Geometrical optics as U(1) local gauge theory

Miftachul Hadi[1, 2]

[1)] *Physics Research Centre, Badan Riset dan Inovasi Nasional (BRIN), Puspiptek, Gd 440-442, Serpong, Tangerang Selatan 15314, Banten, Indonesia.*

[2)] *Institute of Mathematical Sciences, Kb Kopi, Jalan Nuri I, No.68, Pengasinan, Gn Sindur 16340, Bogor, Indonesia. E-mail: instmathsci.id@gmail.com*

We treat the geometrical optics as an Abelian $U(1)$ local gauge theory in a $(3+1)$-dimensional space-time. We formulate the eikonal equation: the refractive index is related to the $U(1)$ gauge potential.

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In a $(3+1)$-dimensional space-time, for the geometrical optics approximation (short wavelength, $\lambda \to 0$[1]), a four-vector potential is replaced by the gauge potential[2–4] below

$$\vec{B}_\mu = a_\mu \; e^{i\psi} \tag{1}$$

where *a phase* (*an eikonal*), $\psi(x, y, z, t)$, and a slowly varying function of coordinates and time, *an amplitude*, a_μ[1], are represented in a $(3+1)$-dimensional space-time. We see from eq.(1), because $e^{i\psi}$ is a scalar, a number, then the amplitude, a_μ, has the same dimension as the displacement from equilibrium[5], the oscillating variable[6], the gauge potential $\vec{B}_\mu$.

In case of *a steady (time-independent) monochromatic wave*, the frequency[7] is constant and the time dependence of the eikonal, ψ, is given by a term $-f_\theta t$ where f_θ is a notation for (angular) frequency[1]. Let us introduce ψ_1, a function, which is also called *eikonal*[1]. The relation between ψ_1 and ψ can be expressed as[1]

$$\psi_1 = \frac{c}{f_\theta}\psi + ct \tag{2}$$

where the eikonal, ψ_1, is *a function of coordinates only (without time)*[1] and c is the speed of light in vacuum. The 3-dimensional eikonal is denoted by $\psi_1(x, y, z)$.

The equation of ray propagation in a transparent medium[8] can be written in relation with the refractive index, n, as below[1,9]

$$|\vec{\nabla}\psi_1| = |\vec{n}| = n \tag{3}$$

where n is *a scalar*, $\vec{\nabla}$ is a notation for gradient. Because ψ_1 is a function of coordinates only, then the refractive index is also a function of coordinates only. More precisely, the refractive index is *a smooth continuous function* of the position[10]. The 3-dimensional refractive index is denoted by $n(x, y, z)$. The equation (3) is called *the eikonal equation*[1,9], i.e. *a type of the first order linear partial differential equation.* The analysis of a partial differential equation for steady state is very important e.g. for formulating the Atiyah-Singer index theorem, an effort for finding the existence and uniqueness of solutions to linear partial differential equations of elliptic type on closed manifold[11,12].

Let us formulate the eikonal, ψ_1, in a $(3+1)$-dimensional space-time. Because the eikonal, ψ_1, is a function of coordinates only, so it becomes the 3-dimensional eikonal which "lives" in a $(3+1)$-dimensional space-time. The gradient operator, $\vec{\nabla}$, in eq.(3) is replaced by the covariant four-gradient, ∂_μ. So, eq.(3) becomes

$$|\partial_\mu \psi_1| = |\vec{n}_\mu| = n \tag{4}$$

where μ runs from 1 to $3+1$ by considering that the time components of ψ_1 and n are zero. We see from eq.(4), the refractive index is *a scalar, a real number. The zeroth rank tensor (a scalar)* of the refractive index describes *an isotropic linear optics*[13]. But, the refractive index can be not simply a scalar[14]. The refractive index can also be *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[14]. The second rank tensor of the refractive index describes *an anisotropic linear optics*[13].

The eikonal equation of the geometrical optics can be derived from the Maxwell equations[9]. Because the Maxwell's theory is an Abelian $U(1)$ local gauge theory, so we treat the geometrical optics as an Abelian $U(1)$ local gauge theory. It has a consequence that *the field strength of the geometrical optics and the Maxwell's theory in principle are the same i.e. both are fields*[15]. In turn, it has a consequence that *the gauge potential of the geometrical optics and the Maxwell's theory are also the same, i.e. both are $U(1)$ gauge potential $\vec{B}_\mu^{\,U(1)}$*. So, we can rewrite eq.(1) as

$$\vec{B}_\mu^{\,U(1)} = a_\mu \; e^{i\psi} \tag{5}$$

Eq.(5) expresses *the $U(1)$ gauge potential of the geometrical optics* in a $(3+1)$-dimensional space-time.

