On the refractive index-curvature relation

Miftachul Hadi[1,2]
[1)] Physics Research Centre, Badan Riset dan Inovasi Nasional (BRIN), Puspiptek, Serpong, Tangerang Selatan 15314, Banten, Indonesia.
[2)] Institute of Mathematical Sciences, Kb Kopi, Jalan Nuri I, No.68, Pengasinan, Gn Sindur 16340, Bogor, Jawa Barat, Indonesia. E-mail: instmathsci.id@gmail.com

In a two-dimensional space, a refractive index-curvature relation is formulated using the second rank tensor of Ricci curvature. A scalar refractive index describes an isotropic linear optics. In a fibre bundle geometry, a scalar refractive index is related to an Abelian (a linear) curvature form. The Gauss-Bonnet-Chern theorem is formulated using a scalar refractive index. Because the Euler-Poincare characteristic is the topological invariant then a scalar refractive index is also a topological invariant.

Keywords: *geometrical optics, Abelian gauge theory, refractive index, Riemann-Christoffel curvature, curvature form, curvature matrix, connection matrix, pfaffian, Gauss-Bonnet-Chern theorem, Euler-Poincare characteristic, topological invariant.*

In the geometrical optics, the refractive index-curvature relation derived from the Fermat's principle describes ray propagation in a steady (time-independent) state[1]. The refractive index-curvature[2] relation can be written as[1,3–5]

$$\frac{1}{R} = \hat{N}\ .\ \vec{\nabla}\ \ln\ n(r) \tag{1}$$

where R is a radius of curvature[1], $\hat{N}$ is an unit vector along the principal normal or has the same direction with $\vec{\nabla}\ \ln\ n(r)$, $\vec{\nabla}\ \ln$ n(r) means the gradient of a function ln n at *a point* r and $n(r)$ is *a space-dependent refractive index, a scalar function of the coordinates only (a smooth continuous function of the position*[6]*)*. We see eq.(1) is a dot product of two vectors, so the result gives a scalar quantity, $\hat{N}\ .\ \vec{\nabla}\ \ln\ n(r) = \sum_{i=1}^{dim} N_i \nabla_i\ \ln\ n(r)$, dim is a number of dimension of space. *Eq.(1) tells us that the rays are therefore bent in the direction of increasing refractive index*[1].

In a 2-dimensional space[7], we write eq.(1) as

$$R_{\mu\nu} = g_{\mu\nu}\ N_{(\mu}\ \partial_{\nu)} \ln n \tag{2}$$

where $R_{\mu\nu}$ is the second rank tensor of Ricci curvature[8,9], $R_{\mu\nu} = g_{\mu\nu} \frac{R_{1212}}{g}$[10], a function of the metric tensor $g_{\mu\nu}$, $g = |(\det\ g_{\mu\nu})|$ is a scalar density[10], a real number, and μ, ν run from 1 to 2. We write $N_{(\mu}\ \partial_{\nu)}$ in eq.(2) to accomodate the symmetry property of the second rank tensor of Ricci curvature, $R_{\mu\nu} \equiv R_{\nu\mu}$, where $N_{(\mu}\ \partial_{\nu)} = \frac{1}{2}(N_\mu\ \partial_\nu + N_\nu\ \partial_\mu)$.

The zeroth rank tensor (a scalar, a real number) of the refractive index (1), (2) describes an isotropic linear optics[11]. But, the refractive index can be not simply a scalar[12]. The refractive index can also be *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[12]. *The second rank tensor of the refractive index describes an anisotropic linear optics*[11].

The geometrical optics can be derived from the Maxwell's theory, an Abelian $U(1)$ local gauge theory[13]. That is why, in this article we also treat the geometical optics as an Abelian $U(1)$ local gauge theory[5]. We will formulate a curvature in a fibre bundle. *Is there a relationship between a fibre bundle and a gauge theory?* Originally, a fibre bundle and a gauge theory are developed independently. Until it was realized that *the curvature (in a fibre bundle) and the field strength (in Yang-Mills theory) are identical*[14].

