Fun with Lagrange Points

John R. Berryhill

Abstract

Earlier this century, when SOHO and WMAP were in the news, Lagrange points L1 and L2 on the Sun-Earth axis were topics of interest, in addition to L3, the supposed location of mythical Planet X. Now that the James Webb Telescope has been successfully deployed, there is comparable interest in the off-axis points L4 and L5.

The relevant orbital mechanics is that of the *restricted three-body problem,* in which two massive objects are orbiting each other, and a third body, of negligible mass, is introduced. The present note is an exercise in numerically integrating the relevant equations of motion. This approach results in physically realistic depictions of orbits and other features of interest.

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Earlier this century, when SOHO and WMAP were in the news, Lagrange points L1 and L2 on the Sun-Earth axis were topics of interest, in addition to L3, the supposed location of mythical Planet X. Now that the James Webb Telescope has been successfully deployed, there is comparable interest in the off-axis points L4 and L5.

The relevant orbital mechanics is that of the *restricted three-body problem,* in which two massive objects are orbiting each other, and a third body, of negligible mass, is introduced. A comprehensive mathematical theory for this case exists, but it is expressed in terms of gravitational and dynamic quasi-potentials. And it provides no visual tutorial aids, beyond contour plots of equipotential surfaces. The present note takes the alternative approach of numerically integrating the equations of motion directly. This method produces physically realistic depictions of orbits and other features of interest.

METHOD

The computations take place in the center-of-mass (CM) system, whose origin lies on the line joining masses m_1 and m_2. $m_0 = 0$. At each time-step of the computation, the acceleration of each particle m_i is calculated by summing the contributions $\sum_{j \neq i} G\, m_j / R_{ij}^2$ of the other particles, at their current, updated, relative positions. Here, G is the gravitational constant, scaled appropriately for the model.

The method of integration is fourth-order Runge-Kutta, similar to the code in reference NR. Initial values of 2-D position and velocity are required for each particle. Then the program propagates the system forward in time, taking tiny steps. The size and number of steps is a technical issue that does not concern us here.

A model is defined by the values of m_1 and m_2, and the initial x_i and y_i coordinates of every particle, relative to the CM. Each velocity then is set to be proportional to, and perpendicular to, a radius joining that body to the CM. Finally, the scale model value of G is determined. The criterion is that the total potential energy shall equal negative twice the total kinetic energy. This produces perfectly circular orbits centered on the CM.

As the computation steps along, several checks are available to detect possible errors: The CM cannot move; total momentum must remain zero; total angular momentum must remain constant; and, of course, total energy must be conserved.

EXAMPLES

Fig. 1 displays the orbits for a model having a mass ratio $m_2 / m_1 = 4$, perhaps like a binary star system. Rotation is clockwise, and the computation stopped just short of one complete revolution. Red denotes the path of m_2, blue is m_1, and green is the L5 trajectory. The star marks the CM. The interval between any two dots represents one time-step. It might be a surprise that L5 is not a point, but a distinct orbit that lies entirely outside the orbit of the lesser mass.

Fig. 2 presents the same orbits, but as seen from the viewpoint of m_0 as it moves along its path. The green dot is the coordinate origin, on m_0. The red and blue orbits of m_2 and m_1 appear as perfect circles, superimposed. We can check that the radius of the circles is equal to the initial separation between m_1 and m_2 on Fig. 1. For that matter, the radius equals the initial separation of any two of the masses. We understand that, as the particles move, they maintain the same relative separations that they had initially. That is, all the orbits are synchronized so that the entire system rotates rigidly.

A second model amplifies this conclusion. Fig. 3 presents the result for a system with $m_2/m_1=10$. The color-coding is more festive here, with red for m_1, dark blue for m_2. As the larger mass becomes more dominant, the difference between the red and green orbits becomes smaller. In the limit, they will coincide. Fig. 4 re-runs this same model for just the first half-revolution. The red triangle connects the initial positions of the three bodies. The green triangle connects their final positions. The triangles are equilateral, as suggested by Fig. 2. The motion of the system rotates the red triangle into the position of the green.

