Knot in geometrical optics

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We treat the geometrical optics as a $U(1)$ gauge theory the same as the $U(1)$ Maxwell's gauge theory. We propose there exists a knot in a 3-dimensional Euclidean flat space of the geometrical optics (eikonal equation) as a consequence there exists a knot in the Maxwell's theory in a vacuum. We formulate Chern-Simons integral using an eikonal. We obtain the relation between the knot (the geometrical optics helicity, an integer number) and the refractive index. We consider this relationship as a topological quantum condition.

Keywords: *knot, geometrical optics, eikonal equation, Maxwell's theory, Chern-Simons integral, helicity, refractive index.*

It is commonly believed there exists no topological object in the linear theory, such as the Maxwell's theory. It is because of *a topological theory must be a non-linear theory*[1]. The existence of topological object, *a knot*, in the Maxwell's linear theory so far has not been well known[2]. So, *how to obtain a knot from the Maxwell's linear theory?* Roughly speaking, the electromagnetic fields (the set of the solutions of Maxwell equations) in *vacuum* has *a subset field with a topological structure*[1]. Any electromagnetic field is *locally equal* (except in a zero measure set) but *globally different* to a subset field[1]. *The electromagnetic field is a linear field, but a subset field is a non-linear field. Both fields are equal in the case of the weak field*[3]. So, in the weak field we can say that *a non-linear subset field theory* reduces to the Maxwell's linear theory. *A knot could exist in the Maxwell's linear theory because the Maxwell's linear theory is the weak field limit*[3] *of a non-linear subset field theory.*

Why must a topological theory be a non-linear theory? Let us consider *a map* of a subset field (consists of *a complex scalar field*) from a finite radius r to an infinite r which implies from the non-linear subset field to the linear field, the weak field. A scalar field has, by definition, the property that its value for a finite r depends on the magnitude and the direction of the position vector $\vec{r}$, but for an infinite r it is *well defined*[3] (depends on the magnitude of $\vec{r}$ only). In other words, for an infinite r, a scalar field is *isotropic*. *The property of such a scalar field can be interpreted as a map* $S^3 \rightarrow S^2$[1] where S^3 and S^2 are 3-dimensional and 2-dimensional spheres, respectively. We see there exists *(one) dimension reduction* in such map. We consider this dimension reduction related to the isotropic (well defined) property of a scalar field for an infinite r. As maps of this kind *can be classified* in *homotopy classes*, labelled by *a topological invariant* called *the Hopf index*[1], *an integer number.*

What is a knot? A knot is a smooth-embedding of a circle in $\mathbb{R}^3$[4]. Here, $\mathbb{R}^3$ denotes a 3-dimensional Euclidean (flat) space[5]. Two knots are *equivalent* if one knot can be *deformed continuously* into the other *without crossing itself*[4]. In electrodynamics, *a knot could be formed by bending the electric and magnetic field lines* (the geometric concept of *magnetic lines of force* - those lines of force are today designated by the symbol $\vec{H}$, *the magnetic field* - is due to Faraday[6]) so that they could form *closed loops*[2]. A set of closed loops in space forms *a link*[7]. These closed loops can be *linked*[2] (although links do not actually need to be linked[8]), and *this linking could provide the topological structure*[2]. Suppose that we have two closed loops of field lines. These two closed loops have *a non-vanishing Gauss integral (Gauss linking integral)* if they are *linked*. *The self-linking number (an integer number)* i.e. a non-vanishing Gauss integral describes *the knottedness*[2].

Where does the origin of a knot idea come from? In physics, the idea of a knot, *topologically stable matter*, had been proposed in 1868 by Lord Kelvin that *the atoms could be knots or links of vorticity lines of aether*[2]. In mathematics, especially in algebraic topology, *a knot is defined by the Hopf index*[2]. We will see that the Hopf index is related to *the Hopf invariant*[1].

As we mentioned, suppose that we have a scalar field as a function of position vector, $\phi(\vec{r})$, with a property that can be interpreted using *non-trivial Hopf map* as written below[1,3]

$$\phi(\vec{r}) : S^3 \rightarrow S^2 \tag{1}$$

This non-trivial Hopf map is related to the Hopf invariant[9]. The Hopf invariant, $\mathcal{H}$, can be expressed as an integral[9,10] below[9,11]

$$\mathcal{H} = \int_{S^3} \omega \wedge d\omega \tag{2}$$

where ω is *a 1-form* on S^3[9]. The relation between the Hopf invariant and the Hopf index, h, can be written as[1]

$$\mathcal{H} = h\ \gamma^2 \tag{3}$$

where γ is *the total strength of the field*, that is the sum of the strengths of all *the tubes* formed by the integral lines of electric and magnetic fields[1].

