log m}\right) \\ &= \frac{m}{\log m}(\log m - \log\log m) \\ &= m\left(1 - \frac{\log\log m}{\log m}\right). \end{aligned}$$

Now we replace in the latter m by $\frac{m}{2}$ and get

$$\begin{aligned} 2p_{\pi(\frac{m}{2})} &\sim 2\frac{m}{2}\left(1-\frac{\log\log\frac{m}{2}}{\log\frac{m}{2}}\right) \\ &= m\left(1-\frac{\log(\log m-\log 2)}{\log(\log m-\log 2)}\right) \\ &\sim m\left(1-\frac{\log\log m}{\log m}\right). \end{aligned}$$

□

Obviously holds with the variable denotations from the previous section

$$w = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \ \mid u \in \mathbb{P}\cup 2\mathbb{P}\} = p_{\pi(m)} \tag{4.1}$$

provides $w \in \mathbb{P}$ and

$$w' = \max\{u \in ||\mathcal{C}(m,\mathbb{N})|| \ \mid u \in \mathbb{P}\cup 2\mathbb{P}\} = 2p_{\pi(\frac{m}{2})} \tag{4.2}$$

provides $w' \in 2\mathbb{P}$.

Lemma 4.2 (Bertrand's postulate)**.** *There exists a prime number in the interval $(k, 2k)$ for all $k > 1$.*

The formula in Lemma 4.1 obviously suggests that the $\pi(m)^{th}$ prime number satisfies and implies the asymptotic relation $p_{\pi(m)} \sim m$. While this is valid in practice, it does not actually help in measuring the asymptotic of the discrepancy between the maximum prime number less than m and m. It gives the misleading impression that this discrepancy has absolute difference tending to zero in the very large. We reconcile this potentially nudging flaw by doing things slightly differently.

Lemma 4.3 (The first little lemma)**.** *Let $\mathbb{P}$ and $2\mathbb{P}$ be the set of all prime numbers and their doubles, respectively, and $m \in \mathbb{N}$ be **sufficiently** large such that $\mathcal{C}(m,\mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$. Then for all $x' \in 2\mathbb{P}$ with $x' = 2x$ for $x \in \mathbb{P}$ satisfying*

$$\frac{m\log\log m}{2\log m} < x < \frac{m\log\log m}{\log m}$$

the asymptotic and inequalities

$$m - w \sim \frac{m\log\log m}{\log m}$$

and

$$0 \lesssim |w - (m + t - x')| \lesssim t$$

hold for $t \geq 4$.

Proof. Due to Lemma 4.2 there is a prime between $\dfrac{m\log\log m}{2\log m}$ and $\dfrac{m\log\log m}{\log m}$. Appealing to the prime number theorem, we obtain with (4.1) the asymptotic inequalities

$$\begin{aligned} m - w &= m - p_{\pi(m)} \\ &\sim m - m\left(1 - \frac{\log\log m}{\log m}\right) \\ &= \frac{m\log\log m}{\log m} \end{aligned}$$

for all sufficiently large $m \in 2\mathbb{N}+1$ and

$$\begin{aligned} m + t - x' = m + t - 2x > m + t - \frac{2m\log\log m}{\log m} \\ &= m\left(1 - \frac{\log(\log m)^2}{\log m}\right) + t \\ &\sim m + t \geq p_{\pi(m)} = w \end{aligned}$$

and

$$\begin{aligned} |w - (m + t - x')| &= |m + t - x' - p_{\pi(m)}| \\ &< |m + t - \frac{m\log\log m}{\log m} - p_{\pi(m)}| \\ &\sim \left| m + t - \frac{m\log\log m}{\log m} - m\left(1 - \frac{\log\log m}{\log m}\right)\right| \\ &= t \end{aligned}$$

for $t \geq 4$. □

Lemma 4.4 (The second little lemma)**.** *Let $\mathbb{P}$ be the set of all prime numbers and their doubles, respectively, and $m \in \mathbb{N}$ be* ***sufficiently*** *large such that $\mathcal{C}(m, \mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$. Then for all $x \in \mathbb{P}$ satisfying*

$$\frac{m\log\log m}{\log m} < x < \frac{m\log(\log m)^2}{\log m}$$

the asymptotic and inequalities

$$m - w' \sim \frac{m\log\log m}{\log m}$$

and

$$0 \lesssim |w' - (m + t - x)| \lesssim t$$

hold for $t \geq 4$.

