A Formula for the Function $\pi(x)$ to Count the Number of Primes Exactly if $25 \leq x \leq 846$ with Python Code to Test It v. 2.0

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February 3, 2022

Abstract

This paper shows a very elementary way of counting the number of primes under a given number with total accuracy. Is the function $\pi(x)$ if $25 \leq x \leq 846$.

Keywords— prime, number, pi, composite, formula, function

1 Introduction

The function $\pi(x)$ is very known to every well documented mathematician interested in number theory. Here we present not an approximation of the function but instead an exact solution. The key idea was born in the February 26 of 2021 at morning when I was thinking about how to test the primality of a given number. By some years I was studying the possibilities of the composite numbers of the form $6k + 1$ and $6k - 1$. Because the prime numbers bigger than 3 has the same structure, I tried to figure how to filter primes from composites. I discover that the composite numbers has a very regular structure beginning from $(2a + 1)(2b + 1) = 4ab + 2a + 2b + 1$, the base of every odd composite number. So the numbers with the form $6k + 1$ has the structure $(6m + 1)(6n + 1)$ or $(6m - 1)(6n - 1)$ and the numbers with the form $6k - 1$ has the structure $(6m - 1)(6n + 1)$. These structures are totally predictable if we choose any pair of numbers but the primes are unpredictable. The set of composite numbers are placed with regularity but the prime numbers are the tiny holes that escapes from the structure of the composites, truly randomly placed if we want. The key idea takes advantage of the regularity of the structures of the composite numbers of the form $6k + 1$ and $6k - 1$. If we can count the number of composites of the form

$6k+1$ and $6k-1$ under a given number we can know the number of primes under that number only making a very basic math: $\pi(x) = Numbers(6k+1) + Numbers(6k-1) - Numbers((6m+1)(6n+1)) - Numbers((6m-1)(6n-1)) - Numbers((6m-1)(6n+1)) + 2$. The number 2 at the end represents the additional count of primes 2 and 3. So in this paper we develop the formula to count every set of numbers involved in some interval. On April 3 of 2021 we derived the necessary theorems and the first version of $\pi(x)$. On April 9 of 2021 we derived the partial version with and interval of $25 \leq x \leq 538$. On January 22 of 2022 we derived the proofs of the structures of the composites that has repetitions.

2 Prime Numbers of the Form $6k+1$ and $6k-1$

Because we want to know the quantity of the primes under x, first we note that every prime greater than 3 has the form $6k+1$ or $6k-1$, the next theorem shows that.

Theorem 2.1. *(Aurelio Baldor, 1985) [1] Every prime number $N > 3$ has the form $N = 6k+1$ or $N = 6k-1$*

Proof. Let $N > 3$ a prime number, we will show that $N = 6k+1$ or $N = 6k-1$. Divide N between 6, q is the quotient and R the residue. We have $N = 6q + R$, $R < 6$. R can not be zero because N is not a multiple of 6 (N is prime!). R must be 1, 2, 3, 4 or 5. R can not be 2 because we would have $N = 6q+2$ and the number would be divisible by 2 (N is prime!). R can not be 3 because we would have $N = 6q+3$ and the number would be divisible by 3 (N is prime!). R can not be 4 because we would have $N = 6q+4$ and the number would be divisible by 2 (N is prime!). So, if R can not be 2, 3 or 4, then R is 1 or 5. We conclude that N is of the form $N = 6k+1$ or $N = 6m+5 = 6k-1$.

Quod erat demonstrandum (Q.E.D). □

3 Composite Numbers of the Form $6k+1$ and $6k-1$

To calculate the quantity of primes lesser or equal than x, we need to subtract the composites that has the forms $6k+1$ or $6k-1$ and later add 2 (because we need to take in account the primes 2 and 3). Here we show two theorems that present us the composites with that forms.

