ON THE LENGTH OF ADDITION CHAINS PRODUCING $2^n - 1$

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Abstract. Let $\delta(n)$ denotes the length of an addition chain producing n. In this paper we prove that the exists an addition chain producing $2^n - 1$ whose length satisfies the inequality

$$\delta(2^n - 1) \lesssim n - 1 + \iota(n) + \frac{n}{\log n} + 1.3 \log n \int\limits_{2}^{\frac{n-1}{2}} \frac{dt}{\log^3 t} + \xi(n)$$

where $\xi : \mathbb{N} \longrightarrow \mathbb{R}$. As a consequence, we obtain the inequality

$$\iota(2^n - 1) \lesssim n - 1 + \iota(n) + \frac{n}{\log n} + 1.3 \log n \int\limits_{2}^{\frac{n-1}{2}} \frac{dt}{\log^3 t} + \xi(n)$$

where $\iota(n)$ denotes the length of the shortest addition chains producing n.

1. Introduction

An addition chain producing $n \geq 3$, roughly speaking, is a sequence of numbers of the form $1, 2, s_3, s_4, \ldots, s_{k-1}, s_k = n$ where each term is the sum of two earlier terms in the sequence, obtained by adding each sum generated to an earlier term in the sequence. The length of the chain is determined by the number of entries in the sequence excluding n. There are numerous addition chains that result in a fixed number n. The shortest or optimal addition chain produces n. However, given that there is currently no efficient method for getting the shortest addition yielding a given number, reducing an addition chain might be a difficult task. This makes addition chain theory a fascinating subject to study. Arnold Scholz conjectured the inequality by letting $\iota(n)$ denote the length of the shortest addition chain producing n.

Conjecture 1.1 (Scholz)**.** The inequality holds

$$\iota(2^n - 1) \leq n - 1 + \iota(n).$$

It has been shown computationally that the conjecture holds for all $n \leq 5784688$ and in fact it is an equality for all $n \leq 64$ [2]. Alfred Brauer proved the scholz conjecture for the star addition chain, an addition chain where each term obtained by summing uses the immediately subsequent number in the chain. By denoting the shortest length of the star addition chain by $\iota^*(n)$, it is shown that (See,[1])

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Theorem 1.1. *The inequality holds*

$$\iota^*(2^n-1)\leq n-1+\iota^*(n).$$

In this paper we study addition chains producing numbers of the form 2^n-1 and the scholz conjecture. We obtain the inequality

Theorem 1.2. *Let $\delta(n)$ denotes the length of the addition chain producing n. Then there exists an addition chain producing 2^n-1 satisfying the inequality*

$$\delta(2^n-1)\leq n-1+\iota(n)+\frac{n}{\log n}+1.3\log n\int\limits_{2}^{\frac{n-1}{2}}\frac{dt}{\log^3 t}+\xi(n)$$

where $\xi:\mathbb{N}\longrightarrow\mathbb{R}$.

2. Sub-addition chains

In this section we introduce the notion of sub-addition chains.

Definition 2.1. Let $n\geq 3$, then by the addition chain of length $k-1$ producing n we mean the sequence

$$1,2,\ldots,s_{k-1},s_k$$

where each term s_j $(j\geq 3)$ in the sequence is the sum of two earlier terms, with the corresponding sequence of partition

$$2=1+1,\ldots,s_{k-1}=a_{k-1}+r_{k-1},s_k=a_k+r_k=n$$

with $a_{i+1}=a_i+r_i$ and $a_{i+1}=s_i$ for $2\leq i\leq k$. We call the partition a_i+r_i the i th **generator** of the chain for $2\leq i\leq k$. We call a_i the **determiners** and r_i the **regulator** of the i th generator of the chain. We call the sequence (r_i) the regulators of the addition chain and (a_i) the determiners of the chain for $2\leq i\leq k$.

Definition 2.2. Let the sequence $1,2,\ldots,s_{k-1},s_k=n$ be an addition chain producing n with the corresponding sequence of partition

$$2=1+1,\ldots,s_{k-1}=a_{k-1}+r_{k-1},s_k=a_k+r_k=n.$$

Then we call the sub-sequence (s_{j_m}) for $1\leq j\leq k$ and $1\leq m\leq t\leq k$ a **sub-addition** chain of the addition chain producing n. We say it is **complete** sub-addition chain of the addition chain producing n if it contains exactly the first t terms of the addition chain. Otherwise we say it is an **incomplete** sub-addition chain.

2.1. Addition chains of numbers of special forms.

In this section we study addition chains of numbers of special forms. We examine ways of minimizing the length of addition chains for numbers of the form 2^n-1.

Lemma 2.3. *Let $\iota(n)$ denotes the shortest length of an addition chain producing n. Then the lower bound holds*

$$\iota(n)>\frac{\log n}{\log 2}-1.$$

Remark 2.4. We now obtain an inequality related to scholz's conjecture.