Why do we need to formulate the curvature in a fibre bundle instead of the Riemann-Christoffel curvature tensor? As a consequence of the geometrical optics is treated as an Abelian $U(1)$ local gauge theory, so we need to formulate the curvature in a fibre bundle as what we call *an Abelian (a linear) curvature form. A curvature form in a fibre bundle can be an Abelian or a non-Abelian (a non-linear). It differs with the Riemann-Christoffel curvature tensor which has the non-linear form only*[15].

The curvature form, $\Omega_{\alpha\mu}$, in a fibre bundle can be written as[16,17]

$$\Omega_{\alpha\mu} = \sum R_{\alpha\mu\beta\nu}\ du^\beta \wedge du^\nu \tag{3}$$

where $R_{\alpha\mu\beta\nu}$ is the fourth rank tensor of Riemann-Christoffel curvature (which has the algebraic properties as symmetry, anti-symmetry and cyclicity[10]), u^β, u^ν are *local coordinates* and $\wedge$ is a notation of *the exterior (wedge) product* (it satisfies the distributive, *anti-commutative*[18,19] and associative laws)[16,17]. $\Omega_{\alpha\mu}$ *is an anti-symmetric matrix of 2-forms*[20,21].

If we reformulate eq.(3) using eq.(2) and the Ricci-Riemann relation in a 2-dimensional space, $R_{\alpha\mu\beta\nu} = (g_{\alpha\beta}\ g_{\mu\nu} - g_{\alpha\nu}\ g_{\mu\beta}) \frac{R_{\mu\nu}}{g_{\mu\nu}}$, then we obtain

$$\begin{aligned} &\sum\ (g_{\alpha\beta}\ g_{\mu\nu} - g_{\alpha\nu}\ g_{\mu\beta})\ N_{(\mu}\ \partial_{\nu)} \ln n\ du^\beta \wedge du^\nu \\ &= \ \Omega_{\alpha\mu} \end{aligned} \tag{4}$$

Eq.(4) shows the relationship between the scalar refractive index and the curvature form in a 2-dimensional space. We see that the scalar refractive index "lives" in a 2-dimensional space.

Let us introduce a general form of *the curvature matrix*, Ω, which is *a matrix of exterior two-forms*[16,22] below

$$\Omega = d\omega - \omega \wedge \omega \tag{5}$$

where ω is *the connection matrix, one-form*[23,24]. We see that eq.(5) is a non-linear equation due to the second term of the right hand side of eq.(5).

Can the curvature matrix equation (5) be an Abelian, a linear equation? A gauge potential, $\mathcal{A}$, can be regarded a *a local expression for a connection in a principal bundle*[23] as written below

$$\mathcal{A} = \sigma^* \omega \tag{6}$$

where σ is *a local section* defined on a chart U of manifold, base space, M. *The local form of the curvature* is defined by[23]

$$\mathcal{F} \equiv \sigma^* \Omega \tag{7}$$

where $\mathcal{F}$ is identified with *the field strength*. In a general case, from *Cartan's structure equation*, we find[23]

$$\begin{aligned} \mathcal{F} &= \sigma^*(d_p\omega + \omega \wedge \omega) = d\sigma^*\omega + \sigma^*\omega \wedge \sigma^*\omega \\ &= d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \end{aligned} \tag{8}$$

where d is *the exterior derivative on* M. We see from eqs.(7), (8) that

$$\Omega = d_p\omega + \omega \wedge \omega \tag{9}$$

and

$$\sigma^* \; d_p\omega = d\sigma^* \; \omega \tag{10}$$

In a special case, for an Abelian $U(1)$ local gauge theory, using eq.(6) and the fact that *the exterior derivative obeys the Leibniz rule*[25], $\mathcal{F}$ can be expressed in terms of the gauge potential $\mathcal{A}$[23] as below

$$\begin{aligned} \mathcal{F} &= d\mathcal{A} \\ \sigma^* \; d_p\omega &= d(\sigma^*\omega) = d\sigma^* \; \omega + \sigma^* \; d\omega \end{aligned} \tag{11}$$

Eq.(11) implies

$$\Omega = d_p\omega \tag{12}$$

Notation d_p means *the covariant derivative of a vector valued one-form on a principal bundle, $P(M,G)$, G is structure group*[23]. We see that eq.(12) is an Abelian, a linear equation.

