Non-Trivial Zeros of Riemann Zeta Function and Riemann Hypothesis

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Abstract. This paper touches on the part played by the non-trivial zeros of the Riemann zeta function ζ, providing many important information and insights in the process, including some approaches to the Riemann hypothesis. (Published in Bulletin of Pure and Applied Sciences Section - E - Mathematics & Statistics Vol. 41E, No.1, January-June 2022.P.88-99)

MSC. 11A41, 11A99
Keywords. Complex plane; Riemann zeta function; prime number theorem; non-trivial zeros; primes; analogy; series; critical line.

1 Introduction

The Riemann hypothesis is an important problem in mathematics as its validity will affirm the manner of the distribution of the prime numbers. It posits that all the non-trivial zeros of the zeta function ζ lie on the critical strip between Re(s) = 0 and Re(s) = 1 at the critical line Re(s) = 1/2. The important point is whether there would be zeros appearing at other locations on this critical strip, e.g., at Re(s) = 1/4, 1/3, 3/4, or, 4/5, etc., which would disprove the Riemann hypothesis. We would look into this.

The following is the Riemann zeta function ζ with its terms:-

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots \qquad (1.1)$$

where s is the complex number $1/2 + bi$

For the term $1/2^{1/2 + bi}$ above, e.g., whether it would be positive or negative in value would depend on which part of the complex plane this term $1/2^{1/2 + bi}$ would be found in, which depends on 2(n) and b (it does not depend on 1/2 - 1/2 and 2(n) only determine how far the term is from zero in the complex plane). This term could be in the positive half (wherein the term would have a positive value) or the negative half (wherein the term would have a negative value) of the complex plane. Hence some of the terms in the Riemann zeta function ζ would have positive values while the rest would have negative values (which depend on the values of n and b). The sum of the series in the Riemann zeta function ζ is obtained with a formula, e.g., the Riemann-Siegel formula, or, the Euler-Maclaurin summation formula. The Riemann zeta function ζ would turn out a non-trivial zero on the critical line Re(s) = 1/2, as more and more terms are added, when it reaches a point at the critical line Re(s) = 1/2 where the positive terms (in the positive half of the complex plane, as explained above) cancel out the negative terms (in the negative half of the complex plane), i.e., a non-trivial zero indicates the point in the Riemann zeta function ζ wherein the total value of the positive terms equals the total value of the negative terms. There would be an infinitude of such non-trivial zeros at the critical line Re(s) = 1/2, which G. H. Hardy had proved. Whether there would be zeros off this critical line Re(s) = 1/2 on the critical strip bounded by Re(s) = 0 and Re(s) = 1 as more and more terms are added to the Riemann zeta function ζ is still an open question, which Riemann himself had thought highly unlikely though he had no proof.

Evidently Riemann anticipated that there would be an equal, or, almost equal number of primes among the terms in the positive half and the negative half of the complex plane when there is a zero, wherein the distribution of the primes would be statistically fair - the more terms are added to the Riemann zeta function ζ, the fairer or "more equal" would be the distribution of the primes in the positive half and the negative half of the complex plane when there is a zero. This is like the tossing of a coin wherein the more tosses there are the "more equal" would be the number of heads and the number of tails. In other words, in the longer term, with more and more terms added to the Riemann zeta function ζ, more or less 50% of the primes should be found in the positive half of the complex plane and the balance 50% should be found in the negative half of the complex plane, the more terms there are the fairer or "more equal" would be this distribution, when there is a zero, when the positive terms cancel out the negative terms in the Riemann zeta function ζ.

It is evident that through the non-trivial zeros the order or pattern of the distribution of the primes could be observed.

2 Main Results: Distribution of Non-Trivial Zeros of Riemann Zeta Function ζ and Possible Approaches to the Riemann Hypothesis

According to the concepts of fractal geometry, phenomena which appear random when viewed en masse display some orderliness and pattern which could be regarded as a fractal characteristic. For instance, the prime numbers are very random and haphazard entities, yet, when viewed en masse they display a regularity in the way they thin out, whereby it is affirmed that the number of primes not exceeding a given natural number n is approximately $n/\log n$, in the sense that the ratio of the number of such primes to $n/\log n$ eventually approaches 1 as n becomes larger and larger, $\log n$ being the natural logarithm (to the base e) of n (vide the prime number theorem proved in 1896 by Hadamard and de la Vallee-Poussin). In other words, the prime number theorem, which is apparently the direct outcome of the Riemann hypothesis, states that the limit of the quotient of the 2 functions $\pi(n)$ and $n/\log n$ as n approaches infinity is 1, which is expressed by the formula:

$$\lim_{n\to\infty} \pi(n)/(n/\log n) = 1 \qquad (2.1)$$

the larger the number n is, the better is the approximation of the quantity of primes, as is implied by the above formula where $\pi(n)$ is the prime counting function (π here is not the π which is the constant 3.142 used to compute perimeters and areas of circles, but is only a convenient symbol adopted to denote the prime counting function)

All this is in spite of the fact that the primes are scarcer and scarcer as n is larger and larger.

The prime number theorem could in fact be regarded as a weaker version of the Riemann hypothesis which posits that all the non-trivial zeros of the zeta function ζ on the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ would be at the critical line $\mathrm{Re}(s) = 1/2$. For a better understanding of the close connection between the prime number theorem and the Riemann hypothesis, it should be noted that Hadamard and de la Vallee Poussin had in 1896 independently proven that none of the non-trivial zeros lie on the very edge of the critical strip, on the lines $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) = 1$ - this was enough for deducing the prime number theorem. The locations of these non-trivial zeros on the critical strip could be described by a complex number $1/2 + bi$ where the real part is 1/2 and i represents the square root of -1. It had already been proven that there is an infinitude of non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$. The important question is whether there would be any zeros off the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, e.g., at $\mathrm{Re}(s) = 1/4$, 1/3, 3/4, or, 4/5, etc., the presence of any of which would disprove the Riemann hypothesis. So far, no such “off-the-critical-line” zeros has been found.