The Hopf invariant have deep relationships with *the Abelian Chern-Simons action*[9] *(the Chern-Simons integral)* and *self-helicity* in magnetohydrodynamics[9]. We will see that the Abelian Chern-Simons action is related to *electromagnetic helicity*.

In this article, *we propose there exists a knot in the geometrical optics, as a solution of the eikonal equation*. The reason is, in fact, there exists a knot in the Maxwell's theory[1–3] and the geometrical optics (the eikonal equation) can be derived from the Maxwell's theory (the Maxwell equations)[12]. We treat the geometrical optics as the $U(1)$ gauge theory[13,14], the same as the $U(1)$ Maxwell's gauge theory. *To the best of our knowledge, the formulation of a knot in geometrical optics has not been done yet*[1,2,4,15].

How does a knot appear in the geometrical optics? The Abelian Chern-Simons integral related to the electromagnetic helicity[2], h_{em}, can be written as[16,17]

$$h_{em} = S_{CS} = \int_{\mathbb{R}^3} \varepsilon^{\alpha\mu\nu}\ \vec{A}_\alpha\ \vec{F}_{\mu\nu}\ d^3x \tag{4}$$

where $\varepsilon^{\alpha\mu\nu}$ is the Levi-Civita symbol, $\alpha, \mu, \nu = 1, 2, 3$[17] denote a 3-dimensional space, $\vec{A}_\alpha$ is *the* $U(1)$ *gauge potential*[17] and $\vec{F}_{\mu\nu}$ is *the* $U(1)$ *gauge field tensor*[17] (field strength tensor) or *its curvature*[16] written below[18]

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu \tag{5}$$

The $U(1)$ field strength tensor is related to the scalar field, ϕ, as[1]

$$\vec{F}_{\mu\nu} = f_{\mu\nu}(\phi) = \frac{1}{2\pi i}\frac{\partial_\mu\phi^*\ \partial_\nu\phi - \partial_\nu\phi^*\ \partial_\mu\phi}{(1+\phi^*\phi)^2} \tag{6}$$

where[1]

$$\phi = a\ e^{i2\pi\sigma} \tag{7}$$

We consider a as *a scalar amplitude* as a consequence ϕ is a scalar field. *The relation (6) is valid for the weak field only*. It means that *in the case of the weak field,* $f_{\mu\nu}(\phi)$ *is linear*. But *in general,* $f_{\mu\nu}(\phi)$ *is non-linear. The non-linearity of* $f_{\mu\nu}(\phi)$ *(6) is shown by* $\phi^*\phi$. If the field is weak then $\phi^*\phi << 1$ so the denominator in eq.(6) *can be taken* as being equal to one and $f_{\mu\nu}(\phi)$ is *equivalent* to the Maxwell theory[1].

In the case of the geometrical optics, the relation between the $U(1)$ gauge potential[13] and *the eikonal (phase)*[19], ψ_1, is given by

$$\vec{A}_\alpha = \vec{A}_\alpha^{\,U(1)} = \vec{a}_\alpha\ e^{i\psi} = \vec{a}_\alpha\ e^{i\frac{f_\theta}{c}(\psi_1 - ct)} \tag{8}$$

$$\psi_1 = \int_{x_1}^{x_2} n\ dx \text{ (1-dimensional space)} \tag{9}$$

where $\vec{a}_\alpha$ is *four-vector amplitudo*, c is the speed of light in vacuum and n denotes *the refractive index, a number*. Eq.(8) shows explicitly that *we treat the geometrical optics as the* $U(1)$ *gauge theory*[13], *the same as the* $U(1)$ *Maxwell's gauge theory*. The phase, ψ_1, obey the *Fermat's principle* $\delta\psi_1 = 0$.