Let us consider $d\mathcal{A}$ in eqs.(8), (11). $d\mathcal{A}$ in such both equations should be the same or in other words *as a consequence of eq.(10), $d\omega$ in eq.(11) should be zero*

$$d\omega = 0 \tag{13}$$

It means that the connection matrix, one-form, ω, is *closed* if $d\omega = 0$[23,26,27].

Can we see something interesting in eq.(10)? We see that *eq.(10) is analog with the Stokes theorem* which can be written roughly[17] as

$$\int_D d\omega = \int_{\partial D} \omega \tag{14}$$

So, we could say that $d\omega = 0$ *is a consequence of the Stokes theorem. Using the Stokes theorem (14), we see that $d\omega = 0$ has the same meaning with ω is closed, i.e. $\partial D = 0$. What does $d\omega = 0$ imply in physics? Can $d\omega = 0$ be related to a conserved quantity in physics?*

Is there a relationship between the curvature matrix, Ω (5), and the curvature form, $\Omega_{\alpha\mu}$ (3)? Yes (there is)[28]. If $\Omega_{\alpha\mu}$ and $\omega_{\alpha\mu}$ denote *the components* of curvature and connection matrices, Ω and ω, respectively then we can write[16]

$$\Omega = (\Omega_{\alpha\mu}), \quad \omega = (\omega_{\alpha\mu}) \tag{15}$$

So, the curvature matrices in eqs.(9), (12) can be written using the curvature form[17] respectively as below

$$\Omega_{\alpha\mu} = d_p\omega_{\alpha\mu} - \omega_\alpha{}^\tau \wedge \omega_{\mu\tau} \tag{16}$$

and

$$\Omega_{\alpha\mu} = d_p\omega_{\alpha\mu} \tag{17}$$

We call eq.(17) as *an Abelian (a linear) curvature form equation.*

As we mentioned that we treat the geometrical optics as an Abelian $U(1)$ local gauge theory, so we choose the curvature form (17) to describe the geometrical optics. By substituting eq.(4) into eq.(17), we obtain

$$\begin{aligned} &\sum \; (g_{\alpha\beta} \; g_{\mu\nu} - g_{\alpha\nu} \; g_{\mu\beta}) \; N_{(\mu} \; \partial_{\nu)} \ln n \; du^\beta \wedge du^\nu \\ &= \; d_p\omega_{\alpha\mu} \end{aligned} \tag{18}$$

We call eq.(18) as *an Abelian curvature form-scalar refractive index relation.*

Let us define *the pfaffian*[29] *of the curvature matrix*, pf Ω, as below[16,30]

$$\text{pf } \Omega \equiv \sum \; \epsilon_{\alpha_1\mu_1 \ldots \alpha_{2q}\mu_{2q}} \; \Omega_{\alpha_1\mu_1} \wedge \ldots \wedge \Omega_{\alpha_{2q}\mu_{2q}} \tag{19}$$

where *the curvature matrix, Ω, is any even-size complex $2q \times 2q$ anti-symmetric matrix* (if Ω is *an odd-size* complex anti-symmetric matrix then the corresponding pfaffian is defined to be *zero*), $\epsilon_{\alpha_1\mu_1 \ldots \alpha_{2q}\mu_{2q}}$ is the $2q$-th rank Levi-Civita tensor which has value +1 or -1 according as its indices form an even or odd permutation of $1, ..., 2q$, and its otherwise zero, and the sum is extended over all indices from 1 to $2q$, *q is a natural number.* Here, $\alpha_1 < \mu_1, \; \ldots \;, \; \alpha_{2q} < \mu_{2q}$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_{2q}$[16,30]. Shortly, the pfaffian of Ω (19) can be rewritten as

$$\text{pf } \Omega = \sum \; \epsilon_{\alpha\mu} \; \Omega_{\alpha\mu} \tag{20}$$