We conclude that if the Sun-Earth system orbits were perfect circles, L4 and L5 would share Earth's orbit, but lead or lag by ± 60 degrees, or two months.

NOTES & REFERENCES

NR: *Numerical Recipes in C,* Press *et al.*, 1995.

The Lagrange Points, Neil J. Cornish, WMAP Education and Outreach, 1998.

Plotting program: *Veusz 1.24,* Jeremy Sanders *et al.,* GNU Public License, 2016.

L5 and L4 are mirror images of each other.

Draftman's dividers is the tool of choice for comparing distances.

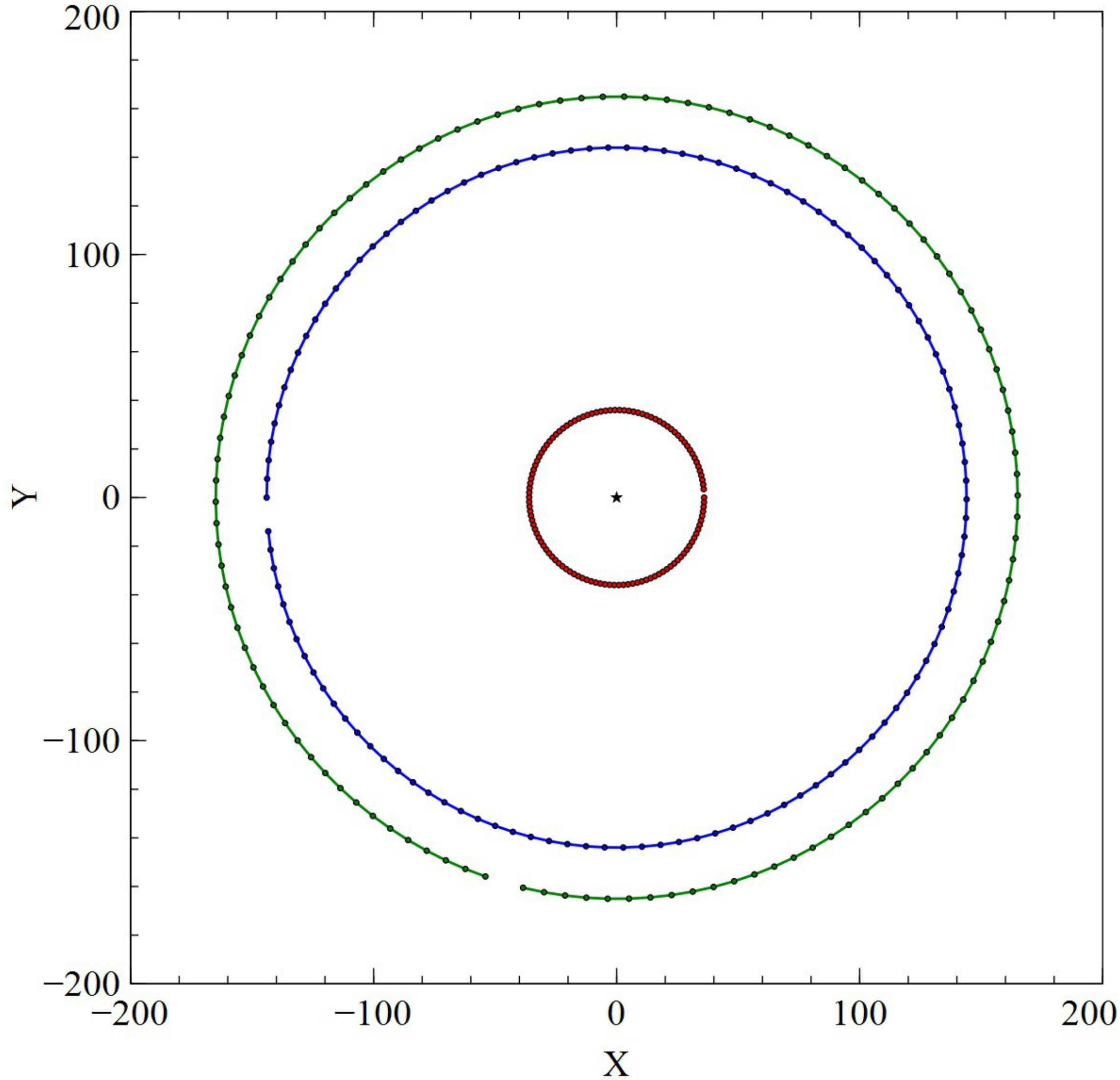

Fig. 1 Orbits in the CM system for a mass ratio of 4:1. The green orbit is L5.

