Can von Neumann's theory be consistent even for commuting observables?

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Nagata and Nakamura claim to derive an inconsistency in von Neumann's theory [K. Nagata, Int. J. Theor. Phys. **48**, 3532 (2009)] and [K. Nagata and T. Nakamura, Int. J. Theor. Phys. **49**, 162 (2010)]. Barros discusses [J. A. de Barros, Int. J. Theor. Phys. **50**, 1828 (2011)] the inconsistencies do not come from von Neumann's theory, but from extra assumptions about the reality of observables. The quantum theory is equivalent to classical theory when we consider only commuting observables. Using this fact, we discuss there is an inconsistency within von Neumann's theory. We can omit extra assumptions about the reality of observables when we consider only commuting observables. Finally, we present main theorem concerning von Neumann's theory.

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I. INTRODUCTION

von Neumann's theory (cf. [1–5]) is a physical theory. Recently, Nagata and Nakamura claim [6, 7] to derive an inconsistency in von Neumann's theory. Barros discusses [8] that the inconsistencies do not come from von Neumann's theory, but from extra assumptions about the reality of observables. We discuss the inconsistency come from von Neumann's theory, without extra assumptions about the reality of observables. We show here the inconsistency by using commuting observables in an arbitrary dimensional unitary space ($d \geq 1$).

We notice that von Neumann's mathematical model is quite logically successful. And the axiomatic system for the mathematical model is a very consistent one. Thus, we cannot say that von Neumann's mathematical model has an inconsistency. What is the inconsistency to be discussed in this paper? We cannot expand the von Neumann's beautiful mathematical model more in handling real experimental data. Mathematically, von Neumann's model is logically very consistent, which fact is true. However, von Neumann's theory is questionable in the sense that the mathematical model does not always explain real experimental data. And there is the inconsistency if we apply the von Neumann's model to explaining even a simple physical situation. In short, von Neumann's mathematical model might not be useful in that case.

Here, we discuss there is an inconsistency within von Neumann's theory. We do not accept extra assumptions about the reality of observables. We suppose the two measured observables are commutative. We introduce a supposition that the sum rule and the product rule commute with each other and we have an example of an inconsistency.

II. VON NEUMANN'S THEORY FOR TWO COMMUTING OBSERVABLES

Let A_1, A_2 be two Hermitian operators, where they are also supposed to be commutative. Though doing later, we dare to introduce firstly a supposition that the sum rule and the product rule commute with each other for the purpose of showing our interesting objective obtained here [9]. A supposition that the sum rule and the product rule commute with each other means a supposition that the two operations Addition and Multiplication commute with each other (see [10]). In other words, the operation Addition is equivalent to the operation Multiplication.

They are defined respectively as follows:

$$A_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A_2 \equiv \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1}$$

Let us consider an input state $|\Psi\rangle$ such that

$$\langle\Psi|A_1|\Psi\rangle = +1, \quad \langle\Psi|A_2|\Psi\rangle = -1. \tag{2}$$

Thus, the measured results of trials are either $+1$ or -1.

The functional rule is defined as follows:

$$f(g(O)) = g(f(O)), \tag{3}$$

where O is an Hermitian operator. On the one hand, the sum rule is defined as follows:

$$f(A_1 + A_2) = f(A_1) + f(A_2). \tag{4}$$

On the other hand, the product rule is defined as follows:

$$f(A_1 \cdot A_2) = f(A_1) \cdot f(A_2). \tag{5}$$

This fact above is based on the property of these two Hermitian operators themselves. This leads to the proposition that it is valid even for the real numbers of the diagonal elements of the two Hermitian operators.

We may have [3, 11, 12] the following relation between the three rules which are valid for the commuting observables:

$$\begin{aligned}
&\text{The functional rule}\\
&\Leftrightarrow \text{The sum rule}\\
&\Leftrightarrow \text{The product rule} \qquad (6)
\end{aligned}$$

The Kochen-Specker theorem says the situation that some quantum observables do not commute [13]. Our argumentations can be based on the two commuting observables. We may introduce a hidden variable theory in handling real experimental data [14]. This fact does not mean a hidden variable theory must be necessary for our purpose. We can introduce the supposition that the sum rule and the product rule commute with each other if and only if all the measured observables commute simultaneously. Notice we consider only the two commuting observables here.

III. VON NEUMANN'S THEORY IS NOT CONSISTENT

We are in an inconsistency when the first result is $+1$ by the measured observable A_1, the second result is -1 by the measured observable A_2 and then $[A_1, A_2] = 0$.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either $+1$ or -1. We suppose the number of -1 is equal to the number of $+1$. If the number of trials is 2, then we have

$$V = (+1) + (-1) = 0. \qquad (7)$$

By using r_1 and r_2, we can depicture experimental data as follows: $r_1 = +1$ and $r_2 = -1$.

Let us write V as follows:

$$V = (\sum_{l=1}^{2} r_l). \qquad (8)$$

The possible values of the measured results r_l are either $+1$ or -1.

In the following, we evaluate a value $(V \times V)$ and derive a necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1, A_2] = 0$.

