ON THE SOLUTION OF EINSTEIN'S EQUATIONS, GENERAL METRIC FOR A FLUID AND HOMOGENEOUS STAR

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Abstract :

From the mathematical aspects of the Schwarzschild's metric, we present different methods of changing variables which tend to prove that the central singularity hypothesis does not exist. The geometric interpretation of the black hole is not mathematically convincing from the Schwarzschild metric. We recall the different mathematical approaches of several authors, the examples of the Painleve metric and its complex variant, then of the Schwarzschild metric, to deduce a metric with a throat sphere which leads to a mirror spacetime. Subsequently, we deduce the possibility of a bi-metric tangent to the Schwarzschild metric's throat sphere. We will also show that a false interpretation of the variables of the Schwarzschild metric can lead to false physical deductions and in particular to the concept of singularity. We compute the general solution of Einstein's equations in the presence of a non-zero energy tensor, i.e., for a homogeneous fluid ball with energy conditions. Our method of resolution involves a reformulation of the Einstein equation and the integration of the differential system. The metrics found are asymptotic to the Schwarzschild metric outside the fluid ball. We will present assumptions for the pressure inside the fluid ball and derive the corresponding metrics. Then, by solving the continuity equation of the energy-impulse tensor, we deduce an expression for the pressure inside the star which permits the expressing of the interior and exterior metrics.

A- Introduction

The Schwarzschild metric has become popular since the beginning of the 20th century for two main reasons.

First, is that it was announced as one of the first explicit solutions of Einstein's equations in the vacuum, which was demonstrated significant advancement for general relativity. The enthusiasm of cosmologists about the Schwarzschild metric was later undermined by 20th century mathematicians and physicists Painlevé and Eddington (1921) [1,2] and then in the 21st century Vankov (2011), Mizony and Crothers (2015) [8, 21,26,27] who have addressed the subject. The physical interpretation of variables in the metric cannot be subject to untested theories of time reversion, singularity, or even spacetime rupture. However, the popular science press, eager for sensationalism, and the science fiction's movies have contributed to make people forget the basis of Einstein's equations and consequently of the role of variables in a metric.

The second reason for this popularity is that the gravitational field deduced from the metric answers the problem of the "advance of Mercury's perihelion" and the gravitational lensing effects. The scientific basis of this discovery has been poorly disseminated and misunderstood. The cinema and the popular press, as well as the computational simulation playing, under the pretext of educating and entertaining the general public, have made erroneous representation of the phenomena. Until very recently, the radiophotography of the M87* black hole into galactic center of the same designation has still confused both the public and some serious physicists. This image reconstructed from the contrast of radio radiation is certainly of great interest but cannot confirm that it is a black hole. It may be a very massive neutron star or "strange star" for a redshift magnitude of 5 to 6 surrounded by a disk of hot material.

Also, many mathematicians and physicists have warned about the interpretation of the Schwarzschild metric. Even if it seems to meet some physical assumptions such as stability, asymptotic convergence at infinity to a Minkowski metric and spherical symmetry, the questions raised by this metric and especially by the description of the black hole it describes are of three kinds.

First, this metric is deduced from the Einstein tensor expressed in a vacuum. The equations deduced from it express both the vacuum and the presence of a central mass! This contradiction is still not resolved.

Secondly, the Schwarzschild radius, the limit beyond which space-time seems to be no longer real, does not have a coherent physical interpretation. The various papers on the black hole problem seem to omit the reality of the 4-dimensional topology and continue to explain a 3-dimensional time-dependent phenomenon, which is not the same physical phenomenon.

Thirdly, Birkhoff's theorem is often misused by some physicists, who instead of considering a 4D spherical symmetry, continue to solve Einstein's equations with the assumption of a 3D central symmetry, which leads to misinterpretation of Penrose and Hawking's theorems (1965-1970) [9, 11] on the emergence of the singularity by gravitational collapse.

In this work, we present, from mathematical calculations, alternative physical interpretations of the Schwarzschild metric and the probable nature of the singularity. Also, we study the general inner and outer solution of the Einstein equations for a homogeneous fluid star.

B- Reminder on the Schwarzschild metric in vacuum.

We search a metric that is expressed in the following form:

$$ds^2 = Ad\xi^2 - Bd\mathcal{R}^2 - Edw^2 \qquad (1)$$

with $dw^2 = (d\theta^2 + \sin^2\theta\, d\varphi^2)$

A, B, E are functions of the variable $\mathcal{R}$ and they must verify the equations of Einstein in a vacuum.

$$G_{ik} = R_{ik} - \frac{1}{2}Rg_{ik} = 0 \qquad (2)$$

From the physical point of view, the problem is less obvious and has raised many questions (Painlevé 1921, Chazy 1930, Edditon 1960, , Mizony 2015, Crothers 2015) [1,4, 8, 26,27].

We recall that Albert Einstein, was the first to raise the Schwarzschild problem and gave an approximate solution before Schwarzschild gave an exact solution in 1916 through two remarkable publications.

One can, in a quick way, solve the mathematical problem in vacuum, then interpret the constants by the presence of a central mass, by prejudging the interpretation of the variables of the metric. The latter is adapted to the physical problem posed a posteriori, that of the black hole.

This method, used in many articles and which is also taught, has often been contested.

We first define the variable $(\xi, \mathcal{R}, \theta, \varphi)$ of the space-time where the metric will be calculated.

The variable «$\mathcal{R}$ » is not the radial distance but is a monotonic function of the radial distance $OM = r$, and when r becomes very large, we can assimilate r to $\mathcal{R}$.

Then we have: $\mathcal{R}(r) \approx r$ very far from the star.