By substituting eqs.(17), (18) into eq.(20) we obtain

$$\begin{aligned} &\sum \epsilon_{\alpha\mu} \; \sum \; (g_{\alpha\beta} \; g_{\mu\nu} - g_{\alpha\nu} \; g_{\mu\beta}) \; N_{(\mu} \; \partial_{\nu)} \ln n \; du^\beta \wedge du^\nu \\ &= \; \text{pf } \Omega \end{aligned} \tag{21}$$

Using the pfaffian of Ω, *the Gauss-Bonnet-Chern theorem*[31–33] says that[16,32]

$$(-1)^q \frac{1}{2^{2q}\pi^q q!} \int_{M^{2q}} \text{pf } \Omega = \chi(M^{2q}) \tag{22}$$

where $\chi(M^{2q})$ is *the Euler-Poincare characteristic*[34,35] *(a topological invariant*[16]*, a global invariant*[31]*) of the even dimensional oriented compact Riemannian manifold,* M^{2q}. We consider q in M^{2q} is the same as q in the description of pf Ω. We interpret that the size (ordo) of curvature matrix of the corresponding pfaffian is related to the number of a dimension of space (manifold). *The size of curvature matrix is the same as the number of a dimension of space.*

By substituting eq.(21) into eq.(22), we obtain the Gauss-Bonnet-Chern theorem related to the scalar refractive index as below

$$(-1)^q \frac{1}{2^{2q}\pi^q q!} \int_{M^{2q}} \sum \epsilon_{\alpha\mu} \sum \left(g_{\alpha\beta}\, g_{\mu\nu} - g_{\alpha\nu}\, g_{\mu\beta}\right) N_{(\mu}\, \partial_{\nu)} \ln n\; du^\beta \wedge du^\nu = \chi(M^{2q}) \quad (23)$$

In case of a 2-dimensional space, i.e. for $q = 1$, eq.(23) becomes

$$-\frac{1}{4\pi} \int_{M^2} \sum \epsilon_{\alpha\mu} \sum \left(g_{\alpha\beta}\, g_{\mu\nu} - g_{\alpha\nu}\, g_{\mu\beta}\right) N_{(\mu}\, \partial_{\nu)} \ln n\; du^\beta \wedge du^\nu = \chi(M^2) \quad (24)$$

We see from eqs.(23), (24), the scalar refractive index is related to the Euler-Poincare characteristic. *Because the Euler-Poincare characteristic is the topological invariant*[36,37] *(the global invariant*[31]*) we consider that the scalar refractive index is also the topological invariant (the local invariant). Eqs.(22), (23), (24) show that the integral of a local topological invariant gives result a global topological invariant.*

The pfaffian of the curvature matrix (20) is defined to be zero or non-zero if the curvature matrix is an odd-size or an even-size complex anti-symmetric matrix respectively. In turn, the zero or non-zero curvature form (3) has a consequence that the Riemann-Christoffel curvature tensor is vanish or not vanish respectively. *The vanishing Riemann-Christoffel curvature tensor means vacuum space. In other words, the Riemann-Christoffel curvature tensor must vanish in vacuum space*[38]*. So, does it mean that the zero or non-zero curvature form is related to vacuum or non-vacuum space (in turn a vanishing or a non-vanishing field strength?)*

The zero or non-zero Euler-Poincare characteristic (22) is a consequence of the zero or non-zero pfaffian of the curvature matrix respectively. *Does it mean that the zero or non-zero Euler-Poincare characteristic is related to vacuum or non-vacuum space? What is the existence of a topological invariant of the zero Euler-Poincare characteristic or vacuum space?*

We see from eq.(13) that the connection matrix, one-form, ω, is closed. *What is the meaning of a closed one-form physically? Could we interpret $d\omega = 0$ related to a conserved quantity (conservation law) in physics, especially in the geometrical optics? What is such conserved quantity in the geometrical optics?*

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[1] L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Butterworth-Heimenann, Oxford, 1975.