The validity of the Riemann hypothesis would apparently imply the validity of the prime number theorem, which is apparently the offspring and weaker version of the Riemann hypothesis, though the validity of the prime number theorem does not imply the former. Nevertheless, both of them have one thing in common in that they are both concerned with the estimate of the quantity of primes less than a given number, with the Riemann hypothesis positing a more exact estimate of the quantity of primes less than a given number. But, on the other hand, what would be the result if the Riemann hypothesis were false? We would return to this later.

Meanwhile, we would bring up more interesting points about the non-trivial zeros of the zeta function $\zeta(s)$ defined by a power series shown below:

$$\zeta(s) = \sum_{n=1}^{\infty} 1/n^s = 1 + 1/2^s + 1/3^s + 1/4^s + 1/5^s + \ldots$$

At the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ all the non-trivial zeros would be found on an oscillatory sine-like wave which oscillates in spirals, there being an infinitude of these spirals, which represent the complex plane. All the properties of the prime counting function $\pi(n)$ are in some way coded in the properties of the zeta function ζ, evidently resulting in the primes and the non-trivial zeros being some sort of mirror images of one another - the regularity in the way the primes progressively thin out and the progressively better approximation of the quantity of primes towards infinity by the prime counting function $\pi(n)$ mirror or reflect the regularity in the way the non-trivial zeros of the zeta function ζ line up at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, the non-trivial zeros becoming progressively closer together there, with no zeros appearing anywhere else on the critical strip, and, all this has been found to be true for the first 10^{13} non-trivial zeros.

Riemann had posited that the margin of error in the estimate of the quantity of primes less than a given number with the prime counting function $\pi(n)$ could be eliminated by utilizing the following J function which is a step function involving the non-trivial zeros expressed in terms of the zeta function ζ, which has been shown to be effective (2 steps are involved here - first, the prime counting function $\pi(n)$ is expressed in terms of the $J(n)$ function, then the $J(n)$ function is expressed in terms of the zeta function ζ, with the $J(n)$ function forming the link between the counting of the prime counting function $\pi(n)$ and the measuring (involving analysis and calculus) of the zeta function ζ, which would result in the properties of the prime counting function π(*n*) somehow encoded in the properties of the zeta function ζ):

$$J(n) = Li(n) - \sum_{p} Li(n^{p}) - \log 2 + \int_{n}^{\infty} dt/(t(t^{2} - 1) \log t) \qquad (2.2)$$

where the first term *Li*(*n*) is generally referred to as the "principal term" and the second term $\sum_{p} Li(n^{p})$ had been called the "periodic terms" by Riemann, *Li* being the logarithmic integral

The above formula may look fearsome but is actually not. The third term log 2 is a number which is 0.69314718055994… while the fourth term $\int_{n}^{\infty} dt/(t(t^{2} - 1) \log t)$ which is an integral representing the area under the curve of a certain function from the argument all the way out to infinity can only have a maximum value of 0.1400101011432869…. Since these 2 terms taken together (and minding the signs) are limited to the range from -0.6931… to -0.5531…, and since the prime counting function π(*n*) deals with really large quantities up to millions and trillions they are much inconsequential and can be safely ignored. The first term or principal term *Li*(*n*), where *n* is a real number, should also be not much of a problem as its value can be obtained from a book of mathematical tables or computed by some math software package such as *Mathematica* or *Maple*. However, special attention should be given to the second term $\sum_{p} Li(n^{p})$ which concerns the sum of the non-trivial zeros of the zeta function ζ (*p* in this second term is a "rho", which is the seventeenth letter of the Greek alphabet, and it means "root" - a root is a non-trivial zero of the Riemann zeta function ζ - a root here is a solution or value of an unknown of an equation which could be factorized). Riemann had evidently called the second term "periodic terms" as the components there vary irregularly.

The prime number theorem asserts that π(*n*) ~ *Li*(*n*) (technically $Li(n) = \int_{2}^{n} dx/\log(x)$) which also implies the weaker result that π(*n*) ~ *n*/log *n*. However, with *Li*(*n*) the prime count estimate would have a margin of error. The Riemann hypothesis asserts that the difference between the true number of primes *p*(*n*) and the estimated number of primes *q*(*n*) would be not much larger than √*n*. With the above *J*(*n*) function we could eliminate this margin of error and obtain an exact estimate of the quantity of primes less than a given number:

$$J(n) = \text{exact quantity of primes less than a given number}$$

Since the third and fourth terms of the *J*(*n*) function are inconsequential and can be safely ignored, as is described above, deducting the second term from the first term should be sufficient:

$$J(n) = Li(n) - \sum_{p} Li(n^{p}) = \text{exact quantity of primes less than a given number}$$

The above in brief shows the intimate relationship between the primes and the non-trivial zeros of the zeta function ζ, the primes and the non-trivial zeros being some sort of mirror images of one another as is described above, with the distribution of the non-trivial zeros being regarded as the music of the primes by mathematicians.

