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Fotonics: Fractional dynamics of optical and photonic phenomena in complex media

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Abstract

Fotonics can be considered as a generalization of the conventional optics and photonics which deals with wave propagation through natural and engineered artificial complex media in the framework of fractional dynamics. In this short note we present an introduction to this new field and then consider its possible applications in the physics of random lasers and metamaterials with anomalous and exotic light-matter interaction behaviors. We have also proposed a new modeling approach for the complex nonlinear metamaterials which exhibit nonlocal and memory effects based on fractional order elements integrated into the underlying circuit of each unit cell.

Keywords: Fotonics; Fractional dynamics; Random lasers; Metamaterials; Nonlocality; Memory effect

1. Introduction

Electromagnetic wave propagation through the complex media has found many applications in different branches of physics and engineering [1-3]. Recently the fractional dynamics is proposed as a reliable framework to study such these phenomena [4-8]. Based on these studies we want here to suggest the new field of fotonics which is in fact a generalization of the conventional optics and photonics and uses the fractional calculus as its modeling tool for describing electromagnetic and optical phenomena occurring in complex structures which can also have fractal properties. The main advantages of fractional operators are that by using them we can consider memory effects, space nonlocalities, fractality, being out of equilibrium and dissipations in the investigation of the physical phenomena in natural and engineered artificial complex media. For this purpose, in the next section we briefly present two important applications of the new field of fotonics. And finally in the Sec. (3) we present our conclusion.

2. Applications of fotonics

In this section we present some new applications of fotonics. First, we present an application in the physics of random lasing and then we propose the notion of fractional order metamaterials. These two areas can be completely connected to each other in the possible future metamaterial based lasing physics.

2-1. Fractional order random laser

As a first application we investigate the application of fractional calculus in the physics of random laser. We propose new theoretical models for the possible applications of complex optical gain media for future novel lasers. In conventional lasers, the optical cavity determines essential characteristics of the lasing modes however in spite this fact that random lasers work on the same principles, but the modes are determined by multiple scattering and not by a laser

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cavity [1, 9]. In recent years the powerful framework of fractional dynamics has found many applications in science and engineering and in particular in description of wave propagation in complex media with disorder [4,5,10-13]. More recently the idea of fractional order random laser (FORL) has been proposed in [6] in which the authors used the classical Letokhov model of a non-resonant random laser that is formulated in terms of the reaction-diffusion equation obeyed by the optical energy density $W(\vec{r},t)$ as:

$$\frac{\partial W(\vec{r},t)}{\partial t} = D\nabla^2 W(\vec{r},t) + \frac{v}{l_g} W(\vec{r},t) \tag{1}$$

where D is the diffusion constant of photons given by $D = \frac{vl_t}{2n}$, where l_t is the transport mean free path and n is the dimensionality of the problem, v is the speed of light in the medium, and l_g ($l_g \gg l_t$) is its characteristic gain length and then based on time fractional generalization of Letokhov diffusion model as:

$$ {}^{C}D_{\tilde{t}}^{\alpha} \tilde{W}(x,\tilde{t}) = K_\alpha \tau_d{}^{\alpha} \frac{\partial^2 \tilde{W}(x,\tilde{t})}{\partial x^2} + (\frac{\tau_d}{\tau_g})^{\alpha} \tilde{W}(x,\tilde{t}) \tag{2}$$

wherein we will consider the scaling $\tau_d = \frac{l_t}{v}$ and $\tau_g = \frac{l_g}{v}$, which are the characteristic time for the scattering and the amplification time of a photon, respectively, l_t is the transport mean free path, the scaled time variable is $\tilde{t} = \frac{t}{\tau_d}$, $K_\alpha = \frac{\Gamma(\alpha+1)\langle x^2(t)\rangle}{2t^\alpha}$ ($\langle x^2 \rangle$ is the average square displacement and $\Gamma(\cdot)$ denotes the Gamma function) is the expression of the generalized diffusion coefficient that has units $m^2 s^{-\alpha}$, ${}^{C}D_{\tilde{t}}^{\alpha}$ is the Caputo fractional derivative operator in dimensionless time of order α $(0<\alpha<1)$ which is defined as:

$$ {}_{0}^{c}D_t^{\alpha} f(t) = \frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(\tau)}{\partial \tau^n} d\tau \tag{3}$$