Proof. Due to Lemma 4.2 there is a prime between $\frac{m\log\log m}{\log m}$ and $\frac{m\log(\log m)^2}{\log m}$. Appealing to the prime number theorem, we obtain with (4.2) the asymptotic inequalities

$$\begin{aligned} m - w' &= m - 2p_{\pi(\frac{m}{2})} \\ &\sim m - m\left(1 - \frac{\log\log m}{\log m}\right) \\ &= \frac{m\log\log m}{\log m} \end{aligned}$$

for all sufficiently large $m \in 2\mathbb{N}+1$ and

$$\begin{aligned} m+t-x &> m+t-\frac{m\log(\log m)^2}{\log m} \\ &= m\left(1-\frac{\log(\log m)^2}{\log m}\right)+t \\ &\sim m+t \geq p_{\pi(m)} \sim 2p_{\pi(\frac{m}{2})} = w' \end{aligned}$$

and

$$\begin{aligned} |w'-(m+t-x)| &= |m+t-x-2p_{\pi(\frac{m}{2})}| \\ &< |m+t-\frac{m\log\log m}{\log m}-2p_{\pi(\frac{m}{2})}| \\ &\sim \left|m+t-\frac{m\log\log m}{\log m}-m\left(1-\frac{\log\log m}{\log m}\right)\right| \\ &= t \end{aligned}$$

for $t \geq 4$. $\square$

We are now ready to prove the Lemoine conjecture for all **sufficiently** large odd numbers. It is a case-by-case argument and a culmination of ideas espoused in this paper.

Theorem 4.5 (Asymptotic Lemoine theorem)**.** *Every* ***sufficiently*** *large odd number can be written as a sum of a prime number and a double of a prime number.*

Proof. The claim is equivalent to the statement:

For every sufficiently large odd number $n \in 2\mathbb{N}+1$ holds $\mathcal{C}(n, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$ since only the sum of an odd and an even number provides an odd number and therefore each axis $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m, \mathbb{P} \cup 2\mathbb{P})$ has an odd and an even axis point.

It is known that there are infinitely many odd numbers $m > 0$ with $\mathcal{C}(m, \mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$. Let us choose $m \in 2\mathbb{N}+1$ **sufficiently** large such that $\mathcal{C}(m, \mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$ and choose $t \geq 4$ such that $\mathcal{C}(m+t, \mathbb{P}\cup 2\mathbb{P}) \neq \emptyset$. Now, we distinguish and examine two special cases as below:

- The case
$$w = \max\{u \in ||\mathcal{C}(m, \mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = p_{\pi(m)}$$
- The case
$$w' = \max\{u \in ||\mathcal{C}(m, \mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = 2p_{\pi(\frac{m}{2})}$$

In the case

$$w = \max\{u \in ||\mathcal{C}(m, \mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = p_{\pi(m)}$$

then we choose a prime number $x < \dfrac{m\log\log m}{\log m}$ such that $x > \dfrac{m\log\log m}{2\log m}$, since by Bertrand's postulate (Lemma 4.2) there exists a prime number x such that

$x \in (k, 2k)$ for every $k > 1$ and set $2x = x' \in 2\mathbb{P}$. Then we get for the axis partner $[y]$ of the axis point $[x']$ of $\mathbb{L}_{[x'],[y']} \hat{\in} \mathcal{C}(m+t, \mathbb{N})$ the inequality

$$\begin{aligned} y' = m + t - x' &> m + t - \frac{m \log(\log m)^2}{\log m} \\ &= m\left(1 - \frac{\log(\log m)^2}{\log m}\right) + t \\ &\sim m + t \geq p_{\pi(m)} = w \end{aligned}$$

for $t \geq 4$ and by appealing to Lemma 4.3 also the following asymptotic inequalities

$$m - w \sim \frac{m \log\log m}{\log m} < x'$$

and

$$|y' - w'| = |(m + t - x') - w| = |m - w + t - x'| \lesssim |x' + t - x'| = t.$$

