The Zeta Function as a particular case of Euler-Maclaurin formula

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Abstract

One of the most famous functions, full of enigmas, is the Riemann zeta function, since it is the basis for one of the most amazing hypotheses because of its relation with the prime counting function. But in the present paper, we will discover some of those enigmas, using the Euler-Maclaurin formula.

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1 Introduction.

1.1 Definition of the Riemann Zeta function.

The Riemann Zeta Function is defined as a complex variable function $\mathbb{C} \to \mathbb{C}$ as follows:

$$\zeta(s) = \sum_{m=1}^{\infty} \frac{1}{m^s} \tag{1}$$

Where: $s \in \mathbb{C}$ and $Re(s) > 1$

And:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

Is the Analytic Continuation of the Riemann Zeta Function, with $\zeta(1) = undetermined$

1.2 Some formulas for certain cases of the Riemann Zeta Function.

We know two formulas to calculate Zeta for some special cases:

$$\zeta(2k) = \frac{(-1)^{k-1}(2\pi)^{2k} B_{2k}}{2(2k)!} \tag{3}$$

$$\zeta(-k) = -\frac{B_{k+1}}{k+1} \tag{4}$$

Where: $k \in \mathbb{N}$

2 Sums of Powers and the Bernoulli Numbers.

In mathematics, the Bernoulli numbers B_j is a set of successive rational numbers with relevant importance in number theory. They appear in *Combinatorics*, in the expansion of the tangent functions and the hyperbolic tangent by Taylor series, and the Bernoulli formula to calculate the sum of powers.

Today, there is a known way to obtain the equation for the sum of powers, and it is the following recursive formula:

$$\sum_{m=1}^{n} m^k = S_k(n) = \sum_{p=1}^{1+k} A_p n^p \tag{5}$$

Where: $k \in \mathbb{N} \cup \{0\}$, *and* S_k *is function of* n.

A_p is obtained by:

$$A_p = \frac{(-1)^{1+k-p}}{1+k} \binom{1+k}{p} B_{1+k-p} \tag{6}$$

And B_{1+k-p} *are the Bernoulli Numbers obtained by:*

$$B_j = -\frac{1}{1+j} \sum_{m=0}^{j-1} \binom{1+j}{m} B_m \tag{7}$$

3 Sums of Powers as sum of higher-order derivatives.

Factorizing $\frac{1}{1+k}$ and developing the summation of Equation (5) to find $S_k(n)$:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^k(k+1)!}{1!k!}B_k n + \frac{(-1)^{k-1}(k+1)!}{2!(k-1)!}B_{k-1}n^2 + \frac{(-1)^{k-2}(k+1)!}{3!(k-2)!}B_{k-2}n^3 + \right.$$
$$\left. \frac{(-1)^{k-3}(k+1)!}{4!(k-3)!}B_{k-3}n^4 + \cdots + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^0(k+1)!}{(k+1)!(0)!}B_0 n^{k+1}\right]$$

Rearranging terms and accommodating the factorials in order to simplify:

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)!}{k!1!}B_1 n^k + \frac{(-1)^2(k+1)!}{(k-1)!2!}B_2 n^{k-1} + \cdots\right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0(k+1)!}{(k+1)!0!}B_0 n^{k+1} + \frac{(-1)^1 k!(k+1)}{k!1!}B_1 n^k + \frac{(-1)^2(k-1)!k(k+1)}{(k-1)!2!}B_2 n^{k-1} + \cdots\right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

$$S_k(n) = \frac{1}{1+k}\left[\frac{(-1)^0}{0!}B_0 n^{k+1} + \frac{(-1)^1(k+1)}{1!}B_1 n^k + \frac{(-1)^2 k(k+1)}{2!}B_2 n^{k-1} + \cdots\right.$$
$$\left. \cdots + \frac{(-1)^k(k+1)!}{1!k!}B_k n\right]$$