Theorem 3.1. *(Danilo Chávez, April 3, 2021) If a composite number N is of the form $N = 6k+1$, then $N = (6m+1)(6n+1)$ or $N = (6m-1)(6n-1)$.*

Proof. Every odd composite number N is of the form $N = (2a+1)(2b+1) = 4ab+2a+2b+1$. If $N = 6k+1$ we have $4ab+2a+2b+1 = 6k+1$ then $2ab+a+b = 3k$. As this expression is a multiple of 3, if we suppose a as a multiple of 3, we conclude that b also is a multiple of 3, therefore $N = (6m+1)(6n+1)$. As m and n are integers, it takes negative values, so $N = (-6p+1)(-6q+1) = (6m-1)(6n-1)$, therefore, if $N = 6k+1$, it takes the form $N = (6m+1)(6n+1)$ or $N = (6m-1)(6n-1)$.

Quod erat demonstrandum (Q.E.D). □

Theorem 3.2. *(Danilo Chávez, April 3, 2021) If a composite number N is of the form $N = 6k-1$, then $N = (6m-1)(6n+1)$*

Proof. Every odd composite number N is of the form $N = (2a+1)(2b+1) = 4ab+2a+2b+1$. If $N = 6k-1$ we have $4ab+2a+2b+1 = 6k-1$ then $2ab+a+b+1 = 3k$. As this expression is a multiple of 3, if we suppose a as a multiple of 3, we conclude that $b+1$ also is a multiple of 3, we say $b+1 = 3s$ therefore $N = (6m+1)(6n-1)$. As m and n are integers, it takes negative values, so $N = (-6p+1)(-6q-1) = (6m-1)(6n+1)$. As N can take the form $N = (6m+1)(6n-1)$ and $N = (6c-1)(6d+1)$ we can conclude that if $N = 6k-1$, it has the unique form $N = (6m-1)(6n+1)$.

Quod erat demonstrandum (Q.E.D). $\square$

4 The Number of Primes Under a Given Number, $\pi(x)$, between an interval of the variable

FUNCTION $\pi(x)$ BETWEEN AN INTERVAL OF THE VARIABLE
(Danilo Chávez, April 3, 2021)

If $25 \leq x \leq 846$ and

$$C_{6k+1}(x) = \left\lfloor \frac{x-1}{6} \right\rfloor$$

$$C_{6k-1}(x) = \left\lfloor \frac{x+1}{6} \right\rfloor$$

$$C_{10}(x,m) = \left\lfloor \frac{x-(6m+1)}{6(6m+1)} \right\rfloor - m + 1$$

$$C_{11}(x,m) = \left\lfloor \frac{x-7(6m+1)}{42(6m+1)} + 1 \right\rfloor - \left\lceil \frac{m-1}{7} \right\rceil$$

$$C_{12}(x,s) = \left\lfloor \frac{x-7(6s+1)}{42(6s+1)} \right\rfloor - 7s$$

$$C_{13}(x,s) = \left\lfloor \frac{x-49(6s+1)}{294(6s+1)} \right\rfloor - s + 1$$

$$C_{14}(x) = \left\lfloor \frac{x-7}{42} \right\rfloor$$

$$C_{20}(x,m) = \left\lfloor \frac{x+(6m-1)}{6(6m-1)} \right\rfloor - m + 1$$

$$C_{21}(x,m) = \left\lfloor \frac{x-5(6m-1)}{30(6m-1)} + 1 \right\rfloor - \left\lceil \frac{m-1}{5} \right\rceil$$

$$C_{22}(x,s) = \left\lfloor \frac{x+5(6s+1)}{30(6s+1)} \right\rfloor - 5s$$

$$C_{23}(x,s) = \left\lfloor \frac{x-25(6s+1)}{150(6s+1)} \right\rfloor - s + 1$$

$$C_{24}(x) = \left\lfloor \frac{x+5}{30} \right\rfloor$$

$$C_{121}(x,m) = \left\lfloor \frac{x+5(6m+1)}{30(6m+1)} \right\rfloor - \left\lfloor \frac{m}{5} \right\rfloor$$

$$C_{122}(x,s) = \left\lfloor \frac{x-5(6s-1)}{30(6s-1)} \right\rfloor - 5s + 2$$

$$C_{123}(x,s) = \left\lfloor \frac{x+25(6s-1)}{150(6s-1)} \right\rfloor - s + 1$$