After mathematical resolution of the equations (2) in vacuum, we finally find the expression of the following metric:

$$ds^2 = \left(1 + \frac{C}{\mathcal{R}}\right)d\xi^2 - \left(1 + \frac{C}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 - \mathcal{R}^2 dw^2 \qquad (3)$$

whose determinant is equal to:

$$det(g) = -\mathcal{R}^4 \sin^2(\theta)$$

The constant C will be determined according to the physics laws compelled by system’s limit conditions.

Whit the correct constant, this metric is a spherical symmetric, static, and asymptotically equivalent to the Minkowski metric at infinity. It is written as the Schwarzschild metric, with

$C = -R_s = -\frac{2MG}{c^2};$

M being the mass of the star, G the gravitational constant, c the speed of light which is taken to be equal to 1 in the following and R_s called the Schwarzschild radius or the black hole horizon. In many scientific papers, the variables ξ and $\mathcal{R}$ are arbitrarily and respectively assimilated to time t and radial distance r.

We have:

$$ds^2 = \left(1 - \frac{R_s}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{R_s}{r}\right)}dr^2 - r^2 dw^2 \qquad (4)$$

We note that equation (4) shows the presence of 2 singularities, in $r = 0$ and in $r = R_s$ the horizon of the black hole.

In this paper, we consider that far from the star of mass M and radius "a" , $r \gg a$, the variables ξ and $\mathcal{R}$ are respectively assimilated to t the time and r the radial distance. Close to the star and inside the star, we cannot consider these variables to time and radial distance.

C- On the changes of variables in the metric

By making changes in variables, i.e. without changing the physical cause, what impact do we observe on the metric and curvature?

Let us posit: $\mathcal{R} = \mathcal{R}(r) \implies d\mathcal{R} = \frac{\partial \mathcal{R}}{\partial r} dr$

the Schwarzschild metric (3) is written :

$$ds^2 = \left(1 + \frac{C}{\mathcal{R}(r)}\right) d\xi^2 - \left(\frac{\partial \mathcal{R}}{\partial r}\right)^2 \left(1 + \frac{C}{\mathcal{R}(r)}\right)^{-1} dr^2 - \mathcal{R}^2(r) dw^2 \quad (5)$$

It satisfies the physical compatibility conditions if and only if it is asymptotically flat for large r , that is

$$\begin{cases} \left(\frac{\partial \mathcal{R}}{\partial r}\right)^2 \longrightarrow 1 \ when\ r \longrightarrow \infty \\ \mathcal{R}(r) \longrightarrow r \ \ when\ r \longrightarrow \infty \end{cases}$$

Exemples :
$$\mathcal{R}(r) = (r^n + a^n)^{\frac{1}{n}}, \quad \mathcal{R}(r) = r + a, \quad \mathcal{R}(r) = r + f(r) \quad (5.1)$$

with $f(r)$ a function derivable and monotone on $[0\,; +\infty]$, asymptotically flat at infinity. The examples are infinitely numerous.

All these metrics (5.1) verify the Einstein equation in vacuum $G_{ik} = 0$, are spherically symmetric, are static, and are asymptotically flat. By change of variables, there are infinitely many metrics equivalent to the Schwarzschild metric.

Particular metric

We can also make the change of variable on the variable ξ by posing $\xi = t - \int \psi(\mathcal{R})$ which is translated by

$$d\xi = dt - \psi d\mathcal{R}$$

ψ is a continuous and differentiable function on $[0\,; +\infty]$.

Therefore, the Schwarzschild metric is written :

$$ds^2 = \left(1 + \frac{2C}{\mathcal{R}}\right) dt^2 - \left(\left(1 + \frac{2C}{\mathcal{R}}\right)^{-1} - \psi^2 \left(1 + \frac{2C}{\mathcal{R}}\right)\right) d\mathcal{R}^2 - 2\psi(\mathcal{R}) \left(1 + \frac{2C}{\mathcal{R}}\right) dt d\mathcal{R} - \mathcal{R}^2 dw^2$$

with $\lim\limits_{\mathcal{R} \to \infty} \psi(\mathcal{R}) = 0$

Discussion on sign of the constant

If $C \le 0$, with $C = -R_s$

We choose $\psi(\mathcal{R}) = \left(\frac{R_s}{\mathcal{R}}\right)^{\frac{1}{2}} \left(1 - \frac{R_s}{\mathcal{R}}\right)^{-1}$, and :

$$d\xi = dt - \sqrt{\frac{R_s}{\mathcal{R}}} \left(1 - \frac{R_s}{\mathcal{R}}\right)^{-1} d\mathcal{R}$$

With this change of variable, the metric is written :

$$ds^2 = \left(1 - \frac{R_s}{\mathcal{R}}\right) dt^2 - d\mathcal{R}^2 - 2\sqrt{\frac{R_s}{\mathcal{R}}} dt d\mathcal{R} - \mathcal{R}^2 dw^2 \quad (6)$$

This formulation is called the Painleve-Gullstrand metric, Gullstrand (1922), Fric (2013), Crothers (2015)[3, 23,26].

It can also be written:

$$ds^2 = dt^2 - \left(d\mathcal{R} + \sqrt{\frac{R_s}{\mathcal{R}}} dt\right)^2 - \mathcal{R}^2 dw^2 \quad (7)$$

If $C \ge 0$

We choose $\psi(\mathcal{R}) = i\left(\frac{R_s}{\mathcal{R}}\right)^{\frac{1}{2}} \left(1 + \frac{2C}{\mathcal{R}}\right)^{-1}$

And the metric be written :

$$ds^2 = \left(1 + \frac{2C}{\mathcal{R}}\right) dt^2 - d\mathcal{R}^2 - 2i\sqrt{\frac{2C}{\mathcal{R}}} dt d\mathcal{R} - \mathcal{R}^2 dw^2 \quad (8)$$

It is a Riemannian metric on a holomorphic fibration tangent to the space $\mathbb{C}^2$ which is isomorphic to $\mathbb{R}^4$ Dumitrescu (2001) [17].