We return to the question of the consequence of the falsity of the Riemann hypothesis. Let us here assume that the Riemann hypothesis is false, i.e., there are also zeros found off the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$, e.g., at $\mathrm{Re}(s) =$ 1/4, 1/3, 3/4, or, 4/5, etc., and see the consequence. What would be the significant implication of this assumption? The falsity of the Riemann hypothesis would imply that the distribution of the zeros of the zeta function ζ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ has lost the regularity of pattern which is characteristic of the non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ and which is described above, and is now disorderly and irregular. This would in turn imply that the distribution of the primes is also similarly disorderly and irregular since the primes and the non-trivial zeros of the zeta function ζ are intimately linked and are some sort of mirror images of one another - any changes in one of them would be reflected in the other on account of their intimate link - note that the zeta function ζ has the property of prime sieving encoded within it (comparable to the sieve of Eratosthenes), the properties of the prime counting function $\pi(n)$ being somehow encoded in the properties of the zeta function ζ, so that if the zeros generated were disorderly and irregular it would mean that the distribution of the primes were also similarly disorderly and irregular - the characteristic of the primes on the input side of the function determines the characteristic of the zeros on the output side of the function (i.e., the distribution of the primes determines the distribution of the zeros, so that from a study of the distribution of the zeros the distribution of the primes could be deduced and vice versa), which is expected for a function. The overall result would be that the more orderly the distribution of the zeros is the more orderly would be the corresponding distribution of the primes, the more disorderly the distribution of the zeros is the more disorderly would be the corresponding distribution of the primes, and, vice versa. But, according to the prime number theorem, or, prime counting function $\pi(n)$, which is apparently closely connected with the Riemann hypothesis itself being an apparent offspring and weaker version of it as is described above, there is instead actually a regularity in the way the primes thin out, with the prime counting function $\pi(n)$ even providing a progressively better estimate of the quantity of primes towards infinity - this progressively better estimate would not be possible if the primes behave really badly and are really highly disorderly and irregular - there is no such really great disorder or irregularity among the primes, a state of affair which is evidently affirmed by the fact that the corresponding non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ display regularity in the way they line up at the critical line $\mathrm{Re}(s) = 1/2$, the non-trivial zeros becoming progressively closer together there with no zeros appearing anywhere else on the critical strip (all of which has been found to be true for the first 10^{13} non-trivial zeros - an important point to note is that though the non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ become more and more closely packed together the farther along we move up this critical line while the primes occur farther and farther along the number line, the density of the one is approximately the reciprocal of the density of the other wherein the complementariness, regularity, symmetry is evident), this regularity of the distribution of the non-trivial zeros mirroring the regularity of the distribution of the primes as is explained above. Our assumption of the falsity of the Riemann hypothesis has thus resulted in a contradiction of the actual distribution of the primes and the actual distribution of the corresponding non-trivial zeros at the critical line $\mathrm{Re}(s) = 1/2$ on the critical strip between $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$. If our assumption that the Riemann hypothesis is false is correct, the prime number theorem would be false as there would be great disorder and irregularity among the primes with no regularity in the way the primes thin out and without the prime counting function $\pi(n)$ providing a progressively better estimate of the quantity of primes towards infinity (this progressively better estimate of the quantity of primes actually implies some regularity in the distribution of the primes). However, as is explained just above the prime number theorem is not false; it had in fact been proven through both non-elementary methods by Hadamard and de la Vallee Poussin, and, elementary methods by Erdos and Selberg later, and is indubitably true. Hence, our assumption of the falsehood of the Riemann hypothesis is at fault. This implies that the Riemann hypothesis is true, since the hypothesis cannot be false; this is a reasoning by contradiction which may be interesting but may not be viewed a very strong or convincing reasoning as the reasoning may be too subtle to be fully grasped and make great sense, even possibly causing misunderstanding, though, at least, it shows the close connection between the Riemann hypothesis and the prime number theorem; a stronger reasoning would be forwarded below. The close link between the Riemann hypothesis and the prime number theorem is thus evident.

The Riemann hypothesis posits that all the non-trivial zeros of the zeta function ζ on the critical strip bounded by Re(s) = 0 and Re(s) = 1 will always be at the critical line Re(s) = 1/2. This has been observed to be true for the first 10^{13} non-trivial zeros. The locations of these non-trivial zeros on the critical strip are described by a complex number $s = 1/2 + bi$ where the real part is 1/2 and i stands for the square root of -1. It should be noted that the mathematical operations and logic of the complex numbers $a + bi$, where a and b are real numbers and i is the imaginary number square root of -1, are practically the same as for the real numbers and are even more versatile. For the zeta function $\zeta(s)$ to be zero, its series would have to have both the positive terms and negative terms cancelling each other out, though the positive or "+" signs in the series may indicate positive values only which is misleading. We would here consider the possibility of any non-trivial zeros being off the critical line Re(s) = 1/2 on the critical strip between Re(s) = 0 and Re(s) = 1, e.g., at Re(s) = 1/4, 1/3, 3/4, 4/5, etc.

It had been proven that there will not be zeros at Re(s) = 0 and Re(s) = 1. The first 10^{13} non-trivial zeros are found only at the critical line Re(s) = 1/2. Nature appears to demand that these zeros must appear only at Re(s) = 1/2, exactly mid-way in the critical strip bounded by Re(s) = 0 and Re(s) = 1 wherein the symmetry is perfect. "1/2" in the complex number 1/2 + bi, which is "square root", also appears to be compatible with and work fine with "i", which is "square root of -1" - both of them are square roots. 1/2 + bi has what is called a complex conjugate 1/2 - bi so that when 1/2 + bi and 1/2 - bi are added together the terms bi in both 1/2 + bi and 1/2 - bi will cancel out one another - in this way the troublesome i which does not actually make mathematical sense will be got rid of. 1/2 is also the reciprocal of the smallest prime and the smallest even number 2, which is significant. But there is a much more compelling reason why all the non-trivial zeros must appear on the critical line Re(s) = 1/2 and it is due to some important similarity to Fermat's last theorem.