[2] In one dimension, the curvature tensor R_{1111} always vanishes. In other words, a curved line should have zero curvature. The Riemann-Christoffel curvature tensor reflects only *the inner properties of the space*, not how it is embedded in a higher dimensional space (Steven Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, 1972.)

[3] Soma Mitra, Somenath Chakrabarty, *Fermat's Principle in Curved Space-Time, No Emission from Schwarzschild Black Hols as Total Internal Reflection and Black Hole Unruh Effect*, https://arxiv.org/pdf/1512.03885.pdf, 2015.

[4] Miftachul Hadi, Andri Husein, Utama Alan Deta, *A refractive index in bent fibre optics and curved space*, IOP Conf. Series: Journal of Physics: Conf. Series **1171** (2019) 012016, https://iopscience.iop.org/article/10.1088/1742-6596/1171/1/012016/pdf.

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[6] G. Molesini, *Geometrical Optics*, Encyclopedia of Condensed Matter Physics, https://www.sciencedirect.com/topics/physics-and-astronomy/geometrical-optics, 2005.

[7] The dimension of the curvature in eq.(1) can be extended to any arbitrary number of dimensions (see Moshe Carmeli, *Classical Fields: General Relativity and Gauge Theory*, John Wiley and Sons, Inc., 1982.)

[8] Moshe Carmeli, *Classical Fields: General Relativity and Gauge Theory*, John Wiley and Sons, Inc., 1982.

[9] Richard L. Faber, *Differential Geometry and Relativity Theory: An Introduction*, Marcel Dekker, Inc., 1983.

[10] Steven Weinberg, *Gravitation and Cosmology*, John Wiley & Sons, 1972.

[11] Roniyus Marjunus, *Private communication.*

[12] Karsten Rottwitt, Peter Tidemand-Lichtenberg, *Nonlinear Optics: Principles and Applications*, CRC Press, 2015.

[13] Max Born, Emil Wolf, *Principles of Optics*, Pergamon Press, 1993.

[14] Chen Ning Yang, *Topology and Gauge Theory in Physics*, International Journal of Modern Physics A, Vol. 27, No. 30 (2012) 1230035.

[15] The Christoffel symbol does not transform as a tensor, but rather as an object in the jet bundle (Wikipedia, *Christoffel symbols*). If the non-linear term (non-Abelian term) of the Christoffel symbol happens to be zero in one coordinate system, it will in general not be zero in another coordinate system[5].

[16] Shiing-Shen Chern, *What is Geometry?* Amer. Math. Monthly 97, 1990.

[17] Shiing-Shen Chern, Wei-Huan Chen, Kai Shue Lam, *Lectures on Differential Geometry*, World Scientific, 2000.

[18] *Anticommutativity* is a specific property of some *non-commutative operations*. In mathematical physics, where symmetry is of central importance, these operations are mostly called *antisymmetric operations* (Wikipedia, *Anticommutative property*).

[19] We consider an anti-commutative (anti-symmetric) property of the wedge product as $du^\beta \wedge du^\nu \neq du^\nu \wedge du^\beta$.

[20]WolframMathWorld, *Antisymmetric Matrix*, https://mathworld.wolfram.com/AntisymmetricMatrix.html

[21]An antisymmetric matrix is a square matrix that satisfies the identity $A = -A^T$ where A^T is the matrix transpose. *All $n \times n$ antisymmetric matrices of odd size (i.e. if n is odd) are singular (determinant of matrix is equal to zero).* Antisymmetric matrices are commonly called "skew symmetric matrices" by mathematicians[20].

[22]In Nakahara[23], the curvature matrix is written as $\Omega = d_p\omega + \omega \wedge \omega$.

[23]Mikio Nakahara, *Geometry, Topology and Physics*, Adam Hilger, 1991.

