On the Number of Twin Primes less than a Given Quantity: An Alternative Form of Hardy-Littlewood Conjecture

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ABSTRACT. I found an alternative form of Hardy-Littlewood Conjecture using a corollary of Mertens' 2nd theorem. This new form would be more useful since it has a theoretical background and is more likely to be proved.

1. Introduction

Though it is not proved yet if there are infinitely many twin primes, here is a proposition stating what the number of twin primes would be.

Proposition 1. (Hardy-Littlewood Conjecture) Let $\pi_2(x)$ denote the number of prime numbers p less than or equal to x such that p+2 is also a prime number. Then, this satisfies

$$\pi_2(x) \sim 2C_2 \frac{x}{(\log x)^2} \tag{1}$$

where C_2 is the twin prime constant, 0.6601618⋯

To make an alternative form of this similarity, the following theorem would be used.

Theorem 1. (Mertens' 2nd Theorem) Let "p≤x" mean all prime numbers not exceeding x, then,

$$\lim_{x \to \infty} \left[\sum_{p \le x} \frac{1}{p} - \log(\log x) \right] = M \tag{2}$$

where M is Meissel-Mertens constant 0.2614972⋯

2. An Alternative Form of Hardy-Littlewood Conjecture

Mertens' 2nd Theorem gives the following corollary.

Corollary 1. For a real number x and prime numbers p, the following limit exists.

$$\lim_{x\to\infty}\left[(\log x)^2 \times \prod_{2<p\le x}\left(1-\frac{2}{p}\right)\right] \tag{3}$$

Proof. Let's consider the logarithm of (3) without limit.

$$\log\left[(\log x)^2 \times \prod_{2<p\le x}\left(1-\frac{2}{p}\right)\right]$$

$$= 2\log(\log x) + \sum_{2<p\le x}\log\left(1-\frac{2}{p}\right)$$

(using Maclaurin's series)

$$= 2\log(\log x) + \sum_{2<p\le x}\left[-\frac{2}{p}-\frac{1}{2}\left(\frac{2}{p}\right)^2-\frac{1}{3}\left(\frac{2}{p}\right)^3-\frac{1}{4}\left(\frac{2}{p}\right)^4-\cdots\right]$$

$$= -2\left[\sum_{2<p\le x}\frac{1}{p}-\log(\log x)\right]-\sum_{2<p\le x}\sum_{r=2}^{\infty}\frac{1}{r}\left(\frac{2}{p}\right)^r$$

$$= -2\left[\sum_{2\le p\le x}\frac{1}{p}-\log(\log x)\right]+1-\sum_{2<p\le x}\sum_{r=2}^{\infty}\frac{1}{r}\left(\frac{2}{p}\right)^r$$

$$\to -2M+1-\sum_{p>2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{r}\left(\frac{2}{p}\right)^r$$

(as $x \to \infty$)

The last term converges since it is a summation of positive terms and has an upper bound.

$$\sum_{p>2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{r}\left(\frac{2}{p}\right)^r < \sum_{p>2}^{\infty}\sum_{r=2}^{\infty}\left(\frac{2}{p}\right)^r = \sum_{p>2}^{\infty}\frac{\left(\frac{2}{p}\right)^2}{1-\frac{2}{p}} = \sum_{p>2}^{\infty}\frac{4}{p(p-2)} < \sum_{p>2}^{\infty}\frac{4}{(p-2)^2} < 4\sum_{n=1}^{\infty}\frac{1}{n^2} = 4\times\frac{\pi^2}{6}$$

Let H be the given limit of Corollary 1.

$$\lim_{x\to\infty}\left[(\log x)^2 \times \prod_{2<p\le x}\left(1-\frac{2}{p}\right)\right] = e^{-2M+1-\sum_{p>2}^{\infty}\sum_{r=2}^{\infty}\frac{1}{r}\left(\frac{2}{p}\right)^r} = H \tag{4}$$

Using equation (4), the right side of Hardy-Littlewood Conjecture can be written as below.

$$2C_2\frac{x}{(\log x)^2} = 2C_2\frac{x}{\left(2\log\sqrt{x}\right)^2} = \frac{C_2}{2}\frac{x}{\left(\log\sqrt{x}\right)^2}$$

$$\sim \frac{C_2}{2}\times\frac{x}{H}\prod_{2<p\le\sqrt{x}}\left(1-\frac{2}{p}\right) = \frac{C_2}{2}\times\frac{x}{H}\times\left(1-\frac{2}{3}\right)\prod_{3<p\le\sqrt{x}}\left(1-\frac{2}{p}\right) = \frac{C_2}{H}\times\frac{x}{6}\prod_{3<p\le\sqrt{x}}\left(1-\frac{2}{p}\right)$$

This is the main theorem.