We here compare Fermat's last theorem with the Riemann hypothesis. As per Fermat's last theorem, the following Diophantine equation which has power n = 2 is the only Diophantine equation with zeros or solutions (zeros and solutions are synonymous):-

$$x^2 + y^2 = z^2 \qquad (2.3)$$

Below is a partial list of Diophantine equations with their zeros:-

[1] $3^2 + 4^2 = 5^2$
$3^2 + 4^2 - 5^2 = 0$

[2] $5^2 + 12^2 = 13^2$
$5^2 + 12^2 - 13^2 = 0$

[3] $7^2 + 24^2 = 25^2$
$7^2 + 24^2 - 25^2 = 0$

[4] $8^2 + 15^2 = 17^2$
$8^2 + 15^2 - 17^2 = 0$

[5] $9^2 + 40^2 = 41^2$
$9^2 + 40^2 - 41^2 = 0$

[6] $11^2 + 60^2 = 61^2$
$11^2 + 60^2 - 61^2 = 0$

[7] $12^2 + 35^2 = 37^2$
$12^2 + 35^2 - 37^2 = 0$

[8] $13^2 + 84^2 = 85^2$
$13^2 + 84^2 - 85^2 = 0$

[9] $16^2 + 63^2 = 65^2$
$16^2 + 63^2 - 65^2 = 0$

[10] $20^2 + 21^2 = 29^2$
$20^2 + 21^2 - 29^2 = 0$

[11] $28^2 + 45^2 = 53^2$
$28^2 + 45^2 - 53^2 = 0$

[12] $33^2 + 56^2 = 65^2$
$33^2 + 56^2 - 65^2 = 0$

[13] $36^2 + 77^2 = 85^2$
$36^2 + 77^2 - 85^2 = 0$

[14] $39^2 + 80^2 = 89^2$
$39^2 + 80^2 - 89^2 = 0$

[15] $48^2 + 55^2 = 73^2$
$48^2 + 55^2 - 73^2 = 0$

[16] $65^2 + 72^2 = 97^2$
$65^2 + 72^2 - 97^2 = 0$

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There is some important similarity between Fermat’s last theorem and the Riemann hypothesis, both of them being involved with series, which will be brought up.

Similar to the series of the Riemann zeta function ζ (1/2), the above Diophantine equations (a few equations with terms that are duplicative are omitted) could be turned into a long series (in fact, an infinitely long series like the series of the Riemann zeta function ζ (1/2)) of positive and negative terms which cancel to give a zero, by adding them together as follows:-

$3^2 + 4^2 - 5^2 + 7^2 + 24^2 - 25^2 + 8^2 + 15^2 - 17^2 + 9^2 + 40^2 - 41^2 + 11^2 + 60^2 - 61^2 + 12^2 + 35^2 - 37^2 + 13^2 + 84^2 - 85^2 + 20^2 + 21^2 - 29^2 + 28^2 + 45^2 - 53^2 + 39^2 + 80^2 - 89^2 + 48^2 + 55^2 - 73^2 + 65^2 + 72^2 - 97^2 = 0$

or, with the same terms re-arranged in numerically ascending order, as follows:-

$3^2 + 4^2 - 5^2 + 7^2 + 8^2 + 9^2 + 11^2 + 12^2 + 13^2 + 15^2 - 17^2 + 20^2 + 21^2 + 24^2 - 25^2 + 28^2 - 29^2 + 35^2 - 37^2 + 39^2 + 40^2 - 41^2 + 45^2 + 48^2 - 53^2 + 55^2 + 60^2 - 61^2 + 65^2 + 72^2 - 73^2 + 80^2 + 84^2 - 85^2 - 89^2 - 97^2 = 0$

The long series above show the very great likeness between Fermat's last theorem and the Riemann hypothesis.

In the above Diophantine equations, the regularity of the powers $n = 2$ is evident. If any of these equations are raised to powers $n > 2$ the regularity will be lost and there will not be zeros, a truth which had been proven by Andrew Wiles as per Fermat's last theorem.

We would show why there are no zeros for the Riemann zeta function ζ for $s < 1/2$ and $s > 1/2$ by bringing up the common underlying principle behind it and Fermat's last theorem, $s = 1/2$ being evidently the optimum or equilibrium power, the only power which brings equilibrium, balance or regularity and thereby the zeros to the Riemann zeta function ζ.

For the case for $x^n + y^n = z^n$ above for Fermat's last theorem which asserts that there are no solutions for $n > 2$, we first explain why there are no solutions for $n > 2$. We begin by selecting example [1] from the list of Diophantine equations above, which has the smallest odd prime number 3 and the smallest composite number 4 (which is the square of the smallest prime number 2) in the series on the left, i.e., the smallest Diophantine equation which has 2 as the power, for illustration:-

$$3^2 + 4^2 = 5^2$$

If the power of 2 in the series on the left above were increased to 3, which is the next, consecutive whole number, e.g., the sum on the right would not be a whole number anymore, which is in accordance with Fermat's last theorem:-

$$3^3 + 4^3 = 4.49795^3$$

The regularity of the power of 2 is now lost, which is for the smallest Diophantine equation which initially had 2 as the power. For the larger Diophantine equations with initial powers of 2 the irregularity after increasing their powers to 3, which is the next, consecutive whole number, or, higher powers, could be expected to be worse.

In the next step we bring up the values of, say, 100, of consecutive whole number powers n, say, 2 to 5, this quantity 100 being representative of the terms of the equation $x^n + y^n = z^n$ as per Fermat's last theorem, to explain the reason for this irregularity, which is as follows:-

[1] $100^2 = 10{,}000$ (The terms of the series of Fermat's last theorem fall under this category. All zeros will be found under this category only.)

[2] $100^3 = 1{,}000{,}000$ (This quantity represents an increase of 9,900% compared to [1] above while the increase in power from $n = 2$ to $n = 3$ is only 50%.)

[3] $100^4 = 100{,}000{,}000$ (This quantity represents an increase of 999,900% compared to [1] above while the increase in power from $n = 2$ to $n = 4$ is only 100%.)

[4] $100^5 = 10{,}000{,}000{,}000$ (This quantity represents an increase of 99,999,900% compared to [1] above while the increase in power from $n = 2$ to $n = 5$ is only 150%.)