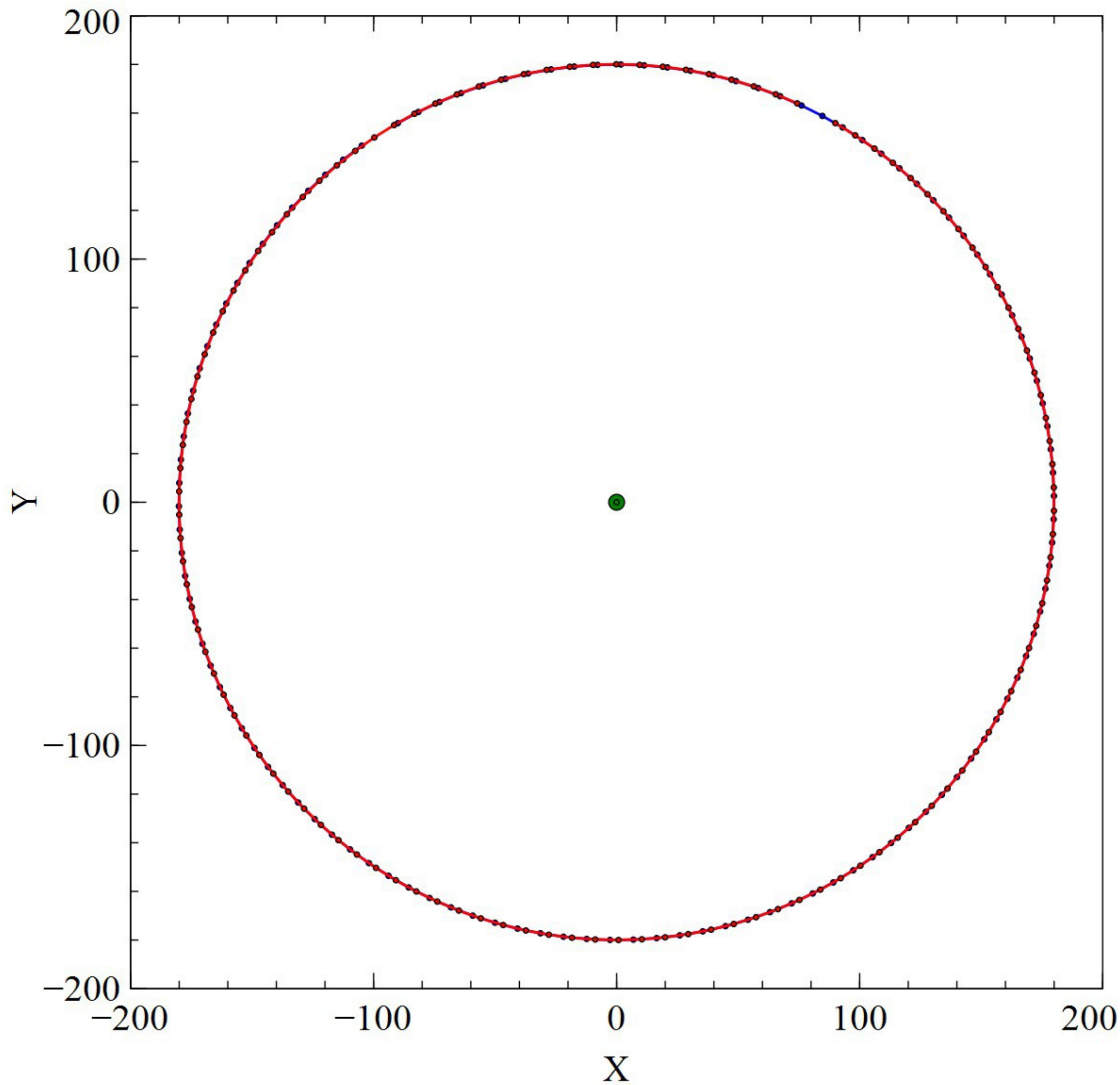

Fig. 2 Orbits of m_1 (blue) and m_2 (red) in relation to the L5 orbit of m_0 .