Theorem 2. (An alternative form of Hardy-Littlewood Conjecture)

$$2C_2\frac{x}{(\log x)^2} \sim \frac{C_2}{H}\times\frac{x}{6}\prod_{3<p\le\sqrt{x}}\left(1-\frac{2}{p}\right) \tag{5}$$

Now, the meaning of this alternative form will be stated including why it is square root of x, which is larger or equal to p, instead of x, and why p starts from 5 instead of 3 giving x divided by 6.

3. Significance of the Alternative Form

All prime numbers except 2 and 3 are of form 6k-1 or 6k+1, so all twin primes except (3, 5) are of form $6k \pm 1$. Tables below consist of numbers of the form 6k-1, 6k, and 6k+1 with multiples of each prime numbers highlighted by blue.

5	11	17	23	29	35	41	47	53	59	65	71	77	83	89	95	101	107
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108
7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103	109

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6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108
7	13	19	25	31	37	43	49	55	61	67	73	79	85	91	97	103	109

47	53	59	65	71	77	83	89	95	101	107	113	119	125	131	137	143	149
48	54	60	66	72	78	84	90	96	102	108	114	120	126	132	138	144	150
49	55	61	67	73	79	85	91	97	103	109	115	121	127	133	139	145	151

65	71	77	83	89	95	101	107	113	119	125	131	137	143	149	155	161	167
66	72	78	84	90	96	102	108	114	120	126	132	138	144	150	156	162	168
67	73	79	85	91	97	103	109	115	121	127	133	139	145	151	157	163	169

There is a pattern which composition numbers appear. This can be examined in two cases.

Case 1 : p is a prime number of form 6k-1

...	6(np-k)-1	...	6np-1	...	6(np+k)-1 = (6n+1)p	...
...	6(np-k)	...	6np	...	6(np+k)	...
...	6(np-k)+1 = (6n-1)p	...	6np+1	...	6(np+k)+1	...

Case 2 : p is a prime number of form 6k+1

...	6(np-k)-1 = (6n-1)p	...	6np-1	...	6(np+k)-1	...
...	6(np-k)	...	6np	...	6(np+k)	...
...	6(np-k)+1	...	6np+1	...	6(np+k)+1 = (6n+1)p	...

Here, n is an arbitrary natural number. Both cases give same conclusion.

Theorem 3. $\forall\ m \in N$, a pair of two numbers 6m-1 and 6m+1 are not twin primes if and only if $m = np \pm k$ for some $n \in N$ and prime number p. (k depends on p by $k = \text{round}\left(\frac{p}{6}\right)$)

Regarding the tables above, the number of columns under a given quantity x is $\frac{x}{6}$ and for all prime number p > 3 (since multiples of 2 and 3 are already excluded considering only numbers of form $6x \pm 1$), two columns among every-continuous-p-columns are not twin primes. In addition, it is enough to consider prime numbers less than or equal to a given quantity x. Therefore, we can compute the number of twin primes under a given quantity x by

$$\frac{x}{6}\prod_{3<p\le\sqrt{x}}\left(1-\frac{2}{p}\right)$$

This is how the new form of Hardy-Littlewood Conjecture has a theoretical background. The only left point is the constant in front of this term, $\frac{C_2}{H}$. The value is about 0.793. It seems that the Twin Prime Conjecture and the Hardy-Littlewood Conjecture might be solved if we find the meaning or the reason why this constant appears.

4. Conclusion

Here, I suggest a new conjecture stating the number of twin primes less than a given quantity which is equivalent to Hardy-Littlewood Conjecture but more intuitive, convincing, and so more helpful to prove the conjecture.

$$\pi_2(x) \sim \frac{C_2}{H} \times \frac{x}{6}\prod_{3<p\le\sqrt{x}}\left(1-\frac{2}{p}\right)$$

(C_2 is the twin prime constant and H is the constant defined in equation (4))

References

[1] Wikipedia: Twin Prime, Mertens' Theorems, Meissel-Mertens Constant

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