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The quantities from the consecutive whole number powers $n > 2$ above increase progressively compared to [1], the larger the power n is the larger the percentage of increase in the quantity is. The increases in the respective quantities and powers are also disproportionate when compared to one another, with the increases in the respective quantities being evidently much too quick. All this shows that the equilibrium, balance or regularity of $x^n + y^n = z^n$ when $n = 2$ as per Fermat's last theorem cannot be maintained when $n > 2$, when disproportionateness between the increases in the respective quantities and powers sets in as is described above, as the increase in quantity is too quick, and, when $n < 2$, e.g., $n = 5/4, 3/2, 7/4$, etc., as the increase in quantity is too slow as could be extrapolated from the above example. (Refer to Appendix 1 below for an analogous example.) For Fermat's last theorem, $n = 2$ can be regarded as the optimum or equilibrium power, the only power wherein $x^n + y^n = z^n$ is possible. There is also the question of the easier solubility of equations with whole number powers $n = 2$ as compared to equations with powers $n > 2$, e.g., $n = 3, 4, 5$, etc., and $n < 2$, e.g., $n = 5/4, 3/2, 7/4$, etc., which is explained below.

For the case of the Riemann zeta function ζ wherein there are no zeros for powers $s < 1/2$ and $s > 1/2$, we bring up the values of the reciprocals of, say, 100, with consecutive fractional powers s, say, 1/2 to 1/5, these reciprocals being representative of the terms of the Riemann zeta function ζ, to explain the reason for the irregularity for powers $s < 1/2$ and $s > 1/2$, which is as follows:-

[1] $1/100^{1/2} = 1/10 = 0.100$ (The terms of the series of the Riemann zeta function ζ as per the Riemann hypothesis fall under this category. 10^{13} zeros have been found under this category only.)

[2] $1/100^{1/3} = 1/4.6416 = 0.215$ (This quantity represents an increase of 115% compared to [1] above while the decrease in power from $s = 1/2$ to $s = 1/3$ is only 33.33%.)

[3] $1/100^{1/4}$ = 1/3.1623 = 0.316 (This quantity represents an increase of 216% compared to [1] above while the decrease in power from *s* = 1/2 to *s* = 1/4 is only 50%.)

[4] $1/100^{1/5}$ = 1/2.5119 = 0.398 (This quantity represents an increase of 298% compared to [1] above while the decrease in power from *s* = 1/2 to *s* = 1/5 is only 60%.)

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As can be seen above, the smaller the power of the reciprocal/denominator is the larger will be the result after division with 1 (or, the larger the power of the reciprocal/denominator is the smaller will be the result after division with 1). The quantities from the reciprocals with consecutive fractional powers $s < 1/2$ above increase progressively compared to [1], the smaller the power *s* is the larger the percentage of increase in the quantity is, the increases in the quantities being similar to the case above for Fermat's last theorem - this shows a similarity between Fermat's last theorem and the Riemann hypothesis. The increases in the respective quantities and the decreases in the respective powers are also disproportionate when compared to one another, with the increases in the respective quantities being evidently much too quick, which is similar to the case above for Fermat's last theorem - this shows another similarity between Fermat's last theorem and the Riemann hypothesis. All this implies that the equilibrium, balance or regularity of the Riemann zeta function ζ when s = 1/2 cannot be maintained when $s < 1/2$, when disproportionateness between the increases and decreases in the respective quantities and powers sets in as is described above, as the increase in quantity is too quick, and, when $s > 1/2$, e.g., *s* = 3/4, 4/5, 5/6, etc., as the increase in quantity is too slow as could be extrapolated from the above example. (Refer to Appendix 1 below for the full details.) For these reciprocals, *s* = 1/2 can be regarded as the optimum or equilibrium power, the only power wherein zeros for the Riemann zeta function ζ are possible. Similar to the case for Fermat's last theorem above, there is also the question of the easier solubility of equations with fractional powers *s* = 1/2 as compared to equations with fractional powers $s < 1/2$, e.g., *s* = 1/3, 1/4, 1/5, etc., and $s > 1/2$, e.g., *s* = 3/4, 4/5, 5/6, etc., which is explained below.

The following list of the first 10 terms of the series of the Riemann zeta function ζ with consecutive fractional powers $s \leq 1/2$ also shows that the sums with smaller powers increase progressively, i.e., the smaller the power *s* is the larger the percentage of increase in the quantity is:-

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + 1/6^{1/2} + 1/7^{1/2} + 1/8^{1/2} + 1/9^{1/2} + 1/10^{1/2} + \ldots =$ 5.03
(The Riemann hypothesis asserts that all zeros will be found in this series only.)

[2] $\zeta(1/3) = 1 + 1/2^{1/3} + 1/3^{1/3} + 1/4^{1/3} + 1/5^{1/3} + 1/6^{1/3} + 1/7^{1/3} + 1/8^{1/3} + 1/9^{1/3} + 1/10^{1/3} + \ldots =$ 6.20
(The sum 6.20 here represents an increase of 23.26% compared to the sum 5.03 in [1] above while the percentage of decrease in power from *s* = 1/2 to *s* = 1/3 is 33.33%.)

[3] $\zeta(1/4) = 1 + 1/2^{1/4} + 1/3^{1/4} + 1/4^{1/4} + 1/5^{1/4} + 1/6^{1/4} + 1/7^{1/4} + 1/8^{1/4} + 1/9^{1/4} + 1/10^{1/4} + \ldots =$

6.97
(The sum 6.97 here represents an increase of 38.57% compared to the sum 5.03 in [1] above while the percentage of decrease in power from s = 1/2 to s = 1/4 is 50%.)

[4] $\zeta(1/5) = 1 + 1/2^{1/5} + 1/3^{1/5} + 1/4^{1/5} + 1/5^{1/5} + 1/6^{1/5} + 1/7^{1/5} + 1/8^{1/5} + 1/9^{1/5} + 1/10^{1/5} + \ldots =$ 7.46
(The sum 7.46 here represents an increase of 48.31% compared to the sum 5.03 in [1] above while the percentage of decrease in power from s = 1/2 to s = 1/5 is 60%.)