where $n-1<\alpha<n$ ($n \in N$) and for the case of $\alpha = 1-\varepsilon$ ($\varepsilon \ll 1$)is equal to [14]:

$$ {}_{0}^{C}D_t^{1-\varepsilon} f(t) = D_t f + \varepsilon(D_t f(0)\ln(t) + \gamma D_t f(t) + \int_0^t D_t^2 f(\tau)\ln(t-\tau)d\tau) + O(\varepsilon^2) \tag{4}$$

where $\gamma = 0.577215664901532...$ is Euler–Mascheroni constant. And its Laplace transform reads as:

$$ L\{{}_{0}^{c}D_t^{\alpha} f(t)\} = s^{\alpha} F(s) - \sum_{m=0}^{n-1} s^{\alpha-m-1} f^{(m)}(0) \tag{5}$$

where, $F(s)$ is the Laplace transform of $f(t)$. So we can easily find the time dependent part of the solution of equation (2), $f(t)$, amplification length, L_α and consequently the lasing volume V_α as:

$$ f(t) = E_\alpha([(\frac{\tau_d}{\tau_g})^\alpha - (\frac{n\pi}{L_\alpha})^2 K_\alpha \tau_d{}^\alpha] t^\alpha), \quad L_\alpha = n\pi\sqrt{K_\alpha (\frac{l_g}{v})^\alpha}, \quad V_\alpha = n^3\pi^3 K_\alpha{}^{\frac{3}{2}} (\frac{l_g}{v})^{\frac{3\alpha}{2}} \tag{6}$$

where E_α is the one-parameter Mittag-Leffler defined as:

$$E_\alpha(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(1+\alpha k)} \quad \alpha>0, z\in\mathbb{C} \tag{7}$$

which the special cases as: (8)

$$E_1(\pm z)=e^{\pm z}, \quad E_2(-z^2)=\cos z, \quad E_2(z^2)=\cosh z, \quad E_{\frac{1}{2}}(\pm z^{\frac{1}{2}})=e^z[1+erf(\pm z^{\frac{1}{2}})]=e^z erfc(\mp z^{\frac{1}{2}})$$

where erf ($erfc$) denotes the error function (complementary error function):

$$erf(z):=\frac{2}{\sqrt{\pi}}\int_0^z e^{-u^2}du, \quad erfc(z):=1-erf(z), \quad z\in\mathbb{C}. \tag{9}$$

They are also proposed the superdiffusive regime of the FORL by considering the reaction-diffusion space-fractional equation as [6]:

$$\frac{\partial W(x,t)}{\partial t}=K_\beta\, {}^R D_{|x|}^{\beta} W(x,t)+\frac{v}{l_g}W(x,t) \tag{10}$$

where ${}^R D_{|x|}^{\beta}$ is the Riesz fractional derivatives of order β ($1<\beta<2$)defined as: (11)

$${}^R D_{|x|}^{\beta} g(x)=-\frac{1}{2\cos(\frac{\pi\beta}{2})\Gamma(m-\beta)}\cdot\frac{d^m}{dx^m}\left(\int_{-\infty}^{t}(x-\chi)^{m-\beta-1}g(\chi)d\chi+(-1)^m\int_{t}^{\infty}(x-\chi)^{m-\beta-1}g(\chi)d\chi\right)$$

where $m-1\le\beta<m\in Z^+$, which in the case of $\beta=2-\varepsilon$ is equal to [14]:

$${}^R D_{|x|}^{2-\varepsilon} g(x)=(D_{|x|})^2 g+\varepsilon(\gamma(D_{|x|})^2 g+...). \tag{12}$$

The above-mentioned approach for the subdiffusive and superdiffusive optical gain media can be considered as a theoretical framework for the FORL. Also, a theoretical model for random lasers based on the superdiffusive optical gain medium has been presented in [15] in which the authors considered the generalized Lambert-Beer law as:

$$p(l)=\sigma^{-\beta}l^{(\beta-1)}E_{\beta,\beta}(-(\frac{l}{\sigma})^\beta) \tag{13}$$

where σ is a scattering characteristic length that decreases as the scattering of the medium becomes stronger, l is the step lengths and $E_{\beta,\beta}$ is the two-parameter Mittag-Leffler function:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)} \quad \alpha,\beta\in\mathbb{C}, \mathrm{Re}(\alpha)>0 \tag{14}$$

where $0<\beta\le 1$, in order to include the case of superdiffusion. The two-parameter Mittag-Leffler function has also the special cases as:

(15)

$$E_{1,1}(z)=e^{z},\quad E_{1,2}(z)=\frac{e^{z}-1}{z},\quad E_{2,1}(z)=\cosh\sqrt{z},\quad E_{2,2}(z)=\frac{\sinh\sqrt{z}}{\sqrt{z}},\quad E_{\frac{1}{2},1}(z)=e^{z^{2}}erfc(-z)$$

We can here generalize the above theoretical models in several possible ways:
Case1: space-time fractional Letokhov model for the FORL:

$$^{C}D_{\tilde{t}}^{\alpha}\tilde{W}(x,\tilde{t})=\left(K_{\alpha,\beta}\tau_{d}{}^{\alpha}\right){}^{R}D_{|x|}^{\beta}\tilde{W}(x,\tilde{t})+(\frac{\tau_{d}}{\tau_{g}})^{\alpha}\tilde{W}(x,\tilde{t}) \tag{16}$$

Case2: variable order space-time fractional Letokhov model for the FORL: because of the interaction between emitted light and the medium and regarding to this fact that the order of fractional operators can be considered as an index of the complexity of the medium we can consider the cases of $\alpha=\alpha(t)$ and $\beta=\beta(x,t)$ which in these cases fractional order derivative reads as [16]:

$${}_{0}^{c}D_{t}^{\alpha(t)}f(t)=\frac{d^{n}}{dt^{n}}\left(\frac{1}{\Gamma(n-\alpha(t))}\int_{0}^{t}(t-\tau)^{n-\alpha(t)-1}f(\tau)d\tau\right) \tag{17}$$

with the following property:

$$\begin{aligned}{}_{a}^{c}D_{t}^{\alpha(t)}(t-a)^{\gamma}&=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+1)}(t-a)^{\gamma-\alpha(t)}\\&-\alpha'(t)\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha(t)+2)}(t-a)^{\gamma-\alpha(t)+1}\times[\ln(t-a)-\Psi(\gamma-\alpha(t)+2)]\end{aligned} \tag{18}$$

where Ψ is the Psi function which is equal to the derivative of the logarithm of the Gamma function i.e.:

$$\Psi(t)=\frac{d}{dt}\ln(\Gamma(t))=\frac{\Gamma'(t)}{\Gamma(t)} \tag{19}$$

Case3: random lasing can be produced using some new fractal media such as fractal perovskite thin films [17]. So we can propose a new model based on the Hausdorff derivative which successfully can describe processes in complex fractal and porous media [18-20] and is defined as:

$$^{H}D_{t}^{\alpha}f(t)=\lim_{t\to t'}\frac{f(t)-f(t')}{t^{\alpha}-t'^{\alpha}}=\frac{1}{\alpha t^{\alpha-1}}\frac{df}{dt} \tag{20}$$

where t and t', respectively, represent the final and internal time instances and α is the time fractal dimensionality. So, a fractal Letokhov model can be considered as a promising theoretical framework for the random lasing due to such fractal structures and future fractal random laser as:

$$^{H}D_{t}^{\alpha}\tilde{W}(x,\tilde{t})=\left(K_{\alpha,\beta}\tau_{d}{}^{\alpha}\right){}^{H}D_{x}^{\beta}\tilde{W}(x,\tilde{t})+(\frac{\tau_{d}}{\tau_{g}})^{\alpha}\tilde{W}(x,\tilde{t}) \tag{21}$$

2-2. Fractional Order Metamaterials

As the second application here we briefly introduce the new concept of fractional order metamaterials (FOMs). FOMs are in fact metamaterials [21,25] composed of unit cells which can be modeled using the fractional order electric and electronic elements. Recently we have proposed the notion of fractional electromagnetic metamaterials [5] based on the framework of fractional electrodynamics [4] for describing nonlocal phenomena in electromagnetic metamaterials. Also, in the last decade fractional calculus has found many applications in physics, electrical and electronic engineering. Among these applications here we can mention fractional order modeling of: signals and systems [26] semiconductor diodes [27,28], anomalous charge transport in nanosystems [29], thermal modeling and temperature estimation of a transistor junction [30], electrical circuits and their applications [31- 40] and so on. In this work we want to propose the notion of using fractional order electric and electronic elements in the modeling new type of complex metamaterials which we call them fractional order metamaterials (FOMs). The following figure (Fig. (1)) shows general structure of the FOM which is composed of unit cells of fractional order elements.