We introduce a supposition that the sum rule and the product rule commute with each other [10, 15, 16]. We have

$$\begin{aligned}
&V \times V\\
&= (\sum_{l=1}^{2} r_l)^2\\
&= (\sum_{l'=1}^{2} r_{l'}) \times (\sum_{l=1}^{2} r_l)\\
&= \sum_{l'=1}^{2} \sum_{l=1}^{2} r_{l'} r_l\\
&\leq \sum_{l'=1}^{2} \sum_{l=1}^{2} |r_{l'} r_l|\\
&= \sum_{l'=1}^{2} \sum_{l=1}^{2} (r_l)^2\\
&= 2((+1)^2 + (-1)^2)\\
&= 4. \qquad (9)
\end{aligned}$$

The inequality (9) can be saturated because the following case is possible:

$$\begin{aligned}
&\|\{l | r_l = +1\}\| = \|\{l' | r_{l'} = +1\}\|,\\
&\|\{l | r_l = -1\}\| = \|\{l' | r_{l'} = -1\}\|. \qquad (10)
\end{aligned}$$

Thus,

$$(V \times V)_{\max} = 4. \qquad (11)$$

Next, we derive another possible value of the product $V \times V$ of the value V.

We have

$$V \times V = 0. \qquad (12)$$

We have the following supposition:

$$(V \times V)_{\max} = 0. \qquad (13)$$

This necessary condition is true.

We cannot assign simultaneously the truth value "1" for the two suppositions (11) and (13) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

In summary, we have been in the inconsistency when the first result is $+1$, the second result is -1, and $[A_1, A_2] = 0$.

IV. GENERAL THEOREM

Let us move ourselves into the more general case. We are in an inconsistency when the first result is x by the measured observable A_1, the second result is not x by the measured observable A_2 and $[A_1, A_2] = 0$.

We consider a value V which is the sum of two data in an experiment. The measured results of trials are either x or $y (\neq x)$. We suppose the number of y is equal to the number of x. If the number of trials is 2, then we have

$$V = x + y. \qquad (14)$$

By using r_1 and r_2, we can depicture experimental data as follows: $r_1 = x$ and $r_2 = y$.

Let us write V as follows:

$$V = (\sum_{l=1}^{2} r_l). \tag{15}$$

The possible values of the measured results r_l are either x or y.

In the following, we evaluate a value $(V \times V)$ and derive a necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1, A_2] = 0$.

We introduce a supposition that the sum rule and the product rule commute with each other [10, 15, 16]. We have

$$\begin{aligned}
&V \times V \\
&= (\sum_{l=1}^{2} r_l)^2 \\
&= (\sum_{l'=1}^{2} r_{l'}) \times (\sum_{l=1}^{2} r_l) \\
&= \sum_{l'=1}^{2} \sum_{l=1}^{2} r_{l'} r_l \\
&\leq \sum_{l'=1}^{2} \sum_{l=1}^{2} |r_{l'} r_l| \\
&= \sum_{l'=1}^{2} \sum_{l=1}^{2} (r_l)^2 \\
&= 2((x)^2 + (y)^2) \\
&= 2(x^2 + y^2).
\end{aligned} \tag{16}$$

The inequality (16) can be saturated because the following case is possible:

$$\begin{aligned}
&\|\{l|r_l = x\}\| = \|\{l'|r_{l'} = x\}\|, \\
&\|\{l|r_l = y\}\| = \|\{l'|r_{l'} = y\}\|.
\end{aligned} \tag{17}$$

Thus,

$$(V \times V)_{\max} = 2(x^2 + y^2). \tag{18}$$

This is a necessary condition under the supposition that the two measured observables are commuting. That is, $[A_1, A_2] = 0$.

Next, we derive another possible value of the product $V \times V$ of the value V.

We have

$$V \times V = (x + y)^2. \tag{19}$$

We have the following supposition:

$$(V \times V)_{\max} = (x + y)^2. \tag{20}$$

The necessary condition is true.

We cannot assign simultaneously the truth value "1" for the two suppositions (18) and (20) when $[A_1, A_2] = 0$. We derive the inconsistency when $[A_1, A_2] = 0$.

Theorem

For an arbitrary dimensional unitary space ($d \geq 1$), von Neumann's theory is not consistent if there exist commuting observables A_1 and A_2 such that $A_1 \neq A_2$.

V. CONCLUSIONS AND DISCUSSIONS

In conclusions, Nagata and Nakamura have claimed [6, 7] to derive an inconsistency in von Neumann's theory. Barros has discussed [8] as follows: The inconsistencies do not have come from von Neumann's theory, but from extra assumptions about the reality of observables. Here we have discussed there is an inconsistency within von Neumann's theory. We do not have accepted extra assumptions about the reality of observables.

By wanting to prove that there is no axiomatic system for the quantum theory, we have to prove this one have to commit to the rules of inference of the formal system and to the appropriate set of valuations for it.

If the problem were simply an inconsistency, there are multiple logical systems that can cope with such a problem with robustness (see [17] (2011)).

We might have just proved that the whole of mathematics is inconsistent, and therefore useless (since, as Aristotle proved a long time ago, first order logics collapse under inconsistencies, and so does all of mathematics currently known, as it is based on first order logic).

Generally Multiplication is completed by Addition. Therefore, we think that Addition of the starting point may be superior to any other case.

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NOTE

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