Eq.(5) can now be written as

$$\vec{B}_\mu^{U(1)} \; \underline{a}^\mu = a_\mu \; \underline{a}^\mu \; e^{i\psi} = a^2 \; e^{i\psi} = e^{i\psi} \tag{6}$$

where $\underline{a}^\mu$ is a complex conjugate of a complex vector amplitude, a_μ, and a is a scalar amplitude[16] which we can take its value as 1. Using Euler's formula, eq.(6) can be written as

$$\cos\psi + i\sin\psi = \vec{B}_\mu^{U(1)} \; \underline{a}^\mu \tag{7}$$

Eq.(7) shows us that $\vec{B}_\mu^{U(1)}\underline{a}^\mu$ is *a complex function.* To simplify the problem, we take the real part[17] of (7) only, we obtain

$$\cos\psi = \mathrm{Re}\ (\vec{B}_{\mu}^{U(1)}\ \underline{a}^{\mu}) \tag{8}$$

where ψ in eq.(8), i.e. phase (eikonal or "gauge") is *an angle*. This angle has value

$$\psi = \arccos\left[\mathrm{Re}\left(\vec{B}_{\mu}^{U(1)}\ \underline{a}^{\mu}\right)\right] \tag{9}$$

By substituting eq.(9) into eq.(2), we obtain

$$\psi_1 = \frac{c}{f_\theta}\arccos\left[\mathrm{Re}\left(\vec{B}_{\mu}^{U(1)}\underline{a}^{\mu}\right)\right] + ct \tag{10}$$

and by substituting eq.(10) into the eikonal equation (4), we obtain

$$\left|\partial_\nu\left\{\frac{c}{f_\theta}\arccos\left[\mathrm{Re}\left(\vec{B}_{\mu}^{U(1)}\underline{a}^{\mu}\right)\right] + ct\right\}\right| = n \tag{11}$$

where n is a dimensionless quantity, a scalar, a real number, i.e. a function of 3-coordinates which "lives" in a $(3+1)$-dimensional space-time. Eq.(11) is a type of the first order linear partial differential equation. As we mentioned the analysis of a partial differential equation for steady state is very important e.g. for finding the existence and uniqueness of solutions to linear partial differential equations. *Does eq.(11) have a solution?*

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[1] L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984.

[2] A.B. Balakin, A.E. Zayats, *Ray Optics in the Field of a Nonminimal Dirac Monopole*, Gravitation and Cosmology, 2008, Vol.14, No.1, pp.86-94.

[3] We treat the four-vector potential, $\vec{B}_\mu$, the same as wave field, ϕ, (any component of $\vec{E}$ or $\vec{H}$) given by a formula of the type $\phi = ae^{i\psi}$, where the amplitude a is a slowly varying function of coordinates and time and phase (eikonal), ψ, is a large quantity which is "almost linear" in coordinates and the time (L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984.). We consider both, $\vec{B}_\mu$ and ϕ, are solutions of the wave equation.

[4] Alexander B. Balakin, Alexei E. Zayats, *Non-minimal Einstein-Maxwell theory: the Fresnel equation and the Petrov classification of a trace-free susceptibility tensor*, https://arxiv.org/pdf/1710.08013.pdf, 2018.

[5] H.J. Pain, *The Physics of Vibrations and Waves*, John Wiley and Sons Limited, 3rd Edition, 1983.

[6] Wikipedia, *Amplitude*.

[7] The time derivative of phase, ψ, gives the angular frequency of the wave, $\partial\psi/\partial t = -\omega$ and the space derivatives of ψ gives the wave vector, $\vec{\nabla}\psi = \vec{k}$, which shows the direction of the ray propagation through any point in space (L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984).

[8] Only transparent media are considered in geometrical optics (L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984).

[9] Max Born, Emil Wolf, *Principles of Optics*, Pergamon Press, 1993.

[10] G. Molesini, *Geometrical Optics*, Encyclopedia of Condensed Matter Physics, https://www.sciencedirect.com/topics/physics-and-astronomy/geometrical-optics, 2005.

[11] Nigel Higson, John Roe, *The Atiyah-Singer Index Theorem.*

[12] Miftachul Hadi, *On the geometrical optics and the Atiyah-Singer index theorem*, https://vixra.org/abs/2108.0006, 2021.

[13] Roniyus Marjunus, *Private communication.*

[14] Karsten Rottwitt, Peter Tidemand-Lichtenberg, *Nonlinear Optics: Principles and Applications*, CRC Press, 2015.

[15] Y.M. Cho, *Private communication.*

[16] See e.g. Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, *Gravitation*, W.H. Freeman and Company, 1973, p.573.

[17] The refractive index is often described as a real value. However, in a lossy material, the attenuation of the electric field is described through an imaginary part of the refractive index (Karsten Rottwitt, Peter Tidemand-Lichtenberg, *Nonlinear Optics: Principles and Applications*, CRC Press, 2015).