# Geometrical optics as U(1) local gauge theory in curved space-time

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We treat the geometrical optics as an Abelian $U(1)$ local gauge theory in vacuum curved space-time. We formulate the eikonal equation in (1+1)-dimensional vacuum centrally symmetric curved space-time using null geodesic of the Schwarzschild metric and obtain mass-the U(1) gauge potential relation.

Keywords: *geometrical optics, eikonal equation, refractive index, Abelian $U(1)$ local gauge theory, gauge potential, vacuum curved space-time, null geodesic, the Schwarzschild metric.*

The geometrical optics corresponds to the limiting case of a very small wavelength of light, $\lambda \to 0$[1], in comparison with the characteristic dimension of the problem[2] or in other words to each of the other scales present, so that the waves can be regarded *locally* as plane waves propagating through space-time[3]. In case of *a steady (constant or unchanging in time*[4]*, time-independent) monochromatic wave*, the frequency[5] is constant and the time dependence of *the eikonal*, $\psi$, *a function of space-time*, is given by a term $-f_\theta t$ (or we can write $\partial\psi/\partial t = -f_\theta$) where $f_\theta$ denotes (angular) frequency[2]. So, $\psi$ is a large quantity due to a very small wavelength. Let us introduce $\psi_1$ which is also called *eikonal*[2]. The relation between $\psi_1$ and $\psi$ can be expressed as[2]

$$\psi_1 = \frac{c}{f_\theta}\psi + ct \tag{1}$$

where the eikonal, $\psi_1$, is *a function of coordinates (space) only*[2], *"a length"*, *a real scalar function*[6] and $c$ is the speed of light in vacuum. We consider that we need to replace $\psi$ to $\psi_1$ because here we concern with a steady monochromatic wave only.

In a 1-dimensional space, the equation of ray propagation in a transparent medium[7] can be written as[2,8–10]

$$|\vec{\nabla}\psi_1(x)| = |\vec{n}(x)| = n(x), \quad x \in \Omega \subset \mathbb{R}^1 \tag{2}$$

subject to $\psi_1(x)|_{\partial\Omega} = 0$ (the solution, $\psi_1(x)$, at the boundary, $\partial\Omega$, is equal to zero), $\Omega$ is *an open set*[9], *bounded*[11]*, with suitably smooth (well-behaved) boundary*[9] *in a 1-dimensional Euclidean space,* $\mathbb{R}^1$, $|.|$ denotes the Euclidean norm, a distance function[10], in 1-dimensional Euclidean space, $\vec{\nabla}$ denotes the gradient, $n(x)$ is *the refractive index*, *a real scalar function with positive values, the slowness (speed*$^{-1}$*)* at $x$ where $x$ lies inside $\Omega$[9]. The function $n(x)$ is typically supplied as *known input, given*, and we seek *the solution*, $\psi_1(x)$, *the shortest time* needed to travel from $x$ to *the boundary*, $\partial\Omega$[9]. Because $\psi_1$ is a function of coordinates only, then the refractive index is also a function of coordinates only (*i.e. a smooth continuous function of the position*[13]). Eq.(2) is called *the eikonal equation*[2,8], i.e. *a type of the first order non-linear partial differential equation*[9,14,15]. The eikonal equation is an approximated version of the wave equation[16], a typical example of steady-state Hamilton–Jacobi equations[17,18]. The eikonal equation can be derived from the Fermat's principle[19], the Euler-Lagrange equation[19] and Maxwell equations[8,9,20]. The Hamilton-Jacobi equations are a type of non-linear *hyperbolic* partial differential equations[21] and Maxwell equations can be formulated as a *hyperbolic* system of partial differential equations[22]. So, we consider *the eikonal equation as the (first order non-linear) hyperbolic partial differential equation.* The analysis of a partial differential equation for a steady state is very important, e.g. in the Atiyah-Singer index theorem (an effort for finding *the existence and uniqueness of solutions* to linear partial differential equations of *elliptic type*[23] on closed manifold[24,25]). *Why is the eikonal equation (2) a non-linear equation?* We consider the eikonal equation (2) as a non-linear[26] equation because there exists the Euclidean norm, $|.|$, in the eq.(2). The Euclidean norm has a non-linear property, $|\vec{v}+\vec{w}| \leq |\vec{v}| + |\vec{w}|$[27], where $\vec{v}$ and $\vec{w}$ are vectors.

