A Proof of Riemann Hypothesis by the Angle Preserving Property of An Analytic Complex Function

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Abstract: The Riemann zeta function(RZF), $\zeta(s)$, is a function of a complex variable $s = x + iy$, which is analytic for $x > 1$. The Dirichlet Eta Function(DEF), $\eta(s)$, is also a function of a complex variable s, which is analytic for $x > 0$. The zeros of RZF and DEF are all same. The Riemann hypothesis(RH) states that the non-trivial zeros of RZF is of the form $s = 0.5 + iy$. The clue of our proof stems from the symmetry properties of RZF zeros, stating that if there exists a zero whose real part is not 0.5, such as $\zeta(\alpha + i\beta) = 0, 0 < \alpha < 0.5$, also $\zeta(1 - \alpha + i\beta) = 0$, called the *critical line symmetry*. Then, the two zeros should be on the two edge lines of a strip $\alpha \le x \le 1 - \alpha$. In the strip there are infinitely many lines that are parallel to the edge lines. Our question was, when that strip is mapped by DEF, will those parallel relationships be kept? If the parallel relationships are kept, RH is true, if not, RH may be false. Because DEF is analytic for all complex numbers for $Re(s) = x > 0$, so, by the angle preserving property of an analytic function, the parallel two edge lines can't intersect at the origin, when mapped by DEF. So, RH is true.

1. Introduction

In this work we studied the implications of the symmetry properties of the zeros of RZF, when the zero deviates the critical line. It is well known that the mapping of an analytic complex function preserves the intersecting angles [1][2].

If there exists a zero whose real part is not 0.5, such as $\zeta(\alpha + i\beta) = 0, 0 < \alpha < 0.5$, the symmetry properties of the zeros of RZF forces $\zeta(1 - \alpha + i\beta) = 0$. This implies that the two zeros should be on the two parallel edge lines of a strip $\alpha \le x \le 1 - \alpha$.

If $\zeta(\alpha + i\beta) = \zeta(1 - \alpha + i\beta) = 0$, the parallel two edge lines must intersect at the origin, when mapped by an analytic complex function, preserving the intersecting angles. But, the two edge lines of the strip can't intersect.

Furthermore, in the strip there are infinitely many lines that are parallel to the edge lines. So, how the two edge lines can be mapped to intersect at the origin, crossing the infinitely many parallel lines inside the strip?

It is almost self-evident that the two parallel edge lines can't intersect at the origin, when mapped by DEF.

2. Symmetry Properties of the Zeros of RZF

RZF [3][4][5][6][7] $\zeta(s)$ and DEF [8] $\eta(s)$ are functions of a complex variable $s = x + iy$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots \tag{2.1}$$

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1 - 2^{1-s})\zeta(s) = \frac{1}{1^s} - \frac{1}{2^s} + \frac{1}{3^s} - \cdots \tag{2.2}$$

It is well known that the following three equations are true, where $\xi(s)$ is the Riemann's Xi-function [10][12].

$$\xi(s) = \frac{1}{2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s)\pi^{\frac{-s}{2}} \tag{2.3}$$

$$\xi(s) = \xi(1-s) \tag{2.4}$$

$$\zeta(\overline{s}) = \overline{\zeta(s)} \tag{2.5}$$

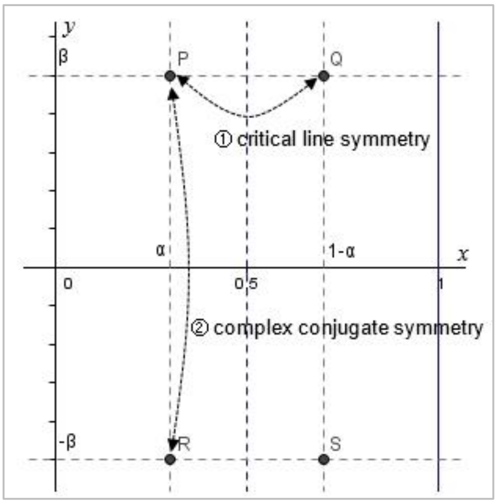
The right side of the equations (2.2) and (2.3) include $\zeta(s)$, so, the zeros of $\zeta(s)$ are also the zeros of $\eta(s)$ and $\xi(s)$.

Lemma 2.1. Equations (2.4) and (2.5) means that there exist two types of symmetries of the zeros of RZF, as in Figure 1.

① **Critical line symmetry:** Symmetry of (2.4), which means that if $s = \alpha + i\beta$ is zero, then $1 - \alpha + i\beta$ is also a zero.

② **Complex conjugate symmetry:** Symmetry of (2.5), which means that if $s = \alpha + i\beta$ is a zero, then $s = \alpha - i\beta$ is also a zero.

Figure 1. Zero symmetries of RZF.



Proof. Let $s = \alpha + i\beta$. First, in (2.5), $\zeta(\alpha - i\beta) = \overline{\zeta(\alpha + i\beta)} = 0$, which is same as $\zeta(R) = \overline{\zeta(P)} = 0$, in Figure 1. So, the complex conjugate symmetry is true. Second, in (2.4), $\xi(\alpha + i\beta) = \xi\{1 - (\alpha + i\beta)\} = 0$, which is same as $\xi(P) = \xi(S) = 0$, in Figure 1. Because of the complex conjugate symmetry, $\xi(S) = \xi(Q) = 0$. So, $\xi(P) = \xi(Q) = 0$, which is the critical line symmetry. ■

3. The Proof of RH

Instead of using RZH, we use DEF, $\eta(x + iy)$, which is analytic for all complex numbers whose real part $x > 0$.

Definition 3.1. Edge lines: Two edge lines of a strip, which are $x = \alpha$ and $x = 1 - \alpha$ or $\eta(\alpha + iy)$ and $\eta(1 - \alpha + iy)$.

Lemma 3.2. To have two zeros such as $\eta(\alpha + i\beta) = \eta(1 - \alpha + i\beta) = 0$, two edge lines should intersect at the origin when $y = \beta$, while y moves from $y = 0$ to $y = \beta$.

Proof. If two edge lines intersect at $(x, y) \neq (0, 0)$, which is not the origin, $\eta(\alpha + i\beta) = \eta(1 - \alpha + i\beta) \neq (0,0)$, so, two edge lines must intersect at the origin. ■

Lemma 3.3. The two edge lines can't intersect at the origin, when mapped by DEF. So, RH is true.

Proof. The intersecting angles are preserved when mapped by an analytic complex function. The two edge lines are parallel and to preserve the intersecting angle, they should not intersect when mapped by DEF, $\eta(s)$, which is analytic for all complex numbers $Re(s) > 0$. So, RH is true. ■

4. Conclusion

In this thesis, we proved RH by the angle preserving properties of an analytic complex function, along with the symmetry properties of the zeros of the RZF.

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