By substituting eqs.(5), (8), (9) into (4), in the case of the 3-dimensional space, we obtain the Abelian Chern-Simons integral expressed in the refractive index related to *the geometrical optics helicity*, $h_{\rm go}$, as follow

$$\int_{\mathbb{R}^3} \varepsilon^{\alpha\mu\nu}\ \vec{a}_\alpha\ e^{i\frac{f_\theta}{c}\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)} \left\{ \partial_\mu \left[\vec{a}_\nu\ e^{i\frac{f_\theta}{c}\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)}\right] - \partial_\nu \left[\vec{a}_\mu\ e^{i\frac{f_\theta}{c}\left(\int_{x_1}^{x_2} n\ d^3x - ct\right)}\right]\right\} d^3x = h_{\rm go} \tag{10}$$

where we replaced the electromagnetic helicity to the geometrical optics helicity or *the geometrical optics knot*. Both, h_{em} and h_{go}, are *integer numbers*.

The Chern-Simons form of a connection was originally used to study secondary characteristic classes before it was *interpreted* as the Lagrangian of a field theory on compact 3-manifold[20]. The characteristic classes play an important role in the index theory of Atiyah-Singer[21] where in the case for *even-dimensional* oriented compact Riemannian manifold, the Gauss-Bonnet-Chern theorem is a special case of the Atiyah-Singer index theory[22]. So, *could the Chern-Simons integral be formulated in even dimensional Euclidean flat space? Could the Chern-Simons integral be formulated in even dimensional oriented compact Riemannian manifold? Does there exist a relation between the Chern-Simons integral and the Gauss-Bonnet-Chern theorem?*

Inspired by the knot solution in the Maxwell's linear theory, roughly speaking, *could every linear equation in physics be constructed as weak field limit of a non-linear equation?*

If the self-linking number and the helicity are related to the knot then *can we show explicitly the relations between the Gauss linking integral, the Hopf invariant and the Chern-Simons integral, the self-linking number and the helicity?*

Eq.(10) shows explicitly the relation between the geometrical optics helicity (the geometrical optics knot) and the refractive index. We consider eq.(10) as *a topological quantum condition*[1]. *What is the consequence to a choice of a number of the refractive index if the geometrical optics helicity is an integer number? Could topological invariant, such as the geometrical optics helicity, be complex integer number? What is the physical interpretation if the geometrical optics knot is complex integer number? Could helicity fluctuate?*[23] *If helicity could fluctuate, how do we choose and interpret a number of the refractive index to accomodate the fluctuation of helicity?*

We see from eqs.(2),(4), both equations are identical where ω, $d\omega$ are identical with $\vec{A}_\alpha$, $\vec{F}_{\mu\nu}$ respectively. *Does it mean that the Hopf invariant (2) is identical with the Abelian Chern-Simons integral (4)? If there exists a non-Abelian Chern-Simons integral, what is its consequence to the form of the Hopf invariant formulation?*

Eq.(6) show us the implicit relation between the $U(1)$ gauge potential, $\vec{A}_\mu$, and the scalar field, ϕ. *Can we* *write the explicit relation between the $U(1)$ gauge vector potential and the scalar field?*

We see from eqs.(7),(8), *the amplitudes*, $\vec{a}_\alpha$ and a, are vector and scalar respectively. *We consider that the amplitude is very important (probably, the most important) quantity in physics.* If the amplitude is zero then the scalar field, the gauge potential are zero and we have no physical information from both. Other example is the most fundamental principle in quantum mechanics, i.e. *the Heisenberg uncertainty principle*, can be written in *the commutation relation of amplitude*[24].

We write explicitly in eq.(8) the gauge potential of geometrical optics as the $U(1)$ gauge potential, $\vec{A}_\alpha^{U(1)}$. It has an interesting consequence if we relate the gauge theory to the fibre bundle (global geometry) and formulate the geometrical optics using the fibre bundle language[14].

So far, we show the theoretical existence of the geometrical optics knot only. *Does the electromagnetic (geometrical optics) knot exist in universe or laboratory? Ball lightning*[25], probably, is an electromagnetic knot in universe[26]. *Tokamaks* and devices constructed to produce *fireball* are two possible laboratory settings to observe ball lightning[26]. *Knot of light* may be generated using *tightly focused circularly polarized laser beams*[27].

Thank to Richard Tao Roni Hutagalung, Idham Syah Alam for fruitful discussions. Special thank to beloved ones, Juwita Armilia and Aliya Syauqina Hadi, for much love and great hope. To Ibunda and Ayahanda, may Allah bless them with Jannatul Firdaus. This research is fully supported by self-funding.

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