In a $(1+1)$-dimensional space-time, the gradient operator, $\vec{\nabla}$, in eq.(2) is replaced by the covariant four-gradient, $\partial_\mu$. So, eq.(2) becomes

$$||\partial_\mu\psi_1(x)|| = n(x) \tag{3}$$

where $\mu$ runs from 1 to 1+1 by considering that the time derivative of $\psi_1$ is equal to zero. We consider that the eikonal equation (3) describes the propagation of wavefronts (field discontinuities) in a (1+1)-dimensional Minkowskian space-time[28], *a flat space-time.* We see from eq.(3), *the zeroth rank tensor (a scalar)* of the refractive index describes *an isotropic linear optics*[29]. *It means that a flat space-time descibes an isotropic linear optics*[30]. But, the refractive index can also be *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[31]. The second rank tensor of the refractive index describes *an anisotropic linear optics*[29].

In a $(1+1)$-dimensional Minkowskian space-time and related to the gauge theory, *a four-vector potential* (a combination of *an electric scalar potential* and *a magnetic vector potential*[32,33]) of the geometrical optics is replaced by *a four-vector field*[34] or *the gauge poten-*

*tial*[3,35–38] (which makes the related field tensor invariant under the gauge transformation) as written below

$$\vec{B}_\mu = \vec{a}_\mu \; e^{i\psi} \tag{4}$$

where $\psi(x,t)$, as we mentioned, is *the eikonal* (*a real phase*[3]) and $\vec{a}_\mu(x,t)$ is *a complex amplitude*[3], a slowly varying function of coordinate and time[2]. We see from eq.(4), $e^{i\psi}$ is *a scalar function* (more precisely, *a complex scalar function, dimensionless*), $\vec{B}_\mu$ is *a complex*[3,39] *quantity (a complex four-vector field)*. $\vec{B}_\mu$ as $\vec{a}_\mu$, can be interpreted as the oscillating variable[40], the displacement from *an equilibrium*[41], a position at infinity where the gauge potential is assumed equal to zero.

The treatment of the geometrical optics as an Abelian $U(1)$ local gauge theory has a consequence that *the gauge potential of the geometrical optics and the Maxwell's theory are the same, i.e. both are the Abelian* $U(1)$ *gauge potential,* $\vec{B}_\mu^{\,U(1)}$. In other words, *the related field strength of the geometrical optics and the Maxwell's theory are, in principle, the same.* So, we can rewrite eq.(4) as

$$\vec{B}_\mu^{\,U(1)} = \vec{a}_\mu \; e^{i\psi} \tag{5}$$

Eq.(5) expresses *the Abelian* $U(1)$ *gauge potential of the geometrical optics* in a $(1+1)$-dimensional Minkowskian space-time. Eq.(5) can be written as[3]

$$\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu = \vec{a}_\mu \; \underline{\vec{a}}^\mu \; e^{i\psi} = (\mathbf{a} \cdot \underline{\mathbf{a}}) \; e^{i\psi} = a^2 \; e^{i\psi} = e^{i\psi} \tag{6}$$

where $\underline{\vec{a}}^\mu$ is *a complex conjugate of* $\vec{a}_\mu$, and $a$ is *a scalar amplitude*[3] which we can take its value as 1.

Using *Euler's formula*, eq.(6) can be written as

$$\cos\psi + i \sin\psi = \vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu \tag{7}$$

Eq.(7) shows us that $\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu$ is *a complex scalar function.* To simplify the problem, we take the real part of (7) only, we obtain

$$\cos\psi = \mathrm{Re}\; (\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu) \tag{8}$$

where $\psi$ in eq.(8), i.e. a real phase ("a gauge") is *an angle.* This angle has value

$$\psi = \arccos\left[\mathrm{Re}\left(\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu\right)\right] \tag{9}$$

By substituting eq.(9) into eq.(1), we obtain

$$\psi_1 = \frac{c}{f_\theta} \arccos\left[\mathrm{Re}\left(\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu\right)\right] + ct \tag{10}$$

and by substituting eq.(10) into the eikonal equation (3), we obtain

$$\left|\left|\partial_\nu \left\{\frac{c}{f_\theta} \arccos\left[\mathrm{Re}\left(\vec{B}_\mu^{U(1)} \; \underline{\vec{a}}^\mu\right)\right] + ct\right\}\right|\right| = n \tag{11}$$

where $n$ is *a dimensionless quantity, a real scalar function of* 1-*coordinate* which "lives" in a $(1+1)$-dimensional Minkowskian space-time.