$$C_{30}(x,m) = \left\lfloor \frac{x-(6m-1)}{6(6m-1)} \right\rfloor$$

$$C_{31}(x,m) = \left\lfloor \frac{x+5(6m-1)}{30(6m-1)} \right\rfloor$$

$$C_{32}(x,m) = \left\lfloor \frac{x-7(6m-1)}{42(6m-1)} \right\rfloor$$

$$C_{33}(x,m) = \left\lfloor \frac{x+35(6m-1)}{210(6m-1)} \right\rfloor$$

$$C_{34}(x,m) = \left\lfloor \frac{x-(6m-1)}{6(6m-1)} - 1 \right\rfloor$$

$$C_{35}(x,m) = \left\lfloor \frac{x-(6m-1)}{6(6m-1)} - 1 \right\rfloor$$

$$C_{36}(x,m) = \left\lfloor \frac{x-(6m-1)}{6(6m-1)} - 1 \right\rfloor$$

$$C_{37}(s) = 35(6s+1)$$

$$C_{(6m+1)(6n+1)}(x) = \sum_{\substack{m\geq 1\\ C_{10}(x,m)>0}} C_{10}(x,m) - \sum_{m\geq 1 C_{11}(x,m)>0} C_{11}(x,m) - \sum_{\substack{s\geq 0\\ C_{12}(x,s)>0}} C_{12}(x,s)$$

$$+ \sum_{\substack{s\geq 0\\ C_{13}(x,s)>0}} C_{13}(x,s) + C_{14}(x)$$

$$C_{(6m-1)(6n-1)}(x) = \sum_{\substack{m\geq 1\\ C_{20}(x,m)>0}} C_{20}(x,m) - \sum_{\substack{m\geq 1\\ C_{21}(x,m)>0}} C_{21}(x,m) - \sum_{\substack{s\geq 0\\ C_{22}(x,s)>0}} C_{22}(x,s)$$

$$+ \sum_{\substack{s\geq 0\\ C_{23}(x,s)>0}} C_{23}(x,s) + C_{24}(x)$$

$$C_{common}(x) = \sum_{\substack{m\geq 1\\ C_{121}(x,m)>0}} C_{121}(x,m) + \sum_{\substack{s\geq 1\\ C_{122}(x,s)>0}} C_{122}(x,s) - \sum_{\substack{s\geq 1\\ C_{123}(x,s)>0}} C_{123}(x,s)$$

$$C_{(6m-1)(6n+1)}(x) = \sum_{\substack{m\geq 1\\ C_{30}(x,m)>0}} C_{30}(x,m) - \sum_{\substack{m\geq 2\\ m\neq 1 mod(5)\\ m\neq -1 mod(7)\\ C_{31}(x,m)>0}} C_{31}(x,m) - \sum_{\substack{m\geq 2\\ m\neq 1 mod(5)\\ m\neq -1 mod(7)\\ C_{32}(x,m)>0}} C_{32}(x,m)$$

$$+ \sum_{\substack{m\geq 2\\ m\neq 1 mod(5)\\ m\neq -1 mod(7)\\ C_{33}(x,m)>0}} C_{33}(x,m) - \sum_{\substack{m\geq 2\\ m\equiv 1 mod(5)\\ C_{34}(x,m)>0}} C_{34}(x,m) - \sum_{\substack{m\geq 2\\ m\equiv -1 mod(7)\\ C_{35}(x,m)>0}} C_{35}(x,m)$$

$$+ \sum_{\substack{m\geq 2\\ m\equiv 1 mod(5)\\ m\equiv -1 mod(7)\\ C_{36}(x,m)>0}} C_{36}(x,m) - \sum_{\substack{s\geq 1\\ C_{37}(s)<x}} 1$$

$$C_2 = 2$$

then, the number of primes lesser or equal to some number x is

$$\pi(x) = C_{6k+1}(x) + C_{6k-1}(x) - C_{(6m+1)(6n+1)}(x) - C_{(6m-1)(6n-1)}(x)$$
$$+C_{common}(x) - C_{(6m-1)(6n+1)}(x) + C_2$$

DEDUCTION OF THE FUNCTION $\pi(x)$ BETWEEN AN INTERVAL OF x

To find the number of primes lesser or equal to some number $x \geq 25$, we calculate the total quantity of numbers with the form $6k+1$ and $6k-1$ (primes or composites), we subtract the quantity of composite numbers with the form $(6m+1)(6n+1)$, we subtract the quantity of composite numbers with the form $(6m-1)(6n-1)$, we subtract the quantity of composite numbers with the form $(6m-1)(6n+1)$ and finally we add 2 (prime numbers 2 and 3).