Rewriting as a summation of a product of factors:

$$S_k(n) = \frac{1}{1+k}n^{k+1} + \sum_{p=2}^{1+k}\frac{(-1)^{p-1}B_{p-1}}{(1+k)(p-1)!}\prod_{m=1}^{p-1}(2+k-m)n^{2+k-p} \tag{8}$$

It can also be expressed as the sum of higher order derivatives:

$$S_k(n) = \int_0^n m^k dm + \sum_{p=1}^{k}\frac{(-1)^p B_p}{(p)!} * \frac{d^{p-1}}{dn^{p-1}}\left(n^k\right) \tag{9}$$

Or:

$$S_{-k}(n) = \int_0^n \frac{1}{m^k}dm + \sum_{p=1}^{k}\frac{(-1)^p B_p}{(p)!} * \frac{d^{p-1}}{dn^{p-1}}\left(\frac{1}{m^k}\right) \tag{10}$$

4 The Euler–Maclaurin formula and the Zeta function.

Given the known Euler-Maclaurin equation [1], for the summation of a given function $f(m), \mathbb{R} \to \mathbb{R}$, and it is q times derivable:

$$\sum_{m=a+1}^{b} f(m) = \int_a^b f(x)dx + \sum_{p=1}^{q}\frac{(-1)^p B_p}{p!}\left[f^{p-1}(b) - f^{p-1}(a)\right] + R_q \tag{11}$$

Where:
B_p are the Bernoulli Numbers, $f^{p-1}(b)$ is the derivative of order $p-1$ of the function f evaluated at b, $f^{p-1}(a)$ is the derivative of order $p-1$ of the function f evaluated at a, and R_q *is the residual error.*

If we define the limits of the sum with $a = 0$ and $b = n$, and let $f(m) = \frac{1}{m^k}$, $k \in \mathbb{R}$, and $f(m)$ be infinitely derivable, one obtains:

$$\sum_{m=1}^{n} \frac{1}{m^k} = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} \left[f^{p-1}(n) - f^{p-1}(0) \right] + R_q$$

Then:

$$\sum_{m=1}^{n} \frac{1}{m^k} = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} \left[f^{p-1}(n) \right] + R_q$$

$$\sum_{m=1}^{n} \frac{1}{m^k} = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^k} \right) + R_q \tag{12}$$

Now let us clear the term R_q as function of k and let's $n \to \infty$:

$$R_q(k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^k} - \int_0^n \frac{1}{m^k} dm - \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^k} \right) \right] \tag{13}$$

Extending the second series of the equation (13) we have:

$$R_q(k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^k} - \left(\frac{1}{1-k} n^{1-k} + \frac{(-1)^1 B_1}{1!} n^{-k} + \frac{(-1)^2 B_2}{2!} (-k) n^{-1-k} + \frac{(-1)^3 B_3}{3!} (-k)(-1-k) n^{-2-k} + \cdots \right) \right] \tag{14}$$

If $k > 1$, and applying the limit, we get:

$$R_q(k) = \left[\sum_{m=1}^{\infty} \frac{1}{m^k} - \left(\frac{1}{1-k} 0 + \frac{(-1)^1 B_1}{1!} 0 + \frac{(-1)^2 B_2}{2!} (-k) 0 + \frac{(-1)^3 B_3}{3!} (-k)(-1-k) 0 + \cdots \right) \right]$$

$$R_q(k) = \sum_{m=1}^{\infty} \frac{1}{m^k} \iff k > 1 \tag{15}$$

Therefore it is concluded that: when $k > 1$ the value of $R_q(k)$ converges, and is equal to the sum of k-th power inverses, and is equal to $\zeta(k)$:

$$R_q(k) = \zeta(k) \iff k > 1, \ k \in \mathbb{R} \tag{16}$$

Now let $k = 1$ and apply the limit in equation (13):

$$R_q(1) = \left[\sum_{m=1}^{\infty} \frac{1}{m^1} - \left(\ln(\infty) - \ln(0) + \frac{(-1)^1 B_1}{1!} 0 + \frac{(-1)^2 B_2}{2!} (-1) 0 + \frac{(-1)^3 B_3}{3!} (-1)(-1-1) 0 + \cdots \right) \right]$$

$$R_q(1) = \infty - (\infty - \infty + 0 + 0 + 0 + \cdots)$$

$$R_q(1) = undetermined$$

Therefore it is concluded that: when $k = 1$ the value of $R_q(1)$ is undetermined, and is equal to $\zeta(1)$:

$$R_q(1) = \zeta(1) \tag{17}$$

Now, if: $k > 1$, $k \in \mathbb{N}$ in equation (13) but for $f(m) = m^k$:

This time let's separate the sum of the derivatives into two parts, the first sum from $p = 1$ to $p = k$, and the second sum from $p = k + 1$ to infinity:

$$R_q(-k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^k - \int_0^n m^k dm - \sum_{p=1}^{q=k} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(n^k\right) - \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(n^k\right) \right] \quad (18)$$

We observed that the integral with the first sum of derivatives is equal to the sum of inverse powers according to equation (9):

$$R_q(-k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} m^k - \sum_{m=1}^{n} m^k - \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(n^k\right) \right]$$

It can be simplified

$$R_q(-k) = \lim_{n\to\infty} \left[- \sum_{p=k+1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(n^k\right) \right]$$

Extending the series of the equation:

$$R_q(-k) = \lim_{n\to\infty} \left[- \left(\frac{(-1)^{k+1} B_{k+1}}{(k+1)!} (k!) n^0 + \frac{(-1)^{k+2} B_{k+2}}{(k+2)!} (k)(k-1) \cdots n^{-1} + \frac{(-1)^{k+3} B_{k+3}}{(k+3)!} (k)(k-1) \cdots n^{-2} + \cdots \right) \right]$$

Applying the limit:

$$R_q(-k) = - \frac{(-1)^{k+1} B_{k+1}}{(k+1)!} (k!)$$

$$R_q(-k) = \frac{(-1)^k B_{k+1}}{k+1}$$

As $B_{2j+1} = 0 \ j \in \mathbb{N}$ we can write:

$$R_q(-k) = - \frac{B_{k+1}}{k+1} \iff k \in \mathbb{N} \quad (19)$$

We conclude that: Equation (19) is equivalent to Equation (4):

$$R_q(-k) = \zeta(-k) \iff , k \in \mathbb{N} \quad (20)$$

Equation (13) is defined for a function $\mathbb{R} \to \mathbb{R}$, but when n tends to infinity, it is verified that the residual term $R_q = \zeta(k)$ According to equations: (16), (17) and (20).

5 Conclusions

We are now ready to write the following Theorem:

Theorem 1 *Let $k \in \mathbb{C}$, from the Euler-Maclaurin formula, for the summation of the function, $f(m) = \frac{1}{m^k}$:*

$$\sum_{m=1}^{n} \frac{1}{m^k} = \int_0^n \frac{1}{m^k} dm + \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left[\frac{1}{n^k}\right] + R_q$$

When the summation is infinite, the residual error $R_q = \zeta(k)$:

$$\zeta(k) = \lim_{n\to\infty} \left[\sum_{m=1}^{n} \frac{1}{m^k} - \int_0^n \frac{1}{m^k} dm - \sum_{p=1}^{q=\infty} \frac{(-1)^p B_p}{p!} * \frac{d^{p-1}}{dn^{p-1}} \left(\frac{1}{n^k}\right) \right] \quad (21)$$

References

[1] Jean-Marie De Koninck and Florian Luca. *Analytic number theory: Exploring the anatomy of integers*, volume 134. American Mathematical Soc., 2012.

Dedication.

"Call to me and I will answer you and tell you great and unsearchable things you do not know"
(Jeremiah 33:3)

I thank God for hearing my prayers, and showing me the way to the resolution of this problem. to him be the glory.

This work is dedicated to all my family who supported me at all times and in the most difficult moments of my life, especially my beloved wife Araceli, my two beautiful children Ocram and Arelys, and my parents who never lost faith in me.