Let us formulate the eikonal equation (11) in (1+1)-dimensional curved space-time using *null geodesic* with the simplest metric, i.e. *the Schwarzschild metric.* Light propagating through curved space-time (gravitational lensing) behaves as if it were traversing *an inhomogeneous medium*[45]. *Why do we treat the geometrical optics as an Abelian* $U(1)$ *local gauge theory in curved space-time? It is because of the eikonal equation can be derived, as we mentioned, from Maxwell equations*[46]*, the classical limit of quantum electrodynamics (QED)*[47] *where QED is an Abelian* $U(1)$ *local gauge theory and the non-vacuum (with charge, with current) Maxwell equations are normally formulated in the local coordinates of curved space-time.* Another reason why the geometrical optics is an Abelian (commutative) is that *the eikonal equation can be derived from the steady state Hamilton-Jacobi equation.* The Hamilton-Jacobi equation, roughly speaking, can be derived using *a canonical transformation* i.e. a special case of a symplectomorphism or symplectic map. The symplectic map is an isomorphism in the category of symplectic manifolds[18]. *This isomorphism preserves commutativity*[18].

Assume that space is *vacuum* and *centrally symmetric*[2]. We consider vacuum here is the same as *empty space*, $R_{\mu\nu} = 0$[48], where *empty* means that *there is no matter present and no physical fields, except the gravitational field which does not disturb the emptyness (other fields than the gravitational field do)*[48]. As a consequence that *space is vacuum i.e. outside*[49] *of the masses producing the gravitational field* and centrally symmetric (due to spherical masses), the gravitational field is automatically *static*[2]. In other words, *the static centrally symmetric gravitational field produced by a spherically symmetric body at rest*[48]. This static spherically symmetric gravitational field is described by *the Schwarzschild metric*[2,48,50–52] below

$$\begin{aligned} ds^2 &= g_{00}(r)\; c^2dt^2 - g_{rr}(r)\; dr^2 \\ &= \left(1 - \frac{2GM}{c^2r}\right) c^2dt^2 - \left(1 - \frac{2GM}{c^2r}\right)^{-1} dr^2 \end{aligned} \tag{12}$$

where $2GM/c^2 = r_s$ is *the Schwarzschild radius,* $M$ is *the mass of the central body (a constant of integration*[48]*, a number*[53,54]*) that is producing the gravitational field*[48], $G$ is the gravitational constant, $c$ is the speed of light, $r$ is the spatial (radial) coordinate (measured as the circumference, divided by $2\pi$, of a sphere centered around the massive body[52]). Eq.(12) is also known as *the Schwarzschild solution*[48]. The Schwarzschild solution, as the Schwarzschild metric, holds outside the surface of the body that is producing the gravitational field, where there is no matter[48].

The world line corresponding to *the propagation of light* is described by *a null geodesic* as below

$$ds^2 = 0 \tag{13}$$

where *a null geodesic is the track of a null vector*[48]. We consider a null geodesic as a consequence of an infinitesimal proper time interval vanishes, $d\tau = 0$.

By substituting eq.(13) into eq.(12), using relations $dr/dt = v$, $c/v = n$, and rearrange the terms, we obtain the space dependent refractive index, $n(r)$, related to the mass of the central body that is producing the gravitational field, $M$, as below[50,55]

$$n = \left(1 - \frac{2G}{c^2 r}\, M\right)^{-1} \tag{14}$$

It means that in curved space-time indicated by the Schwarzschild metric, the refractive index can be related to (as a consequence of null geodesic or the Schwarzschild metric is equal to zero) mass[42] or metric tensor[43]. *The metric tensor is the field, the gravitational field*, describes *the varying geometry of space-time*[44].

By substituting eq.(14) into eq.(11), we obtain the eikonal equation in (1+1)-dimensional curved space-time as below

$$\begin{aligned} &\left\|\partial_\nu \left\{ \frac{c}{f_\theta} \arccos \left[ \mathrm{Re} \left( \vec{B}^{U(1)}_{\mu}\ \underline{\vec{a}}^{\mu} \right) \right] + ct \right\}\right\| \\ &= \ \left(1 - \frac{2G}{c^2 r}\, M\right)^{-1} \end{aligned} \tag{15}$$

As we mentioned, the analysis of a partial differential equation for steady state is very important for finding the existence and uniqueness of solutions to partial differential equations (PDEs). *Related to the existence and uniqueness of solutions to PDEs, does eq.(15) have a solution? In general, what are the characteristics of a partial diferential equation which has a solution? What is a consequence if we treat the eikonal in eq.(15), as a complex scalar function? Roughly speaking, does a solution of a (complex) eikonal equation generate a non-trivial topological configurations*[6,56]?


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