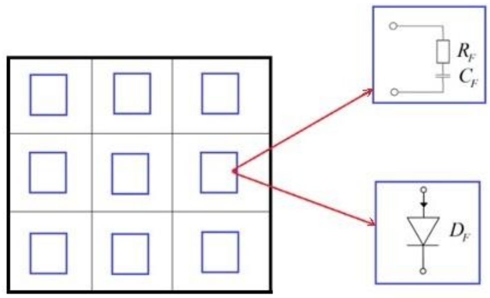

Figure1: schematic of FOM constituent elements.

As the above figure shows a FOM is composed of complex unit cells which are equivalent to the fractional order elements instead of the conventional electrical and electronic elements. Fractional order resistor, capacitor and inductor are elements whose governing equations are respectively as follows [36]:

$$i(t) = \frac{1}{R_F} D^{1-\alpha} v(t) \tag{22}$$

$$q(t) = C_F D^{1-\beta} v(t) \tag{23}$$

$$v(t) = L_F D^{\gamma} i(t) \tag{24}$$

where in the above equations D^{μ} is the fractional derivative of order $0 < \mu \leq 1$, $i(t)$ is the total current, $v(t)$, difference voltage of the element, R_F, resistance of the fractional resistor, $q(t)$, the total charge of the fractional capacitor and C_F is its capacity and finally L_F is inductance of the fractional order inductor. Since the electromagnetic characteristics of the metamaterials depend on their unit cells components in addition to their shape, size, orientation, and alignment of unit cells, we expect completely different physical characteristics in comparison with the conventional metamaterials and metasurfaces. In our previous work [5] we proposed a theoretical framework to study the electromagnetic properties of such new type of metamaterials which exhibit nonlocal properties and those which cause anomalous light transport and show exotic light-matter interactions [41-43].

The above-mentioned fractional order circuit theory modeling approach can provide an accurate model for the future complex and nonlinear metamaterials working in radio-frequency, microwave or even in higher frequency applications and also for some new nonlinear optical metamaterials. As a special example of this approach, we can use it for the modeling of a complex nonlinear metacrystal formed from resonant circuit elements that couple strongly to the magnetic field by the split ring resonator (SRR) medium [44 and Refs therein] with its equivalent inductively driven RLC circuit model shown in Fig. (2).

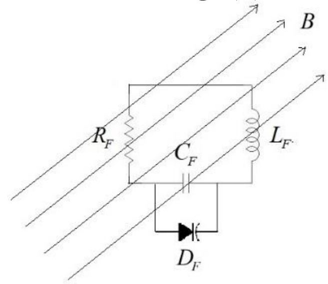

Figure2: Equivalent effective fractional circuit model for SRR.

The produced current $I_\alpha(t)$ due to the induced electromotive force $\varepsilon_\alpha(t)$, satisfies:

$$L_F D^\alpha I_\alpha + R_F I_\alpha + V_{D_F} = \varepsilon_\alpha(t) = -D^\alpha \Phi_m \tag{25}$$

where L_F , R_F , V_{D_F} are, respectively, the distributed inductance, distributed resistance and the induced voltage across the effective capacitor C_F of the circuit.

3. Conclusion

Fotonics as a generalization of the conventional optics and photonics can be considered as new framework when we deal with wave propagation through natural and engineered artificial complex media in the framework of fractional dynamics. we present an introduction to this new field and then some of its possible applications. As the first application we present a theoretical framework for the physics of random lasers. Based on this fact that in the physics of random lasers we deal with the complex optical gain media such as Lévy type glasses and as a result Lévy regime for the light propagation in such media we can expect an important role for fractional dynamics in understanding the physics behind new random lasing phenomena with many possible applications. For the second application we proposed the notion of fractional order metamaterials which are in fact new type of complex nonlinear metamaterials which are composed of and can be modeled by the fractional order elements which will have new optical and photonic properties that are needed to be experimentally examined in future.

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