3. Main result

In this section, we prove an explicit upper bound for the length of the shortest addition chain producing numbers of the form $2^n - 1$.

Theorem 3.1. *Let $\delta(n)$ denotes the length of an addition chain producing n. Then there exists an addition chain producing $2^n - 1$ such that the inequality holds*

$$\delta(2^n - 1) \lesssim n - 1 + \iota(n) + \frac{n}{\log n} + 1.3 \log n \int\limits_{2}^{\frac{n-1}{2}} \frac{dt}{\log^3 t} + \xi(n)$$

where $\xi : \mathbb{N} \longrightarrow \mathbb{R}$.

Proof. First, let us construct the shortest addition chain producing 2^n as $1, 2, 2^2, \ldots, 2^{n-1}, 2^n$ with corresponding sequence of partition

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3 \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}, 2^n = 2^{n-1} + 2^{n-1}$$

with $a_i = 2^{i-2} = r_i$ for $2 \leq i \leq n + 1$, where a_i and r_i denotes the determiner and the regulator of the i th generator of the chain. Let us consider only the complete sub-addition chain

$$2 = 1 + 1, 2 + 2 = 2^2, 2^2 + 2^2 = 2^3 \ldots, 2^{n-1} = 2^{n-2} + 2^{n-2}.$$

Next we extend this complete sub-addition chain by adjoining the sequence

$$2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor}, 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} \ldots, 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \cdots + 2^1.$$

We note that the adjoined sequence contributes at most

$$\lfloor \frac{\log n}{\log 2} \rfloor \leq \iota(n)$$

terms to the original complete sub-addition chain, where the upper bound follows from Lemma 2.3. Since the inequality holds

$$\begin{aligned} 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \cdots + 2^1 &< \sum_{i=1}^{n-1} 2^i \\ &= 2^n - 1 \end{aligned}$$

we make the substitution

$$R_2(n) := 2^{n-1} + 2^{\lfloor \frac{n-1}{2} \rfloor} + 2^{\lfloor \frac{n-1}{2^2} \rfloor} + \cdots + 2^1$$

and extend the addition chain by further adjoining the sequence

$$R_2(n) + 2^{\lfloor \frac{n-1}{3} \rfloor}, R_2(n) + 2^{\lfloor \frac{n-1}{3} \rfloor} + 2^{\lfloor \frac{n-1}{3^2} \rfloor}, \ldots, R_2(n) + 2^{\lfloor \frac{n-1}{3} \rfloor} + 2^{\lfloor \frac{n-1}{3^2} \rfloor} + \cdots + 2^1.$$

We note that the adjoined sequence contributes at most

$$\lfloor \frac{\log n}{\log 3} \rfloor$$

terms to the original complete sub-addition chain. Since

$$R_2(n) + 2^{\lfloor \frac{n-1}{3} \rfloor} + 2^{\lfloor \frac{n-1}{3^2} \rfloor} + \cdots + 2^1 < 2^n - 1$$

we continue the extension of the addition chain by using all the primes $p \leq \frac{n-1}{2}$, so that by induction the number of terms adjoined to the original complete sub-addition chains is the sum

$$\sum_{p\leq \frac{n-1}{2}} \lfloor \frac{\log n}{\log p} \rfloor = \lfloor \frac{\log n}{\log 2} \rfloor + \sum_{3\leq p\leq \frac{n-1}{2}} \lfloor \frac{\log n}{\log p} \rfloor$$
$$\leq \iota(n) + \log n \sum_{3\leq p\leq \frac{n-1}{2}} \frac{1}{\log p}.$$

Now, we obtain the upper bound

$$\sum_{3\leq p\leq \frac{n-1}{2}} \frac{1}{\log p} = \int_{2}^{\frac{n-1}{2}} \frac{d\pi(u)}{\log u}$$
$$= \frac{\pi(\frac{n-1}{2})}{\log(\frac{n-1}{2})} - \frac{\pi(2)}{\log 2} + \int_{2}^{\frac{n-1}{2}} \frac{\pi(t)}{t \log^2 t} dt$$
$$\lesssim \frac{n}{\log^2 n} - \frac{1}{\log 2} + 1.3 \int_{2}^{\frac{n-1}{2}} \frac{1}{\log^3 t} dt.$$

This completes the proof of the inequality. $\square$

Corollary 3.1. Let $\iota(n)$ denotes the length of the shortest addition chain producing n. The the inequality holds

$$\iota(2^n - 1) \lesssim n - 1 + \iota(n) + \frac{n}{\log n} + 1.3 \log n \int_{2}^{\frac{n-1}{2}} \frac{dt}{\log^3 t} + \xi(n)$$

where $\xi : \mathbb{N} \longrightarrow \mathbb{R}$.

4. **Data availability statement**

The manuscript has no associated data.

5. **Conflict of interest statement**

The authors declare no conflict of interest regarding the publication of this manuscript.
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