It can also be written :

$$ds^2 = dt^2 - \left(d\mathcal{R} + i\sqrt{\frac{R_s}{\mathcal{R}}}dt\right)^2 - \mathcal{R}^2 dw^2 \tag{9}$$

We do not wish to make a physical interpretation of the metric. Our approach is to find a mathematical expression of the metrics, by solving a tensor equation. The physical interpretation will be done as needed.

Remark

For the case $\boldsymbol{C \leq 0}$ $\boldsymbol{C = -R_s}$, we can see from equation (6) that the singularity in $r = R_s$ does not exist and equation (6) is written :

$$ds^2 = \tag{10}$$

$$\left(1 - \frac{R_s}{\mathcal{R}(R_s)}\right)dt^2 - d\mathcal{R}^2 - 2\sqrt{\frac{R_s}{\mathcal{R}(R_s)}}dtd\mathcal{R} - \mathcal{R}^2(R_s)dw^2$$

On the Minkowski's vacuum metric

The Minkowski metric which represents an empty spacetime is written :

$$ds^2 = dt^2 - dr^2 - r^2 dw^2$$

The determinant of this metric is equal to :

$$-r^4 sin^2(\theta)$$

From the Minkowski metric, let us apply the following changes of variables:

$$dt = dT + \frac{\Phi}{1 - \Phi^2}dR$$

$$dr = \Phi dT + \frac{1}{1 - \Phi^2}dR$$

With $\Phi = \Phi(R)$ continuous function except at localized points, and the function $\Phi(R)$ tends asymptotically to 0 at infinity.

The change of variable is written in matrix form:

$$(dt; dr) = \mathcal{M}\begin{pmatrix} dT \\ dR \end{pmatrix} = \begin{pmatrix} 1 & \frac{\Phi}{1 - \Phi^2} \\ \Phi & \frac{1}{1 - \Phi^2} \end{pmatrix}\begin{pmatrix} dT \\ dR \end{pmatrix}$$

With $\mathcal{M}$ the matrix of change of variable. It is an isometry since the determinant of this matrix is equal to 1.

The metric is written :

$$ds^2 = (1 - \Phi^2)dT^2 - (1 - \Phi^2)^{-1}dR^2 - r^2\big(\Phi(R)\big)dw^2$$

Even if we posit $\Phi(R) = \sqrt{\frac{R_S}{R}}$, this metric cannot be interpreted as a Schwarzschild metric.

There is indeed a difference between the so-called Schwarzschild solution and the empty space metric.

We will assume that the Schwarzschild metric is the particular solution of the Einstein's equation.

To consider that the energy tensor is zero ($T_\mu^\nu = 0$)and that the solution represents a central mass is nonsensical. We demonstrate that the solution of the Einstein's equation with second term allows to find a general solution in a non-empty space-time, and that the particular solution is asymptotically the Schwarzschild one (see Chapter F)

D- Remarks on singularity

All Schwarzschild metrics seem to have a physical singularity at the center of mass, i.e. at $\mathcal{R}(r) = 0$.

The Kretschmann scalar for these metrics is written using the Riemann tensor:

$$R^{\mu\nu\sigma\sigma}R_{\mu\nu\sigma\sigma} = \frac{48{R_s}^2}{\mathcal{R}^6(OM)}$$

This shows first that the singularity in $r = R_s$ is purely geometric, that the variable $\mathcal{R}$ is not the radial distance, so there is no singularity in OM=0 unless $\mathcal{R}(0) = 0$.

According to equation (5.1), there always exists a metric solution of the Einstein equations such that $\mathcal{R}(0) \neq 0$.

$$\mathcal{R}(0) = (0^n + a^n)^{\frac{1}{n}}$$
$$\mathcal{R}(0) = 0 + a \qquad if\ a \neq 0$$
$$\mathcal{R}(0) = 0 + f(0), \qquad if\ \ f(0) \neq 0$$

According to current definitions, gravitational singularities in general relativity are locations in spacetime where the gravitational field becomes infinite. Some physicists such as Dewitt (1967), Dvali

2014, Farnes 2018, Barrau 2019) and philosophers such as Saint-Ours (2011), [10, 25, 28,32,20] propose that because the density of matter seems to tend towards infinity in the singularity, the laws of behavior of spacetime are no longer compatible with classical physics. This has given rise to a multitude of theories, such as quantum gravity, loop quantum gravity, string theory applied to black holes, space-time reversing, etc...

Yet, regarding the definition of singularities, there on still debates and order general disagreement many the physicists, mathematicians and philosophers community regarding the definition of singularity. (Saint Ours 2011, Fromholz 2014) [20,24].

Although it changes the local geometry, it seems difficult to speak of a singularity as a point that lies at a location in spacetime. Therefore, some physicists and philosophers cautiously propose to speak of "singular space-time" instead of "singularities". The most important definitions refer either to incomplete paths or to the idea of "missing space" in space-time. The idea is often called "singular structure with pathological behavior". (Curiel and Bokulich 2018). [29]

R. Penrose and S. Hawking and [9,11, 13] managed to show the existence of a "singularitý" during gravitational collapse. However, one should be careful with the meaning of this term. In their work, these authors do not prove the existence of a point where the geometry of spacetime would become singular in the mathematical sense. What they have explained is the existence of half-geodesics time-light specific to, which are incomplete. A zone of space-time where the history of the objects that penetrate it stop after a finite time.