[24]For any smooth 1-form, ω, and smooth vector fields, X and Y, on a manifold, the exterior derivative of a 1-form is defined as $d\omega(X,Y) \equiv X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$ (https://idv.sinica.edu.tw/ftliang/diff_geom/*diff_geometry%28II%29/3.11/exterior_derivative_2.pdf, https://math.stackexchange.com/questions/648504/v-vector-field-omega-one-form-v-omegav)

[25]Anonymous, *Lecture 1: Differential Forms*, https://spot.colorado.edu/~jnc/lecture1.pdf

[26]*Exterior derivative of a form and* $d(d\omega) = 0$? https://math.stackexchange.com/questions/1321754/exterior-derivative-of-a-form-and-dd-omega-0 answered Jun 11, 2015 at 18:57 by msteve.

[27]The exterior derivative of a connection can be written as[23] $d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$, where X, Y are *vector fields*. Roughly speaking, the vector fields are *the infinitesimal generators* of the transformation[23]. If $d\omega(X,Y) = 0$, it means that $[X,Y]$ is commute, $XY - YX = 0$ so $X = Y$. We consider closed 1-form, $d\omega = 0$, physically looks like the tendency of the fluid to rotate (see e.g. https://www.quora.com/What-is-irrotational-vector-field). In gauge theory, for each group generator there necessarily arises a coressponding *field* (usually *a vector field*) called *the gauge field*. When such a theory is quantized, the quanta of the gauge fields are called gauge bosons. In case of QED, i.e. an Abelian U(1) local gauge theory, it has one gauge field, *the electromagnetic four potential*, with *the photon* being the gauge boson (Wikipedia, *Gauge theory*).

[28]Shing Tung Yau, *Private communication*.

[29]The Pfaffian (considered as *a polynomial*) is nonvanishing only for $2n \times 2n$ skew-symmetric matrices, in which case it is *a polynomial of degree n* (Wikipedia, *Pfaffian*). In our article, we consider n is the same as q, *a natural number*.

[30]Howard E. Haber, *Notes on antisymmetric matrices and the pfaffian*, http://scipp.ucsc.edu/~haber/webpage/pfaffian2.pdf, January 2005.

[31]Shiing-Shen Chern, *From Triangles to Manifolds*, The American Mathematical Monthly, Vol. 86, No.5. (May, 1979), pp.339-349.

[32]Spalluci E. et al (2004), *Pfaffian*. In: Duplij S., Siegel W., Bagger J. (eds), Concise Encyclopedia of Supersymmetry, Springer, Dordrecht.

[33]*Gauss-Bonnet formula expresses the global invariant, $\chi(M)$, as the integral of a local invariant*, which is perhaps the most desirable relationship between local and global properties[31]. For *even-dimensional* oriented compact Riemannian manifold, M^{2n}, *the Gauss-Bonnet-Chern theorem is a special case of the Atiyah-Singer index theorem*[32].

[34]Milosav M. Marjanovic, *Euler-Poincare Characteristic - A Case of Topological Convincing*, The Teaching of Mathematics, 2014, Vol. XVII, 1, pp. 21–33.

[35]The Euler-Poincare characteristic starts from *Euler's polyhedron formula (a number)* which appeared first in a note submitted by Euler to the Proceedings of the Petersburg Academy of 1752/53. Henri Poincare who defined *an integer* to be *a topological property* of all other geometric objects. *The Euler-Poincare characteristic is a stable topological property*[34]. *What is a stable topological property?*

[36]Topological Invariant. Encyclopedia of Mathematics. https://encyclopediaofmath.org/wiki/Topological_invariant.

[37]Topological invariant is any property of a topological space that is *invariant* under *homeomorphisms*[36]. Homeomorphisms are, roughly speaking, the mappings that preserve all the topological properties of a given space.

[38]Yongmin Cho, *Topology of Classical Vacuum Space-Time*, Progress of Theoretical Physics Supplement No. 172, 2008.