Let $x \geq 25$ be some integer.

A) QUANTITY OF NUMBERS WITH THE FORM $6k+1$ AND $6k-1$

The total quantity of numbers of the form $6k+1$ lesser or equal to x.
If $6k+1 \leq x$ then

$$k \leq \frac{x-1}{6}$$

Now we define

$$C_{6k+1}(x) = \left\lfloor \frac{x-1}{6} \right\rfloor$$

The total quantity of numbers with the form $6k-1$ lesser or equal to x.
If $6k-1 \leq x$ then

$$k \leq \frac{x+1}{6}$$

Because we want integers we define

$$C_{6k-1}(x) = \left\lfloor \frac{x+1}{6} \right\rfloor$$

B) QUANTITY OF COMPOSITE NUMBERS WITH $(6m+1)(6n+1) = (6p+1)(6q+1)$

(6 * m + 1) * (6 * n + 1)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	49	91	133	175	217	259	301	343	385	427	469	511	553	595	637	679	721	763	805	847	889	931
2		169	247	325	403	481	559	637	715	793	871	949	1027	1105	1183	1261	1339	1417	1495	1573	1651	1729
3			361	475	589	703	817	931	1045	1159	1273	1387	1501	1615	1729	1843	1957	2071	2185	2299	2413	2527
4				625	775	925	1075	1225	1375	1525	1675	1825	1975	2125	2275	2425	2575	2725	2875	3025	3175	3325
5					961	1147	1333	1519	1705	1891	2077	2263	2449	2635	2821	3007	3193	3379	3565	3751	3937	4123
6						1369	1591	1813	2035	2257	2479	2701	2923	3145	3367	3589	3811	4033	4255	4477	4699	4921
7							1849	2107	2365	2623	2881	3139	3397	3655	3913	4171	4429	4687	4945	5203	5461	5719
8								2401	2695	2989		3577	3871	4165	4459	4753	5047	5341	5635	5929	6223	6517
9									3025	3355	3685	4015	4345	4675	5005	5335	5665	5995	6325	6655	6985	7315
10										3721	4087	4453	4819	5185	5551	5917	6283	6649	7015	7381	7747	8113
11											4489	4891	5293	5695	6097	6499	6901	7303	7705	8107	8509	8911
12												5329	5767	6205	6643	7081	7519	7957	8395	8833	9271	9709
13													6241	6715	7189	7663	8137	8611	9085	9559	10033	10507
14														7225	7735	8245	8755	9265	9775	10285	10795	11305
15															8281	8827	9373	9919	10465	11011	11557	12103
16																9409	9991	10573	11155	11737	12319	12901
17																	10609	11227	11845	12463	13081	13699
18																		11881	12535	13189	13843	14497
19																			13225	13915	14605	15295
20																				14641	15367	16093
21																					16129	16891
22																						17689
23																						

Figure 1: Composite numbers with the form $(6m+1)(6n+1) = (6p+1)(6q+1)$, the pink numbers are those which are repeated

The next theorem shows the structure of the composites with $(6m+1)(6n+1) = (6p+1)(6q+1)$.

Theorem 4.1. *(Danilo Chávez, January 22, 2022)*
Let be m, n, p, q, r, t, k integers. Let be $1 \leq m, 1 \leq n, 1 \leq p, 1 \leq q, 0 \leq r, 1 \leq t, 1 \leq k$.
When

$$(6m+1)(6n+1) = (6p+1)(6q+1)$$

If

$$p = 7r + 1$$

with $0 \leq r$, then

$$q = 1, 2, 3, 4...$$

or by symmetry, if

$$q = 7r + 1$$

with $0 \leq r$, *then*

$$p = 1, 2, 3, 4...$$

Proof. Let bem, n, p, q, r, t, k integers. Let be $1 \leq m, 1 \leq n, 1 \leq p, 1 \leq q, 0 \leq r, 1 \leq t, 1 \leq k$. We take the line $m = 1$ and we have