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Note: Though the respective percentages of increase in quantity above, namely, 23.26%, 38.57% & 48.31%, are disproportionate with and lower than the respective percentages of decrease in power, namely, 33.33%, 50% & 60%, at a later stage when there are more and more terms in the series, there being an infinitude of terms, when the sums get larger and larger, the percentages of increase in quantity will all be infinitely higher than the percentages of decrease in power, as is evident from **Table 1** below. The same will apply for the quantities when the powers $s > 1/2$, e.g., s = 3/4, 4/5, 5/6, etc., as could be extrapolated from the above list (and is evident from Appendix 2 below).

(The series of the Riemann zeta function ζ with powers $s > 1/2$, e.g., s = 3/4, 4/5, 5/6, etc., will have sums which are all smaller than the sums shown in the above list for powers $s \leq 1/2$ as could be extrapolated from the above list. For the largest power in the critical strip s = 1, which has no zeros, the sum of the first 10 terms is a mere 2.93. Refer to Appendix 1 below for an analogous example.)

It is evident from all the above that when the sum of the series in the Riemann zeta function ζ increases too quickly as is the case when the powers $s < 1/2$, when disproportionateness between the increases and decreases in the respective quantities and powers sets in as is described above, or, too slowly as is the case when the powers $s > 1/2$, e.g., s = 3/4, 4/5, 5/6, etc., as could be extrapolated from the above list, the equilibrium, balance or regularity will be lost and there will not be zeros. (Refer to Appendix 1 below for an analogous example.) Similar to the case of Fermat's last theorem wherein all the zeros will be at the optimum or equilibrium power n = 2 only, all the zeros of the Riemann zeta function ζ will be at the optimum or equilibrium power s = 1/2 only. (The analogue of this optimum or equilibrium power could be that of a shirt or pants which exactly fits a person, e.g., size A could be too small for the person, size C too large, while size B fits just fine.) At least 10^{13} zeros have been found at s = 1/2 while none has been found for $s < 1/2$ and $s > 1/2$.

We bring up an important point here. If more and more terms are added to the series in the list of the sums of the Riemann zeta function ζ above where the consecutive fractional powers $s \leq 1/2$, which presently have 10 terms each, the differences in the sums between that for power s = 1/2 and that for powers $s < 1/2$, e.g., s = 1/3, 1/4, 1/5, etc., and, that for power s = 1/2 and that for powers $s > 1/2$, e.g., s = 3/4, 4/5, 5/6, etc., will be greater and greater, i.e., the differences between these sums will be more pronounced the more terms are added to the series. We can see this point by comparing, e.g., the sums of the first 5 terms of the Riemann zeta function ζ for consecutive fractional powers $s \leq 1/2$ and the sums of the first 10 terms of the Riemann zeta function ζ for consecutive fractional powers $s \leq 1/2$, which is as follows, and extrapolating from there:-

For the comparison, we here compute the sums for the first 5 terms of the series of the Riemann zeta function ζ with consecutive fractional powers $s \leq 1/2$ as follows, after which the results of this computation are incorporated in **Table 1** (shown in bold) below:

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + \ldots = 3.24$
(The Riemann hypothesis asserts that all zeros will be found in this series only.)

[2] $\zeta(1/3) = 1 + 1/2^{1/3} + 1/3^{1/3} + 1/4^{1/3} + 1/5^{1/3} + \ldots = 3.69$
(The sum 3.69 here represents an increase of **13.89%** (the increase here is **23.26%** for the 1st. 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

[3] $\zeta(1/4) = 1 + 1/2^{1/4} + 1/3^{1/4} + 1/4^{1/4} + 1/5^{1/4} + \ldots = 3.98$
(The sum 3.98 here represents an increase of **22.84%** (the increase here is **38.57%** for the 1st. 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

[4] $\zeta(1/5) = 1 + 1/2^{1/5} + 1/3^{1/5} + 1/4^{1/5} + 1/5^{1/5} + \ldots = 4.15$
(The sum 4.15 here represents an increase of **28.09%** (the increase here is **48.31%** for the 1st. 10 terms as is shown in the list above) compared to the sum 3.24 in [1] above.)

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Table 1 below of the above-mentioned **percentage *increases*** for the sums for the first 2 terms to the first 10 terms for ζ(1/3), ζ(1/4) & ζ(1/5) will give a clearer picture:-

	1st. 2 Terms	1st. 3 Terms	1st. 4 Terms	1st. 5 Terms	1st. 6 Terms	1st. 7 Terms	1st. 8 Terms	1st. 9 Terms	1st. 10 Terms	1st. 11 Terms	…
[1] ζ(1/2)	-	-	-	-	-	-	-	-	-	-	
[2] ζ(1/3)	4.68%	8.30%	11.47%	**13.89%**	16.16%	18.11%	20.09%	21.87%	**23.26%**	To Be Extrapolated	
[3] ζ(1/4)	7.60%	13.54%	18.28%	**22.84%**	26.30%	29.78%	32.88%	35.88%	**38.57%**	To Be Extrapolated	
[4] ζ(1/5)	9.36%	16.59%	22.94%	**28.09%**	32.88%	37.22%	41.32%	45.01%	**48.31%**	To Be Extrapolated	

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It is evident that the percentage increases shown above will go up in value continuously to infinity with the infinitude of the terms of the Riemann zeta function ζ. All this indicates more and more bad news for the solubility of the Riemann zeta function ζ for powers $s < 1/2$, and, $s > 1/2$ (as could be extrapolated from the above; refer to Appendix 1 and Appendix 2 (which provides an example) below) when there are more and more terms in the Riemann zeta function ζ, i.e., for powers $s < 1/2$ and $s > 1/2$, the more terms there are in the Riemann zeta function ζ the less soluble it will be. This is a serious irregularity and is another reason why there are no zeros for the Riemann zeta function ζ for powers $s < 1/2$ and $s > 1/2$.