Then the requirements in Theorem 3.5 are fulfilled **asymptotically** in this case with

$$y' \gtrsim w \text{ and } x' \gtrsim m - w \text{ and } 0 \lesssim |y' - w| \lesssim t.$$

In the case

$$w' = \max\{u \in ||\mathcal{C}(m, \mathbb{N})|| \ \mid u \in \mathbb{P} \cup 2\mathbb{P}\} = 2p_{\pi(\frac{m}{2})}$$

then we choose a prime number $x < \dfrac{m \log(\log m)^2}{\log m}$ such that $x > \dfrac{m \log\log m}{\log m}$, since by Bertrand's postulate (Lemma 4.2) there exists a prime number x such that $x \in (k, 2k)$ for every $k > 1$. Then we get for the axis partner $[y]$ of the axis point $[x]$ of $\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(m+t, \mathbb{N})$ the inequality

$$\begin{aligned} y = m + t - x &> m + t - \frac{m \log(\log m)^2}{\log m} \\ &= m\left(1 - \frac{\log(\log m)^2}{\log m}\right) + t \\ &\sim m + t \geq p_{\pi(m)} \sim 2p_{\pi(\frac{m}{2})} = w' \end{aligned}$$

for $t \geq 4$ and by appealing to Lemma 4.4 also the following asymptotic inequalities

$$m - w' \sim \frac{m \log\log m}{\log m} < x$$

and

$$|y - w'| = |(m + t - x) - w'| = |m - w' + t - x| \lesssim |x + t - x| = t.$$

Then the requirements in Theorem 3.5 are fulfilled **asymptotically** in this second case with

$$y \gtrsim w' \text{ and } x \gtrsim m - w' \text{ and } 0 \lesssim |y - w'| \lesssim t.$$

The result follows by arbitrarily choosing $t \geq 4$ so that $\mathcal{C}(m+t, \mathbb{P} \cup 2\mathbb{P}) \neq \emptyset$ and adapting the proof in Theorem 3.5. $\square$

Theorem 4.5 is equivalent to the statement: there must exist some positive constant N such that for all $m \geq N$, then it is always possible to partition every odd number m as a sum of a prime number and a double of a prime number. This result - albeit constructive to some extent - looses its constructive flavour so that we cannot carry out this construction to cover all odd numbers, since we are unable to obtain any quantitative (lower) bound for the threshold N. At least, we are able to get a handle on the conjecture **asymptotically**.

References

1. Lemoine, Émile *L'Intermédiaire des mathematiciens*,1 (1894), 179; ibid 3 (1896), 151.
2. Levi, Hyman, *On Goldbach's Conjecture*, Math. Gaz. 47 (1963): 274.
3. Estermann, Theodor *On Goldbach's problem: Proof that almost all even positive integers are sums of two primes*, Proceedings of the London Mathematical Society, vol. 2:1, Wiley Online Library, 1938, pp. 307–314.
4. Chudakov, Nikolai Grigor'evich, *The Goldbach's problem*, Uspekhi Matematicheskikh Nauk, vol. 4, Russian Academy of Sciences, Steklov Mathematical Institute of Russian , 1938, 14–33.
5. Chen, Jing-run *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, The Goldbach Conjecture, World Scientific, 2002, pp. 275–294.
6. Helfgott, Harald A *The ternary Goldbach conjecture is true*, arXiv preprint arXiv:1312.7748, 2013.
7. Agama, Theophilus and Gensel, Berndt *Studies in Additive Number Theory by Circles of Partition*, arXiv:2012.01329, 2020.
8. Gensel, Berndt and Agama, Theophilus *Complex Circles of Partition - Basics*, ResearchGate, 2022.

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