To illustrate schematically this mathematical object, let us write $t = Cte$ and $\theta = \frac{\pi}{2}$ in the Schwarzschild metric. We have:

$$ds^2 = -\left(1 - \frac{R_s}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 - \mathcal{R}^2 d\varphi^2$$

Let's look for a representation in a space such that:

$$ds^2 = -dz^2 - d\mathcal{R}^2 - \mathcal{R}^2 d\varphi^2$$

$$= -\left(1 + \left(\frac{dz}{d\mathcal{R}}\right)^2\right) d\mathcal{R}^2 - \mathcal{R}^2 d\varphi^2$$

By identifying the two relations we have:

$$\left(1 + \left(\frac{dz}{d\mathcal{R}}\right)^2\right) = \frac{1}{1 - \frac{R_s}{\mathcal{R}}}$$

The visualization of the Schwarzschild space-time is obtained with the help of a 2D surface immersed in a space-time of dimension 3. The Schwarzschild surface thus visualized by the function $z(\mathcal{R})$, for $t = Cte$ and $\theta = \frac{\pi}{2}$ is written:

$$z(\mathcal{R}) = \int_0^{\mathcal{R}} du \sqrt{\frac{\frac{R_s}{u}}{1 - \frac{R_s}{u}}} = 2\sqrt{R_s(\mathcal{R} - R_s)} + z_0$$

i.e

$$z^2 = \left(4R_s(\mathcal{R} - R_s)\right) + 4z_0\sqrt{R_s(\mathcal{R} - R_s)} + {z_0}^2$$

It is a Flamm paraboloid, with a throat circle for $\mathcal{R} = R_s$. This figure represents in 3D space two 2D parabolic layers connected by a 1D throat circle of parameter $\mathcal{R} = R_s$.

For r very large, i.e. far from the center of mass, we have :

$$\left(1 + \left(\frac{dz}{dr}\right)^2\right) \approx 1$$

$$z(r) = z_0$$

At infinity, this 2-dimensional surface immersed in a 3-dimensional space is a visualization of Schwarzschild's asymptotic space-time, i.e. Minkowski's flat space-time.

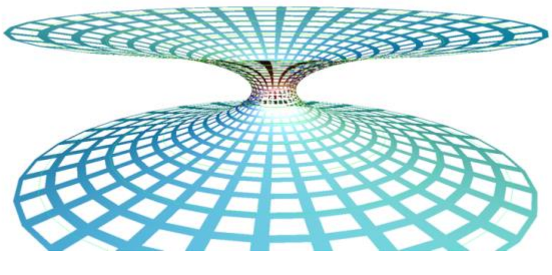

Figure 1. 3D visualization of the Schwarzschild hyper-surface

The physical singularity at $r = 0$ does not exist since the “throat circle” at $r = R_s$ is the physical limit for any object plunging into the Schwarzschild metric. An object coming from the upper sheet which dives towards the center of mass following a parabolic geodesic crosses the gorge then slides towards the lower sheet and disappears forever. For us, its story ends. In these interpretations of the black hole, it is the zone of no return where the history of the object stops.

The hypothesis that consists in presenting the throat circle as impossible to cross because of the pulsation of the circle remains to be verified.

Note : The above visualization of the Schwarzschild space is a sheet of dimension 2 with a hole, but the reality, with 4 dimensions, would rather be a hyperplane of dimension 4 with a throat sphere of dimension 3.

An object coming from the upper sheet and having a slightly oblique trajectory will go around the throat one or more times before passing on the other sheet and disappearing forever. The throat sphere acts as a transition zone between our space-time and another space-time to be determined.

We will call the upper space-time E^+ that of the upper sheet, the one in which we live and the lower space-time E^-, the complementary space-time of the lower sheet.

We will then define a metric on each Space-Time: ${g_{ik}}^+$ and ${g_{ik}}^-$ and an Einstein tensor for each of the metrics such that ${G_{ik}}^+$ and ${G_{ik}}^-$ will have to verify the equations of relativity respecting the continuity on the throat sphere of between the two Space-times.

We have:

$$ {G_{ik}}^+ = {R_{ik}}^+ - \frac{1}{2} R^+ {g_{ik}}^+ $$
$$ {G_{ik}}^- = {R_{ik}}^- - \frac{1}{2} R^- {g_{ik}}^- $$

This defines two metrics .
In our space-time, with positive masses

$$ ds^{2+} = \left(1 - \frac{R_s}{\mathcal{R}}\right) d\xi^2 - \left(1 - \frac{R_s}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 - \mathcal{R}^2 dw^2 \tag{11} $$

And in the mirror space-time with negative masses.

$$ ds^{2-} = \left(1 + \frac{R_s}{\mathcal{R}}\right) d\xi^2 - \left(1 + \frac{R_s}{\mathcal{R}}\right)^{-1} d\mathcal{R}^2 - \mathcal{R}^2 dw^2 \tag{12} $$

This hypothesis of negative mass has been developed and is perfectly accepted by contemporary physics. The consequences on the conception of energy, frequencies, momentum of mechanics are to be reviewed from these new concepts.

E- On the complementary space-times

A. Sakharov (1980) [15] was the first to express a bi-metric representation of space-time. Subsequently, Bondi (1957), Rosen (1973), Hossenfelder (2008) Hassan (2012), Damour (2019), Petit (2021) [7,12,18,22,31,34] developed their models on this basis. Boyle, Finn, and Turok [30] published a cosmological model based on the existence of a mirror universe, populated by antimatter and "going back in time", like the Sakharov'. The scientific literature shows that the absence of negative mass matter in our known universe, supports the hypothesis of a bimetric of space-time that separates the known matter from this negative mass matter.

Many researchers, starting with Dirac, predicted intuitively that the mirror Universe (at the antipodes of our Universe), should be sought not in our space, but rather in a space where the particles have masses and energies of opposite sign. Since the masses of our Universe are positive, those of the mirror Universe will be negative according to Borissova and Rabounski (2009) [19].