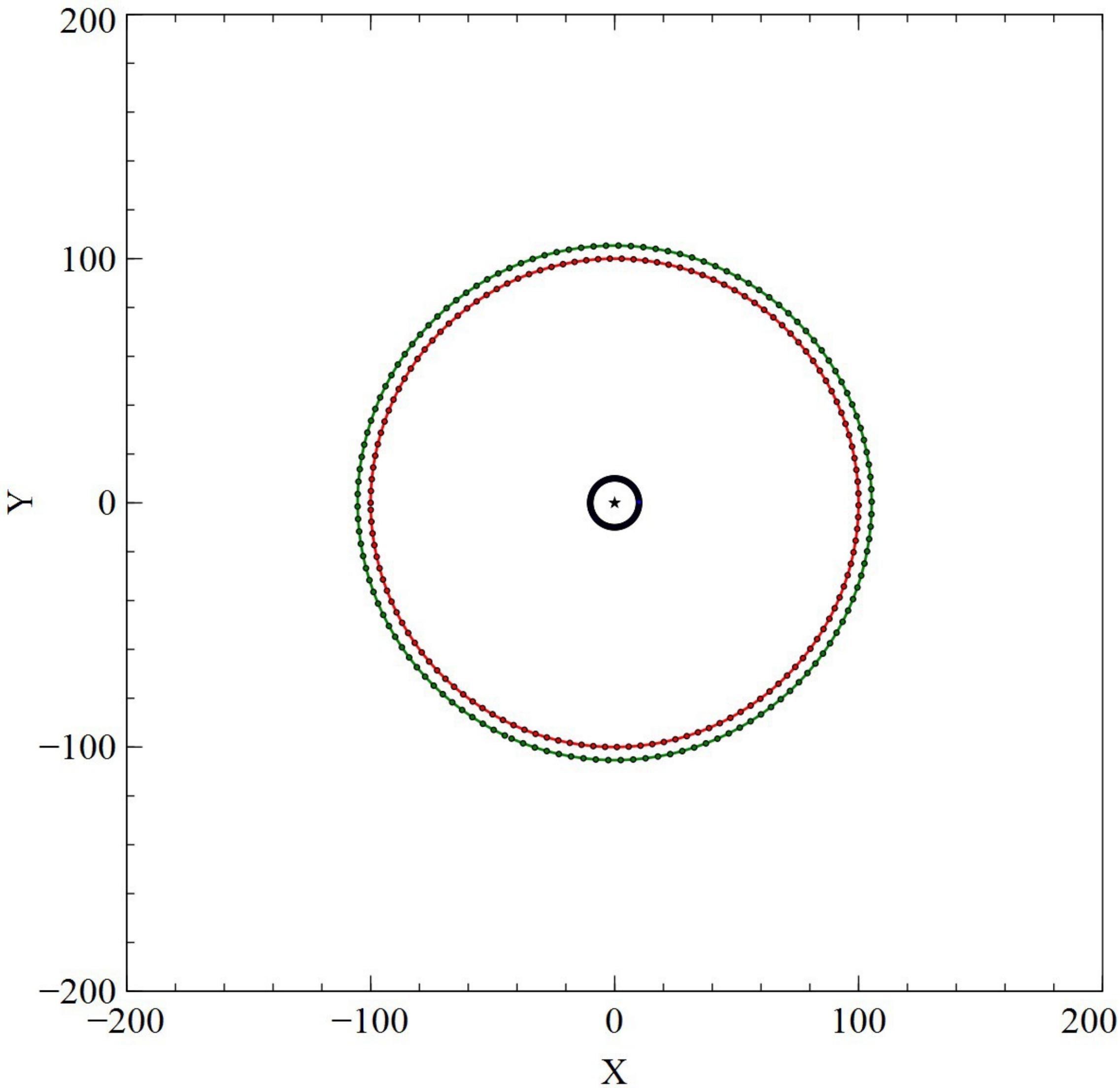

Fig. 3 Orbits in the CM system for a mass ratio of 10:1. Red is the lighter mass. Green is L5.