The similarity between the Riemann hypothesis and Fermat's last theorem is great - they each have an optimum or equilibrium power which is the only power wherein zeros are possible - $s = 1/2$ in the case of the Riemann hypothesis and $n = 2$ in the case of Fermat's last theorem, powers which are all solely responsible for all the zeros. The fact that all these optimum or equilibrium powers are either square root ($s = 1/2$ for the Riemann hypothesis) or square ($n = 2$ for Fermat's last theorem) is significant as they seem some sort of images of 2 which is the smallest prime number and the smallest even number. $s = 1/2$ is the largest root among the roots with 1 as the numerator. As such $s = 1/2$ as a fractional power with 1 as the numerator gives the largest result as compared to the fractional powers with 1 as the numerator $s < 1/2$, e.g., $s = 1/3$, 1/4, 1/5, etc. (but this largest result brings the *smallest increase in quantity* as compared to the results of the fractional powers with 1 as the numerator $s < 1/2$, e.g., $s = 1/3$, 1/4, 1/5, etc., when divided by 1, e.g., $1/2^{1/2} < 1/2^{1/3} < 1/2^{1/4} < 1/2^{1/5}$, etc. - *this is an important similarity to the case for $n = 2$ described below*) - equations with fractional powers $s = 1/2$ would evidently be easier to solve than equations with fractional powers $s < 1/2$ (e.g., in a computation $s = 1/2$ needs only 1 rooting step while $s = 1/5$ needs 4 rooting steps) and $s > 1/2$, e.g., $s = 2/3$, 3/4, 4/5, etc. (e.g., in a computation $s = 1/2$ needs only 1 rooting step, while $s = 4/5$ needs 7 steps - 3 squaring steps for $s = 4$ & 4 rooting steps for $s = 1/5$). $n = 2$ is the smallest whole number power which brings an increase in quantity. As such $n = 2$ is the whole number power which brings the *smallest increase in quantity* as compared to the whole number powers $n > 2$, e.g., $n = 3$, 4, 5, etc., for instance $2^2 < 2^3 < 2^4 < 2^5$, etc. - equations with whole number powers $n = 2$ would evidently be easier to solve than equations with powers $n > 2$ (with general equations with powers $n = 5$ having been proven unsolvable - $n = 2$ needs only 1 squaring step while $n = 5$ needs 4 squaring steps) and $n < 2$, e.g., $n = 5/4$, 3/2, 7/4, etc. (e.g., in a computation $n = 2$ needs only 1 squaring step, while $n = 7/4$ needs 9 steps - 6 squaring steps for $n = 7$ & 3 rooting steps for $n = 1/4$). $n = 2$ and its reciprocal $s = 1/2$ are the opposite of one another but despite this there appears to be complementariness and symmetry between them, as is evident in the cases of Fermat's last theorem and the Riemann hypothesis which involve optimum or equilibrium powers $n = 2$ and its reciprocal $s = 1/2$, the only powers wherein zeros are possible for each of them. $n = 2$ and its reciprocal $s = 1/2$ are evidently important quantities which may be comparable to π (3.14159265) or e (2.71828).

It is evident that the Riemann hypothesis is the analogue of Fermat's last theorem, which points to its validity.

Hence, for the Riemann zeta function ζ, $s = 1/2$ is the optimum or equilibrium power wherein there would be zeros. No zeros would be found in the critical strip bounded by $\mathrm{Re}(s) = 0$ and $\mathrm{Re}(s) = 1$ for $s < 1/2$ and $s > 1/2$ because if $s < 1/2$ the sum of the series in the zeta function ζ increases too fast when more and more terms are added to the series and if $s > 1/2$ the sum of the series in the zeta function ζ increases too slowly when more and more terms are added to the series; $s = 1/2$ is evidently optimum, just fits - evidently the only power conducive for the production of zeros.

We here elaborate more on the apparently subtle points in the above paragraph which may be difficult to grasp. To grasp the point that if $s < 1/2$ the sum of the series in the zeta function ζ increases too fast when more and more terms are added to the series we need to make a close study of and understand **Table 1** above referring also to the computations above this table, and, to grasp the point that if $s > 1/2$ the sum of the series in the zeta function ζ increases too slowly when more and more terms are added to the series we need to study closely and understand **Table 2** in **Appendix 2** below referring also to the computations above this table. A careful study of **Table 1** above would reveal that for $s < 1/2$ all the sums for these series, e.g., for $s = 1/3$, 1/4, 1/5, 1/6, etc., would diverge more and more from the sum for $s = 1/2$ when more and more terms are added to all these series including $s = 1/2$. As evidently only the series for s = 1/2 are conducive for the production of zeros, what this implies is that as more and more terms are added to the series for $s < 1/2$ such as $s = 1/3$, 1/4, 1/5 and 1/6, it would be less and less likely for these series to be able to produce zeros (i.e., these series would be less and less soluble) due to the rate of ***increase*** of their sums becoming greater and greater (in fact too greatly) with more and more terms added to these series. Likewise, a careful study of **Table 2** in **Appendix 2** below would also reveal that for $s > 1/2$ all the sums for these series, e.g., for $s = 2/3$, 3/4, 4/5, 5/6, etc., would diverge more and more from the sum for $s = 1/2$ when more and more terms are added to all these series including $s = 1/2$. Since evidently only the series for s = 1/2 are conducive for the production of zeros, what this also implies is that as more and more terms are added to the series for $s > 1/2$ such as $s = 2/3$, 3/4, 4/5 and 5/6, it would be less and less likely for these series to be able to produce zeros (i.e., these series would be less and less soluble) due to the rate of ***decrease*** of their sums becoming greater and greater (in fact too greatly) with more and more terms added to these series. In other words, for the Riemann zeta function ζ for powers $s < 1/2$ and $s > 1/2$, the more terms there are in the Riemann zeta function ζ the less soluble it will be, which is a serious irregularity. Extrapolating from **Table 1** and **Table 2** it is evident that the non-trivial zeros would not be found in the critical strip bounded by Re(s) = 0 and Re(s) = 1 for $s < 1/2$ and $s > 1/2$.