Both Newton's and Einstein's theories of gravitation predict a non-intuitive behavior of negative masses. For two bodies of equal and opposite masses, the positive mass attracts the negative mass, but the latter repels the positive mass; the two masses pursue each other. The motion along a line joining the centers of mass of the considered bodies, would thus be a motion in constant acceleration.

Between these spacetimes, the throat sphere that separates our Universe from the mirror universe and prevents particles of negative and positive mass from curing into contact, thus prohibiting any particle annihilation, except for the quantum tunneling effect.

From the geometrical point of view, the throat sphere 3D contains particles of null mass is tangent to the regions occupied by particles characterized by $m > 0$, or $m < 0$. It follows those particles of zero mass, can interact both with the particles of our Universe $m > 0$, and with those of the Mirror Universe, $m < 0$.

The throat sphere contains only energy in the form of elementary particles of zero mass described by quantum fields. This energy contributes to generate the gravity field.

F- Solving Einstein's general equation in a nonempty space.

In this chapter we will present the mathematical solution of Einstein's equations in a non-empty space-time. We suppose that a spherical object of radius a, massive, fluid, and homogeneous generates a gravitational field inside and outside the object.

Let $\mathrm{M} = \frac{4\pi}{3}(\mathrm{a}^3 - \mathrm{R}_\mathrm{s}^3)$ be the mass of the homogeneous fluid contained in the sphere of radius a . It is assumed in this paper that the Schwarzschild radius of the star is such that $R_s \ll a$.

The matter in the interior of the star is described by a fluid of energy-impulse tensor T_μ^ν proposed by Schwarzschild, cf J. Haag (1931), Brilloin (1935) [5,6].

The energy tensor is written : $T_\mu^\nu = \rho U_\mu U^\nu - P_\mu^\nu$

where $\rho(\mathcal{R})$ represents the proper density and P_μ^ν the internal pressure tensor, and U^ν the quadratic components of the generalized velocity.

It is also assumed we search the metric of the general form

$$ds^2 = Ad\xi^2 - Bd\mathcal{R}^2 - \mathcal{R}^2 dw^2$$

With $A(\mathcal{R})$ and $B(\mathcal{R})$ two functions of the variable $\mathcal{R}$.

We define the metric tensor by :

$$g_{\xi\xi} = A(\mathcal{R}), \qquad g_{\mathcal{R}\mathcal{R}} = -B(\mathcal{R}), \qquad g_{\theta\theta} = -\mathcal{R}^2,$$

$$g_{\varphi\varphi} = -\mathcal{R}^2 sin^2(\theta),$$

The pressure in the fluid is described by an equation of state $P_\mu^\nu = P_\mu^\nu(\rho, \mathcal{R})$ and because of homogeneity we assume that :$P_\mu^0 = P_0^\nu = 0$

$$P_\mu^\nu = 0 (\mu \neq \nu)$$

$P_1^1 = P(\mathcal{R}); P_2^2 = P_3^3 = Q(\mathcal{R})$; the pressure tensor is then written:

$$P_\mu^\nu = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -P(\mathcal{R}) & 0 & 0 \\ 0 & 0 & -Q(\mathcal{R}) & 0 \\ 0 & 0 & 0 & -Q(\mathcal{R}) \end{pmatrix}$$

The pressure has a normal component $\mathrm{P}(\mathcal{R})$ and a transversal component $\mathrm{Q}(\mathcal{R})$.

The energy tensor in the star is then written:

$$T_\mu^\nu = \begin{pmatrix} \rho(\mathcal{R}) & 0 & 0 & 0 \\ 0 & -P(\mathcal{R}) & 0 & 0 \\ 0 & 0 & -Q(\mathcal{R}) & 0 \\ 0 & 0 & 0 & -Q(\mathcal{R}) \end{pmatrix}$$

Considering the metric, we have:

$$T_{\mu\nu} = \begin{pmatrix} \rho A & 0 & 0 & 0 \\ 0 & PB & 0 & 0 \\ 0 & 0 & Q\mathcal{R}^2 & 0 \\ 0 & 0 & 0 & Q\mathcal{R}^2 sin^2(\theta) \end{pmatrix}$$

With $P(\mathcal{R}) = Q(\mathcal{R}) = \rho(\mathcal{R}) = 0$ located outside the fluid sphere, so $T_\mu^\nu = 0$ located outside the star.
This hypothesis is compatible with the presence of a fluid with density $\rho(\mathcal{R})$ and the structure of the gravity field of the star is thus determined by 4 functions A, B, P, ρ depending on the variable $\mathcal{R}$.

The general equations of Einstein are written:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \aleph T^{\mu\nu}$$

R is the curvature radius of the metric, $g^{\mu\nu}$ the metric tensor, $\aleph = 8\pi G$.

As,

$$g_{i\mu} g_{j\nu} G^{ij} = G_{\mu\nu} \text{ , and } \quad g_{\mu\nu}\left(R^{\mu\nu} - \tfrac{1}{2} R g^{\mu\nu}\right) = -R$$

Then

$$g_{\mu\nu} T^{\mu\nu} = T \ \ then \ \ R = -\aleph T$$

The Einstein equations are written :

$$R_{\mu\nu} = \aleph\left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T\right) \qquad (13)$$

After calculation, the Ricci tensor is written:

$$\begin{cases} R_{\xi\xi} = \dfrac{A}{B}\left[\dfrac{A''}{2A} + \dfrac{A'}{4A}\left(\dfrac{A'}{A} + \dfrac{B'}{B}\right) - \dfrac{A'}{\mathcal{R}A}\right] \\ R_{\mathcal{R}\mathcal{R}} = \dfrac{A''}{2A} - \dfrac{A'}{4A}\left(\dfrac{A'}{A} + \dfrac{B'}{B}\right) - \dfrac{B'}{\mathcal{R}B} \\ R_{\theta\theta} = \dfrac{\mathcal{R}}{B}\left(\dfrac{A'}{A} - \dfrac{B'}{B} + \dfrac{2}{\mathcal{R}} - \dfrac{2B}{\mathcal{R}}\right) \\ R_{\varphi\varphi} = sin^2(\theta) R_{\theta\theta} \end{cases}$$