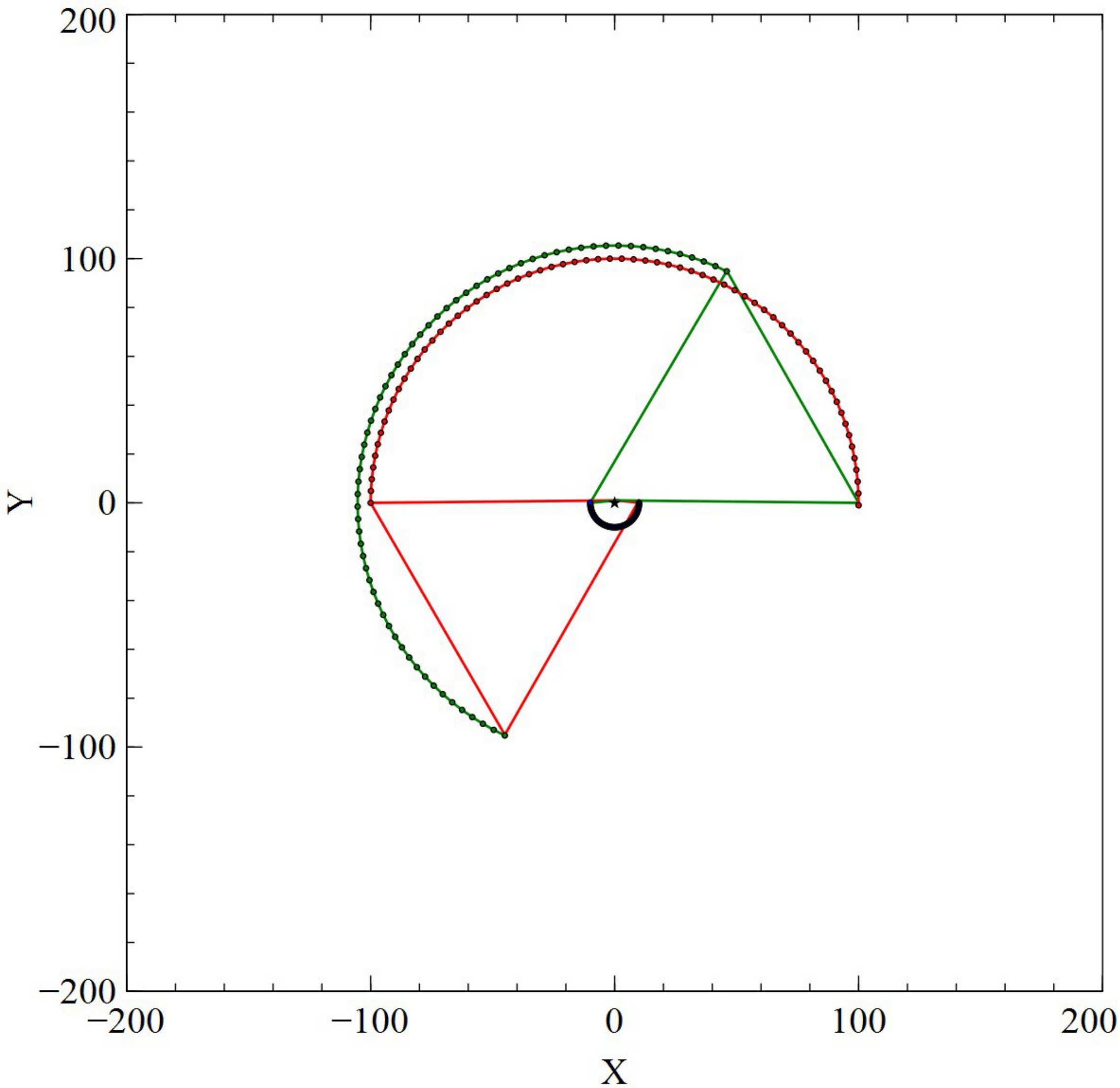

Fig. 4 The first half-revolution. The red triangle connects the initial positions of the three bodies. The green triangle connects their final positions.

Appendix

How It Works

Fig. A. The reader can verify that m_S , m_L, and L_5 define the vertices of an equilateral triangle. The length R of one side is divided at the CM into L and S, and S/L = m_S/m_L. The altitude rising from L_5 to the midpoint of that side bisects the triangle into two 30-60-90 right triangles, whose sides are in the proportions $1:\sqrt{3}:2$. By construction, L = R – S, and the base of the red isosceles triangle is L – S. The tangent of the half-angle φ at the apex of the red triangle is

$$\tan\varphi=\frac{(L-S)/2}{\sqrt{3}(L+S)/2}=\frac{(1-\rho)}{(1+\rho)\sqrt{3}}, \quad \text{where*} \quad \rho=S/L=m_S/m_L.$$

The magnitude of the acceleration $\mathbf{a_S}$ directed toward m_S is ρ times that of $\mathbf{a_L}$ directed toward m_L. The total acceleration at L_5 is the *vector* sum of $\mathbf{a_S}$ and $\mathbf{a_L}$. Proceeding by components, where a is the magnitude of $\mathbf{a_L}$.