$$7(6n + 1) = (6p + 1)(6q + 1)$$

or $(6p+1) = 7t$ or $(6q+1) = 7t$. By symmetry we can take $(6p+1) = 7t$ and the same will happen with $(6q + 1) = 7t$ with inverted values.
Lets take $(6p + 1) = 7t$ and we have

$$p = \frac{7t - 1}{6}$$

We are looking for the values where p is integer and we have that t is of the form $6r + 1$ with $r = 0, 1, 2, 3...$

$$p = \frac{7(6r + 1) - 1}{6} = 7r + 1$$

p takes the values $p = 1, 8, 15, 22...$
Now we are looking for the values of q given p. We start with the equation

$$7(6n + 1) = (6p + 1)(6q + 1)$$

We can see that

$$q = \frac{7n - p + 1}{6p + 1}$$

As we are looking for the integer values given p, we suppose that for $k = 1, 2, 3, 4...$

$$7n - p + 1 = k(6p + 1)$$

giving

$$7n = 6kp + k + p - 1$$

Substituting we have

$$q = \frac{7n - p + 1}{6p + 1} = \frac{6kp + k + p - 1 - p + 1}{6p + 1} = k\left(\frac{6p + 1}{6p + 1}\right) = k$$

that shows that q takes all the values $k = 1, 2, 3, 4...$, for any value of p, remember that $p = 7r + 1$. We conclude that when

$$p = 7r + 1$$

with $0 \leq r$

$$q = 1, 2, 3, 4...$$

or by symmetry if

$$q = 7r + 1$$

then

$$p = 1, 2, 3, 4...$$

Quod erat demonstrandum (Q.E.D).

□

To find the total quantity of composite numbers with the form $(6m+1)(6n+1) = (6p+1)(6q+1)$, we calculate the total composite numbers with that form lesser or equal to x, we subtract the numbers in the columns (it has repetitions), we subtract the numbers in the rows (it has repetition), we need to add the numbers between intersections (it was subtracted twice) and finally we add the numbers in the first row (it was subtracted before).

B.0) To find the total quantity of composite numbers with the form $(6m + 1)(6n + 1) = (6p + 1)(6q + 1)$ lesser or equal to x.

If $(6m + 1)(6n + 1) \leq x$ then

$$n \leq \frac{x - (6m + 1)}{6(6m + 1)}$$

Because we want integers we define

$$C_{10}(x, m) = \left\lfloor \frac{x - (6m + 1)}{6(6m + 1)} \right\rfloor - m + 1$$

where the last term eliminates the number of composite numbers under the main diagonal. Thus, the total quantity of composite numbers with the form $(6m+1)(6n+1)$, lesser or equal to x is

$$\sum_{\substack{m\geq 1\\ C_{10}(x,m)>0}} C_{10}(x,m)$$

B.1) To find the total quantity of composite numbers with the form $(6m+1)(6(7t+1)+1)$ lesser or equal to x in the columns.

$$(6m+1)(6(7t+1)+1) = 7(6m+1)(6t+1)$$

If $7(6m+1)(6t+1) \leq x$ then

$$t \leq \frac{x - 7(6m+1)}{42(6m+1)}$$

but $t \geq 0$ and we want to count the number of columns, so

$$t+1 \leq \frac{x - 7(6m+1)}{42(6m+1)} + 1$$

Because we want integers we define

$$C_{11}(x,m) = \left\lfloor \frac{x - 7(6m+1)}{42(6m+1)} + 1 \right\rfloor - \left\lceil \frac{m-1}{7} \right\rceil$$

where the last term eliminates the number of extra columns under the main diagonal. Thus, the total quantity of composite numbers with the form $(6m+1)(6(7t+1)+1)$, lesser or equal to x is

$$\sum_{\substack{m\geq 1\\ C_{11}(x,m)>0}} C_{11}(x,m)$$

B.2) To find the total quantity of composite numbers with the form $(6(7s+1)+1)(6n+1)$ lesser or equal to x in the rows.