As the trace of the energy tensor is written :

$$T = T_i^i = \rho(\mathcal{R}) - P(\mathcal{R}) - 2Q(\mathcal{R})$$

with

$$T_{\mathcal{R}\mathcal{R}} = P(\mathcal{R})B(\mathcal{R}); \qquad T_{\theta\theta} = Q(\mathcal{R})\mathcal{R}^2$$

$$T_{\varphi\varphi} = Q(\mathcal{R})\mathcal{R}^2 sin^2(\theta)$$

$$T_{\xi\xi} = \rho(\mathcal{R})A(\mathcal{R})$$

Then, equation (13) is expressed by the following system of 4 nonlinear differential equations:

$$\begin{cases} R_{\xi\xi} = \aleph A \dfrac{1}{2}(\rho + P + 2Q) \\ R_{\mathcal{R}\mathcal{R}} = \aleph B \dfrac{1}{2}(\rho + P - 2Q) \\ R_{\theta\theta} = \dfrac{1}{2}\aleph\mathcal{R}^2(P - \rho) \\ R_{\varphi\varphi} = sin^2(\theta)\aleph\mathcal{R}^2\dfrac{1}{2}(P - \rho) \end{cases}$$

i.e

$$\begin{cases} \dfrac{A''}{2A} - \dfrac{A'}{4A}\left(\dfrac{A'}{A} + \dfrac{B'}{B}\right) + \dfrac{A'}{\mathcal{R}A} = \aleph B\dfrac{1}{2}(\rho + P + 2Q) & (a) \\ \dfrac{A''}{2A} - \dfrac{A'}{4A}\left(\dfrac{A'}{A} + \dfrac{B'}{B}\right) - \dfrac{B'}{\mathcal{R}B} = \aleph B\dfrac{1}{2}(2Q - P - \rho) & (b) \\ \left(\dfrac{A'}{A} - \dfrac{B'}{B}\right) - \dfrac{2B}{\mathcal{R}} + \dfrac{2}{\mathcal{R}} = \aleph\mathcal{R}B(P - \rho) & (c) \end{cases}$$

$A(\mathcal{R}), B(\mathcal{R})$ are functions defining the desired interior and exterior metrics. These functions depend on the parameters of the energy tensor $P, Q, \rho, \aleph$.

We calculate

$$(a) - (b) \rightarrow \frac{1}{\mathcal{R}}\left(\frac{A'}{A} + \frac{B'}{B}\right) = \aleph B(\rho + P)$$

i.e.

$$\left(\frac{A'}{A} + \frac{B'}{B}\right) = \aleph\mathcal{R}B(P + \rho) \tag{14}$$

To the equation (c) we show :

$$\left(\frac{A'}{A} - \frac{B'}{B}\right) = \frac{2B - 2}{\mathcal{R}} + \aleph\mathcal{R}B(P - \rho) \tag{15}$$

Differentiating and adding the two equations (14) and (15) , we show :

$$\frac{A'}{A} = \frac{B - 1}{\mathcal{R}} + \aleph\mathcal{R}BP(\mathcal{R}) \tag{16}$$

$$\frac{B'}{B} = \frac{1 - B}{\mathcal{R}} + \aleph\mathcal{R}B\rho(\mathcal{R}) \tag{17}$$

.

To resolve these equations, in a first time, we assume some mathematical hypothesis as density is constant, or the pression is a simple explicit function of the variable R. In a second time we show energy conditions in the fluid will determine the metric. Equations relating the pressure and the density of the fluid determine these energy conditions.

<u>Hypothesis F.1</u> :

We consider in the following that the density $\rho = \rho_0$ is constant in all the fluid. This assumption is compatible with the physics of classical stars.

If the pressure in the fluid is a function of $\mathcal{R}$ (this is a mathematical hypothesis) with :

$P(a) = P(R_s) = 0$. Then it can be written:

$$P(\mathcal{R}) = P_0\left(1 - \frac{\mathcal{R}}{a}\right)\left(\frac{\mathcal{R}}{R_s} - 1\right), \qquad R_s \le \mathcal{R} \le a$$

The general solution of equation (17) is:

$$B(\mathcal{R}) = \frac{3\mathcal{R}}{3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3} \tag{18}$$

For $\rho_0 = 0$, we find the particular solution of the metric outside the star:

$$B_0(\mathcal{R}) = \frac{\mathcal{R}}{\mathcal{R} + C} = \left(1 + \frac{C}{\mathcal{R}}\right)^{-1}$$

Since the density is constant in the star and given the general expression for $B(\mathcal{R})$, we write equation (16):

$$\frac{A'}{A} = \frac{-1}{\mathcal{R}} + \frac{3(1 + \aleph\mathcal{R}^2P(\mathcal{R}))}{3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3} \tag{19}$$

The solution of this equation is:

$$A(\mathcal{R}) = e^{\zeta(\mathcal{R})}\prod_{j=1}^{3}\frac{\left(\mathcal{R} - z_j\right)^{\beta_j}}{\mathcal{R}} \tag{20}$$

$$\zeta(\mathcal{R}) = \frac{3P_0}{2aR_s\rho_0}\mathcal{R}^2 - \frac{3P_0(R_s + a)}{\rho_0 aR_s}\mathcal{R}, \qquad R_s \le \mathcal{R} \le a$$

$K(\mathcal{R}) = 3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3$ is a polynomial, with roots z_j and β_j are coefficients depending on the constants $C, a, R_s, \aleph\rho_0$.