$$A_x=a(1-\rho)/2 \quad \text{and} \quad A_y=a\sqrt{3}(1+\rho)/2.$$

The numerical factors arise from the sine and cosine of 30 degrees. Total acceleration **A** is directed at an angle whose tangent is

$$A_x/A_y=(a(1-\rho)/2)/(a\sqrt{3}(1+\rho)/2)=\frac{(1-\rho)}{\sqrt{3}(1+\rho)}=\tan\varphi.$$

Thus the vector total acceleration **A** is directed from L_5 along the red dotted line toward the CM of the binary pair. The direction is correct. The magnitude of **A** is the square root of

$$A^2=A_x^2+A_y^2=(a^2/4)((1-\rho)^2+3(1+\rho)^2)=a^2(1+\rho+\rho^2).$$

Similarly, the squared length of the red dotted line connecting L_5 to CM is

$$hyp^2=3(L+S)^2/4+(L-S)^2/4=L^2+LS+S^2=L^2(1+\rho+\rho^2).$$

A readout of the initial accelerations confirms the expression for A. If we attribute A to a fictitious equivalent mass located at the CM, so that

$$A=GM_{eq}/hyp^2, \qquad \text{then} \qquad M_{eq}=m_L(1+\rho+\rho^2)^{3/2}.$$

As regards stability, I find that if I change the initial y coordinate of m_0 by a part in a thousand, its orbit does not close, and it leaves the picture before completing a second revolution.

* Short / Long = small / large.

