Representation of the Collatz Graph using adjacency matrix

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Abstract

Let G be a weighted directed graph with node V(G) represented by a positive odd integer. Each edge E(g) directed from node a_n to node a_{n+1} with weigh $e(a_n)$ defined from $a_{n+1} = (3a_n + 1)/2^{e(a_n)}$ where $e(a_n)$ is the highest exponent for which $2^{e(a_n)}$ exactly divide $3a_n + 1$. This graph is called the Collatz weighted directed graph with its unique adjacency matrix. The structure of this adjacency matrix provides new insights into the validity of the Collatz conjecture.

1. **Introduction: the Collatz conjecture**

Denote by $N = \{1,2,3...........\}$, $N_0 = \{ 0,1,2,3,.............\}$, and

$D^+ = 2N_0 + 1$ the set of positive odd number. Define the recursive function introduced by Crandall [1] :

$$a_{n+1} = (3a_n + 1)/2^{e(a_n)} \qquad (1)$$

where $a_n \in D^+$ and $e(a_n) \in N$ is the highest exponent for which $2^{e(a_n)}$ exactly divide $3a_n + 1$. For an initial a_0 , any k iteration on a_0 generate a sequence of odd integer , { a_0 , a_1,.........a_k }. The collatz conjecture asserts that for every positive odd integer a_0 there exists $k \in N$ such that $a_k = 1$

2. **The Collatz weighted directed graph**

Let G be a weighted directed graph with node V(G) represented by a positive odd integer. Each edge E(G) directed from node a_n to node a_{n+1} with weigh $e(a_n)$ defined from

$$a_{n+1} = (3a_n + 1)/2^{e(a_n)} \qquad (2)$$

where $a_n \in D^+$ and $e(a_n)$ is the highest exponent for which $2^{e(a_n)}$ exactly divide $3a_n + 1$.

As an example, an edge from node 1 to node 1, node 5 to node 1 will have a weigh of 2 and 4, respectively. Some part of G is shown in Figure 1.

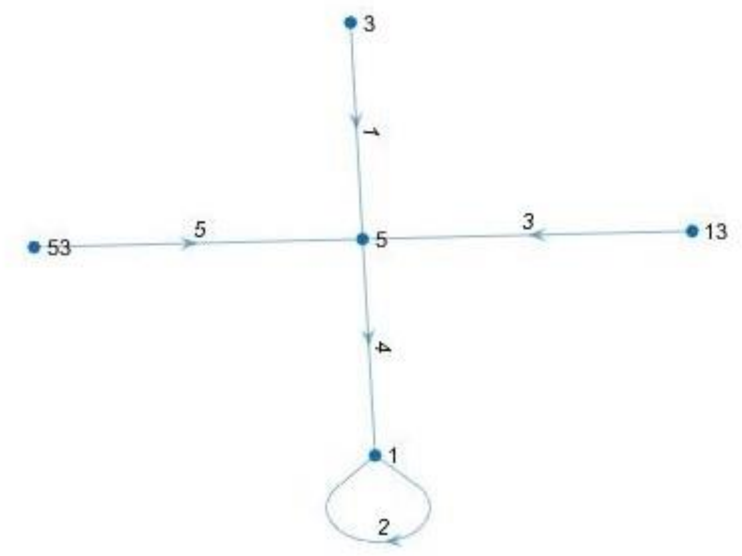

Figure 1. Part of G with node 1, 3, 5, 13, 53

3. Adjacency matrix of G

The well-defined adjacency matrix A for the collatz weighted directed graph is shown in Figure 2.

	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	35	37 . . . .
1	2		4								6								
3																			
5		1					3												
7					2														4
9																			
11				1											3				
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Figure 2 The adjacency matrix A with invisible zero elements for the collatz weighted directed graph

A is an infinite matrix with element a(i ,j) as a weigh directed from node 2(j-1)+1 to node 2(i-1)+1, i.e. a(1,1) = 2, a(1,3) = 4. In each row of A ,there are infinite nonzero elements except at node 6n+3, n= 0, 1, 2, 3,….. which has only zero elements. Let S_i be a set related to node i, i.e.

$S_1 = \{ 1, 5 , 21, 85,\ldots\ldots\}$,

$S_3 = \{ \emptyset \}$,

$S_5 = \{ 3, 13, 53, \ldots\ldots\}$.

All S_k can be divided in three groups [2]:

S_{6n+3} = { ∅ ; $n = 0,1,2, \ldots\ldots$},

S_{6n+1} = { $\left(8n+1+\frac{1}{3}\right)4^k - \frac{1}{3}$; $n = 0, 1, 2, \ldots\ldots ; k = 0, 1, 2, \ldots\ldots$ },

S_{6n+5} = { $\left(4n+3+\frac{1}{3}\right)4^k - \frac{1}{3}$; $n = 0, 1, 2, \ldots\ldots ; k = 0, 1, 2, \ldots\ldots$ },

Also each column of A has only one non-zero element. It means that each odd positive integer is an element in some set S_k, k = 1, 3, 5, 7, …..which implies that the union of all S_k, k = 1, 3, 5, 7……is equal to D^+.

We can see that it takes one step from each element in S_1 to reach 1 and two steps from each element in the union of S_5 , S_{85}, S_{341}, ….. to reach 1.

Let T_i be a set with its element can reach 1 in i steps; and since the union of all T_i equals to D^+ then each a_0 will be element of a particular T_k; $k \in D^+$.

Based on these facts, it is concluded that the Collatz cojecture is true.

References

[1] R . E. Crandall, " On the "3x+1" problem", Math. Of Comp. Vol. 32, N0. 144, October 1978, p. 1281-1292.

[2] Z. B. Batang, "Integer patterns in Collatz sequence", arXiv: 1907.07088v2 [math.GM] 17 Jul 2019.