For $\rho_0 \to 0, P(\mathcal{R} \geq a) = 0$, we find the particular solution of the metric located outside the star:

$$A_0(\mathcal{R}) = \frac{\mathcal{R} + C}{\mathcal{R}} = 1 + \frac{C}{\mathcal{R}}$$

Hypothesis F.2 : If the pressure in the fluid is a function of $\mathcal{R}$ (this is a mathematical hypothesis) with

$P(a) = P(R_s) = 0$. Then it can be written:

$$P(\mathcal{R}) = P_0\left(\frac{a}{\mathcal{R}} - 1\right)\left(1 - \frac{R_s}{\mathcal{R}}\right)$$

We find solutions for the Einstein equations in the form (18) and (20) having made assumptions of continuity for the pressure and of limit conditions ($\mathcal{R} = a$) and ($\mathcal{R} = R_s$).

we have:

$$ds^2 = A(\mathcal{R})d\xi^2 - B(\mathcal{R})d\mathcal{R}^2 - \mathcal{R}^2 dw^2$$

with

$$A(\mathcal{R}) = \prod_{j=1}^{3} \frac{(\mathcal{R} - w_j)^{\gamma_j}}{\mathcal{R}}$$

$$B(\mathcal{R}) = \frac{3\mathcal{R}}{3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3}$$

Then

$$AB = \prod_{j}^{3} \frac{(\mathcal{R} - w_j)^{\gamma_j}}{(\mathcal{R} - z_j)^{\beta_j}}$$

For $\rho_0 \to 0, and\ P(\mathcal{R} \geq a) = 0$, we find the particular solution of the metric located outside the star:

$$A_0 B_0 = 1$$

$$B_0(\mathcal{R}) = \frac{\mathcal{R}}{\mathcal{R} + C} = \left(1 + \frac{C}{\mathcal{R}}\right)^{-1}$$

G- Physics of pressure and density in the star

It is known that the divergence of the energy tensor verifies the following physical property because of Einstein's equations, the energy conditions is:

$$\partial_i T^{ij} = 0$$

i.e.

$$P' + \frac{(P + \rho)}{2}\frac{A'}{A} + \frac{(P - Q)}{\mathcal{R}} = 0$$

In a perfect fluid sphere, we have : $P = Q$

The equation reduces to the following expression:

$$P' + \frac{(P + \rho)}{2}\frac{A'}{A} = 0 \qquad (21)$$

We deduce from this $P(\mathcal{R})$ and $\rho(\mathcal{R})$, and we can calculate $B(\mathcal{R})$ and $A(\mathcal{R})$.

Hypothesis G.1

For $P + \rho = Cte = K \neq 0$ then

$$P' = \frac{-K}{2}\frac{A'}{A}$$

$$P(\mathcal{R}) = ln\left(\frac{1 + \frac{C_0}{a}}{A(\mathcal{R})}\right)^{\frac{K}{2}} ;\ \rho(\mathcal{R}) = K - P(\mathcal{R})$$

The general solution of equation (17) is:

$$B(\mathcal{R}) = \frac{1}{1 - \frac{\aleph}{\mathcal{R}}M(\mathcal{R})}$$

With

$$M(\mathcal{R}) = \int_{R_s}^{\mathcal{R}} u^2 \rho(u) du$$

Equation (14) give

$$\left(\frac{A'}{A} + \frac{B'}{B}\right) = \aleph K B \mathcal{R}$$

then

$$Ln(AB) = \aleph K \int_{R_s}^{\mathcal{R}} uB(u)\, du$$

i.e

$$AB = exp\left(\aleph K \int_{R_s}^{\mathcal{R}} uB(u)\, du\right)$$

For $\rho(\mathcal{R} \geq a) = 0\ and\ P(\mathcal{R} \geq a) = 0$, we find the particular solution of the metric located outside the star:

$$A_0 B_0 = 1$$

$$B_0(\mathcal{R}) = \frac{\mathcal{R}}{\mathcal{R}+C} = \left(1+\frac{C}{\mathcal{R}}\right)^{-1}$$

Hypothesis G.2

The density is constant: $\rho = \rho_0$

With $P(a) = 0$, *and* $\rho = \rho_0$ then equation (21) give:

$$P + \rho_0 = \rho_0 \frac{\sqrt{1+\frac{C_0}{a}}}{\sqrt{A(\mathcal{R})}}$$

Since $A(a) = A_0(a) = 1 + \frac{C_0}{a}$

The condition $\sqrt{1+\frac{C_0}{a}} > \sqrt{A(R_s)} > 0$ will decide on the critical conditions for the density ρ_0. ([5]. Haag J. 1931)

Let's calculate the general expression of the function $A(\mathcal{R})$

The equation (14) can be written:

$$\left(\frac{A'}{A}+\frac{B'}{B}\right) = \aleph B\mathcal{R}\rho_0 \frac{\sqrt{1+\frac{C_0}{a}}}{\sqrt{A(\mathcal{R})}} \qquad (22)$$

We deduce :

$$\frac{A'}{\sqrt{A}} = -\frac{B'\sqrt{A}}{B} + \aleph B\mathcal{R}\rho_0\sqrt{1+\frac{C_0}{a}}$$

Let's $U = A^{1/2}$ then $2U' = A'A^{-1/2}$

then we have :

$$2U' = -U\frac{B'}{B} + \aleph B\mathcal{R}\rho_0\sqrt{1+\frac{C_0}{a}} \qquad (23)$$

In a first case, by choosing $C = 0$ for B in the equation (18) , we find a simple equation easier to solve as the same of Schwarzschild.