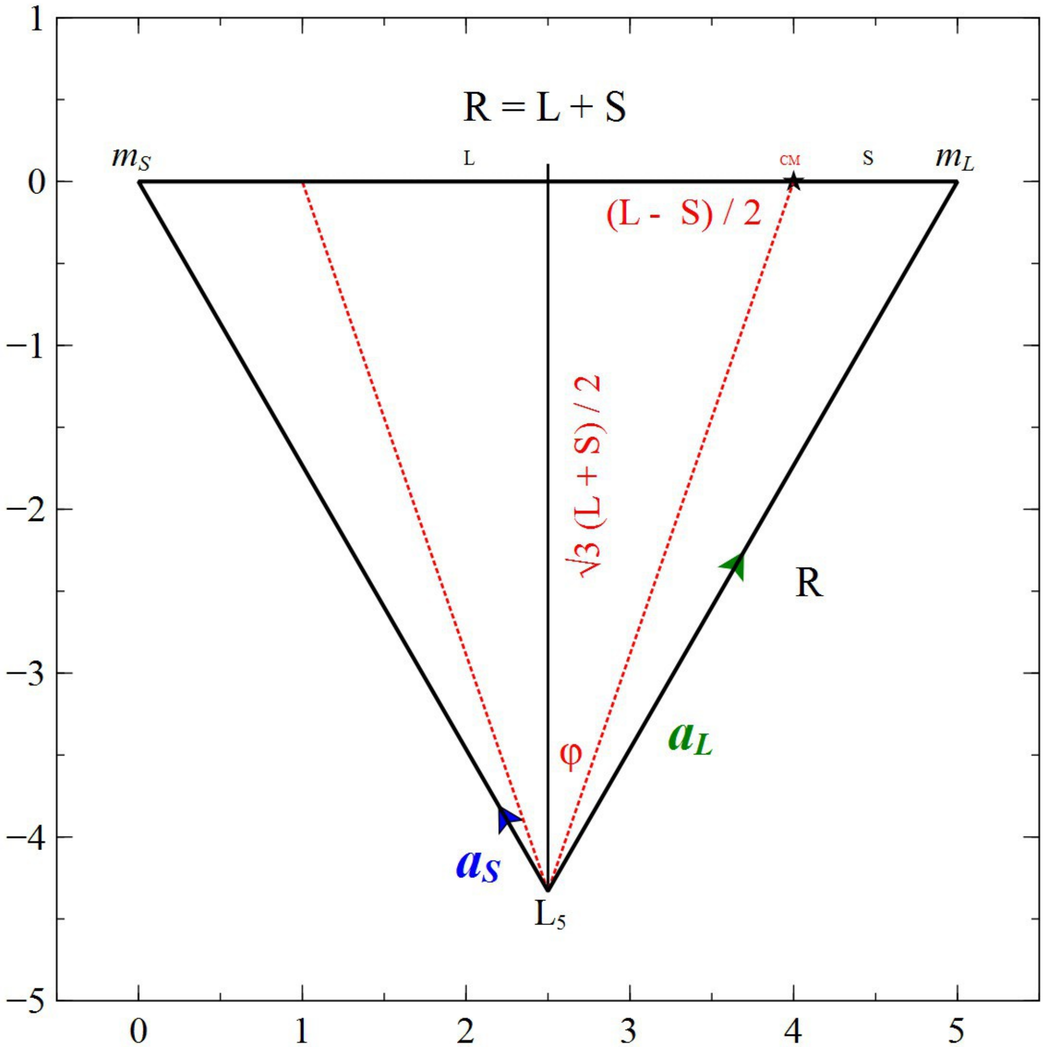

Fig. A: m_S , m_L, and L_5 define the vertices of an equilateral triangle. The length R of one side is divided at the CM into L and S, with S/L = m_S/m_L. The altitude rising from L_5 to the midpoint of the top side bisects the triangle into two 30-60-90 right triangles,.