There is the feeling that for $s < 1/2$ and $s > 1/2$ the Riemann zeta function ζ may yield some non-trivial zero or zeros after innumerable terms, e.g., after many billions, trillions or more terms, have been added to the series, as past experience has shown this could happen. However, extrapolations with **Table 1** above and **Table 2** in **Appendix 2** below would show that this is not possible. It may happen only when the following occur: (a) For $s < 1/2$ all the sums for these series, e.g., for $s = 1/3$, 1/4, 1/5, 1/6, etc., would diverge less and less (instead of more and more), even gradually so, from the sum for $s = 1/2$ when more and more terms are added to all these series including $s = 1/2$. (b) For $s > 1/2$ all the sums for these series, e.g., for $s = 2/3$, 3/4, 4/5, 5/6, etc., would diverge less and less (instead of more and more), even gradually so, from the sum for $s = 1/2$ when more and more terms are added to all these series including $s = 1/2$. As the Riemann hypothesis is shown above to be the analogue of Fermat's last theorem and Fermat's last theorem posits that there are solutions only for $n = 2$ and none for $n > 2$ and $n < 2$, by the same principle there should not be solutions for $s < 1/2$ and $s > 1/2$ and the feeling that for $s < 1/2$ and $s > 1/2$ the Riemann zeta function ζ may yield some non-trivial zero or zeros after innumerable terms have been added to the series appears misplaced.

3 Conclusion

As per the reasons above, all the non-trivial zeros of the Riemann zeta function ζ could be expected to be found on the critical line Re(s) = 1/2 only and not anywhere else on the critical strip bounded by Re(s) = 0 and Re(s) = 1.

Appendix 1

Below are the values of the reciprocals of, say, 100, with consecutive fractional powers $s \leq 4/5$, these reciprocals being representative of the terms of the Riemann zeta function ζ:-

[1] $1/100^{4/5}$ = 1/39.8107171 = 0.025 (This quantity represents a decrease of 75% compared to [4] below while the increase in power from $s = 1/2$ to $s = 4/5$ is only 60%.)

[2] $1/100^{3/4}$ = 1/31.62278 = 0.032 (This quantity represents a decrease of 68% compared to [4] below while the increase in power from $s = 1/2$ to $s = 3/4$ is only 50%.)

[3] $1/100^{2/3}$ = 1/21.5444 = 0.046 (This quantity represents a decrease of 54% compared

to [4] below while the increase in power from $s = 1/2$ to $s = 2/3$ is only 33.33%.)

[4] $1/100^{1/2} = 1/10 = 0.100$ (The terms of the series of the Riemann zeta function ζ as per the Riemann hypothesis fall under this category. 10^{13} zeros have been found under this category only.)

[5] $1/100^{1/3} = 1/4.6416 = 0.215$ (This quantity represents an increase of 115% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/3$ is only 33.33%.)

[6] $1/100^{1/4} = 1/3.1623 = 0.316$ (This quantity represents an increase of 216% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/4$ is only 50%.)

[7] $1/100^{1/5} = 1/2.5119 = 0.398$ (This quantity represents an increase of 298% compared to [4] above while the decrease in power from $s = 1/2$ to $s = 1/5$ is only 60%.)

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Note the disproportionateness between the respective percentages of decrease in quantity and the respective percentages of increase in power for the reciprocals with powers $s > 1/2$, and, between the respective percentages of increase in quantity and the respective percentages of decrease in power for the reciprocals with powers $s < 1/2$.

Appendix 2

The following list of the first 5 terms of the series of the Riemann zeta function ζ with consecutive fractional powers $s \geq 1/2$ shows that the sums with larger powers decrease progressively, i.e., the larger the power s is the larger the percentage of decrease in the quantity is:-

[1] $\zeta(1/2) = 1 + 1/2^{1/2} + 1/3^{1/2} + 1/4^{1/2} + 1/5^{1/2} + \ldots = 3.24$
(The Riemann hypothesis asserts that all zeros will be found in this series only.)

[2] $\zeta(2/3) = 1 + 1/2^{2/3} + 1/3^{2/3} + 1/4^{2/3} + 1/5^{2/3} + \ldots = 2.85$
(The sum 2.85 here represents a decrease of **12.04%** compared to the sum 3.24 in [1] above.)

[3] $\zeta(3/4) = 1 + 1/2^{3/4} + 1/3^{3/4} + 1/4^{3/4} + 1/5^{3/4} + \ldots = 2.68$
(The sum 2.68 here represents a decrease of **17.28%** compared to the sum 3.24 in [1] above.)

[4] $\zeta(4/5) = 1 + 1/2^{4/5} + 1/3^{4/5} + 1/4^{4/5} + 1/5^{4/5} + \ldots = 2.59$
(The sum 2.59 here represents a decrease of **20.06%** compared to the sum 3.24 in [1] above.)

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Table 2 below is a tabulation of the above-mentioned **percentage *decreases*** for the sums for the first 2 terms to the first 5 terms for ζ(2/3), ζ(3/4) & ζ(4/5):-

	1st. 2 Terms	1st. 3 Terms	1st. 4 Terms	1st. 5 Terms	1st. 6 Terms	…
[1] ζ(1/2)	-	-	-	-	-	
[2] ζ(2/3)	4.52%	7.63%	9.98%	**12.04%**	To Be Extrapolated	
[3] ζ(3/4)	6.65%	11.08%	14.37%	**17.28%**	To Be Extrapolated	
[4] ζ(4/5)	7.86%	12.98%	16.78%	**20.06%**	To Be Extrapolated	
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