$$2U' = -\frac{\frac{2\aleph\rho_0}{3}\mathcal{R}}{1-\frac{\aleph\rho_0}{3}\mathcal{R}^2}U + \frac{\aleph\rho_0\mathcal{R}}{1-\frac{\aleph\rho_0}{3}\mathcal{R}^2}\sqrt{1+\frac{C_0}{a}}$$

whose solution is to the nearest constant:

$$U = \frac{-1}{2}\sqrt{1-\frac{\aleph\rho_0}{3}\mathcal{R}^2} + \frac{3}{2}\sqrt{1+\frac{C_0}{a}}$$

This solution was found by Schwarzschild in 1916 , with

$$2m = \frac{\aleph\rho_0}{3}a^3 \text{ } and \text{ } C_0 = -2m$$

$$A(\mathcal{R}) = \left(\frac{3}{2}\sqrt{1-\frac{2m}{a}} - \frac{1}{2}\sqrt{1-\frac{2m}{a^3}\mathcal{R}^2}\right)^2$$

In the second case, by choosing $C \neq 0$, the solution of the inhomogeneous equation to explain the function $A(\mathcal{R})$ is a more difficult problem that involves hyper elliptic integrals.

The general solution of equation (23), is

$$U(\mathcal{R}) = \left(\frac{\aleph\rho_0\sqrt{1+\frac{C_0}{a}}}{2\sqrt{B}}\int_0^{\mathcal{R}} tB^{3/2}dt + \frac{1}{\sqrt{B}}\right)$$

and

$$A(\mathcal{R}) = \frac{1}{B(\mathcal{R})}\left(\frac{\aleph\rho_0\sqrt{1+\frac{C_0}{a}}}{2}\int_0^{\mathcal{R}} tB(t)^{3/2}dt + 1\right)^2$$

Remark :

If $\rho_0 \to 0$, then :

$$A_0(\mathcal{R}) = \frac{1}{B_0(\mathcal{R})}$$

It is the particular solution of the Einstein equations out of the star.

Hypothesis G.3

We assume le star is a special fluid where

$P + \rho = 0$ and $P - Q \neq 0$ as energy condition

For $\rho > 0$ then $P < 0$, i.e energy is negative,

Or For $\rho < 0$ then $P > 0$, i.e mass is negative,

Then the energy conditions can be write

$$P' + \frac{(P-Q)}{\mathcal{R}} = 0$$

The solution of equation (14) is

$$AB = 1$$

Assume that $Q = 2P_0\mathcal{R}$ then

$$P(\mathcal{R}) = P_0\left(\mathcal{R} - \frac{a^2}{\mathcal{R}}\right) and\ P(a) = 0$$

Eq (17) give

$$\frac{B'}{B} = \frac{1-B}{\mathcal{R}} - \aleph B P_0(\mathcal{R}^2 - a^2)$$

The interior solution is a particular De Sitter-Schwarzschild metric :

$$B(\mathcal{R}) = \frac{4\mathcal{R}}{4C + 4\mathcal{R} - 2\aleph P_0 a^2\mathcal{R}^2 + \aleph P_0\mathcal{R}^4}$$

$$A(\mathcal{R}) = \frac{1}{B(\mathcal{R})}$$

And the exterior solution is :

$$A_0(\mathcal{R}) = \frac{1}{B_0(\mathcal{R})} = 1 + \frac{C}{\mathcal{R}}$$

H- Conclusions

We have shown in this article, through mathematical assumptions and a more attentive reading the second article of Schwarzschild (1916) that the physical interpretation of the variable $\boldsymbol{r}$ is not a radial distance when located closer to the star.

We have also addressed the problem of the singularity of this metric by explaining what S. Hawking and R. Penrose have said about gravitational collapse.

Starting from this principle of non-singularity, we have constructed a metric from the Einstein tensor in a vacuum. This metric, asymptotic to a plane metric far away from the star and tangent to a throat sphere near the center, extends to a mirror metric. In the mirror metric, the masses are negative, and the time is reversed.

In the last part of this article, we explained the general solution of the Einstein equation, i.e., with second term, considering assumptions on the pressure and density inside the fluid ball representing the star.

For $\boldsymbol{P + \rho = K \neq 0}$ then

$$A(\mathcal{R})B(\mathcal{R}) = exp\left(\aleph K \int_{R_S}^{\mathcal{R}} uB(u)\,du\right)$$

For $\boldsymbol{P + \rho = 0}$ then

$$A(\mathcal{R})B(\mathcal{R}) = 1$$

With

$$B(\mathcal{R}) = \frac{4\mathcal{R}}{4C + 4\mathcal{R} - 2\aleph P_0 a^2\mathcal{R}^2 + \aleph P_0\mathcal{R}^4}$$

It is a particular De Sitter-Schwarzschild's metric in a special fluid.

For $\boldsymbol{\rho = \rho_0}$ then

$$\boldsymbol{B(\mathcal{R}) = \frac{3\mathcal{R}}{3C + 3\mathcal{R} - \aleph\rho_0\mathcal{R}^3}}$$

$$\boldsymbol{A(\mathcal{R})B(\mathcal{R}) = \left(\frac{\aleph\rho_0\sqrt{1+\frac{C_0}{a}}}{2}\int_0^{\mathcal{R}} tB(t)^{3/2}dt + 1\right)^2}$$

It is the hyper-elliptic interior solution who generalize the interior Schwarzschild's solution.

In a next paper, we will make new physical assumptions about the dimension 3 throat's sphere. We will study the physics and the nature of zero mass and relativistic particles, which can pass through that throat sphere.

The topology of the $\boldsymbol{S^3}$ sphere immersed in a space of dimension 4 is not visualizable easily. Indeed, the stereographic projection of the $\boldsymbol{S^3}$ sphere in our space is the whole $\mathbb{R}^3$ space plus a point at infinity according to our viewpoint. The gorge sphere is then isomorphic to $\mathbb{R}^3$ in which time have been stopped. The sphere $\boldsymbol{S^3}$ is the junction of our paraboloid space -time with the mirror space-time who have property of negative mass.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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