$$(6(7s+1)+1)(6n+1) = 7(6s+1)(6n+1)$$

If $7(6s+1)(6n+1) \leq x$ then

$$n \leq \frac{x - 7(6s+1)}{42(6s+1)}$$

Because we want integers we define

$$C_{12}(x,s) = \left\lfloor \frac{x - 7(6s+1)}{42(6s+1)} \right\rfloor - 7s$$

where the last term eliminates the number of composite numbers before the columns that touches the main diagonal. Thus, the total quantity of composite numbers with the form $(6(7s+1)+1)(6n+1)$, lesser or equal to x and above the main diagonal is

$$\sum_{\substack{s\geq 0\\ C_{12}(x,s)>0}} C_{12}(x,s)$$

B.3) To find the total quantity of composite numbers with the form $(6(7s+1)+1)(6(7t+1)+1)$ lesser or equal to x that are intersections between columns and rows.

$$(6(7s+1)+1)(6(7t+1)+1) = 49(6s+1)(6t+1)$$

If $49(6s+1)(6t+1) \leq x$ then

$$t \leq \frac{x - 49(6s+1)}{294(6s+1)}$$

Because we want integers we define

$$C_{13}(x,s) = \left\lfloor \frac{x - 49(6s+1)}{294(6s+1)} \right\rfloor - s + 1$$

where the last term eliminates the number of composite numbers under the main diagonal. Thus, the total quantity of composite numbers with the form $(6(7s+1)+1)(6(7t+1)+1)$, lesser or equal to x and above the main diagonal is

$$\sum_{\substack{s\geq 0\\ C_{13}(x,s)>0}} C_{13}(x,s)$$

B.4) To find the total quantity of composite numbers with the form $7(6n+1)$ lesser or equal to x that are numbers in the first row.

If $7(6n+1)\leq x$ then

$$n\leq \frac{x-7}{42}$$

Because we want integers we define

$$C_{14}(x)=\left\lfloor \frac{x-7}{42}\right\rfloor$$

Thus, the total quantity of composite numbers with the form $7(6n+1)$, lesser or equal to x in the first row is

$$C_{14}(x)$$

B.Final) The total quantity of composite numbers with the form $(6m+1)(6n+1)$, without repetition is

$$C_{(6m+1)(6n+1)}(x)=\sum_{\substack{m\geq 1\\ C_{10}(x,m)>0}} C_{10}(x,m)-\sum_{\substack{m\geq 1\\ C_{11}(x,m)>0}} C_{11}(x,m)-\sum_{\substack{s\geq 0\\ C_{12}(x,s)>0}} C_{12}(x,s)$$
$$+\sum_{\substack{s\geq 0\\ C_{13}(x,s)>0}} C_{13}(x,s)+C_{14}(x)$$

C) QUANTITY OF COMPOSITE NUMBERS WITH $(6m-1)(6n-1)=(6p-1)(6q-1)$

The next theorem shows the structure of the composites with $(6m-1)(6n-1)=(6p-1)(6q-1)$.

(6 * m – 1) * (6 * n - 1)

	1	2	3	4	5	6	7	8	9	10
1	25	55	85	115	145	175	205	235	265	295
2		121	187	253	319	385	451	517	583	649
3			289	391	493	595	697	799	901	1003
4				529	667	805	943	1081	1219	1357
5					841	1015	1189	1363	1537	1711
6						1225	1435	1645	1855	2065
7							1681	1927	2173	2419
8								2209	2491	2773
9									2809	3127
10										3481
11										
12										
13										
14										
15										
16										
17										
18										
19										
20										
21										
22										

	11	12	13	14	15	16	17	18	19	20	21
1	325	355	385	415	445	475	505	535	565	595	625
2	715	781	847	913	979	1045	1111	1177	1243	1309	1375
3	1105	1207	1309	1411	1513	1615	1717	1819	1921	2023	2125
4	1495	1633	1771	1909	2047	2185	2323	2461	2599	2737	2875
5	1885	2059	2233	2407	2581	2755	2929	3103	3277	3451	3625
6	2275	2485	2695	2905	3115	3325	3535	3745	3955	4165	4375
7	2665	2911	3157	3403	3649	3895	4141	4387	4633	4879	5125
8	3055	3337	3619	3901	4183	4465	4747	5029	5311	5593	5875
9	3445	3763	4081	4399	4717	5035	5353	5671	5989	6307	6625
10	3835	4189	4543	4897	5251	5605	5959	6313	6667	7021	7375
11	4225	4615	5005	5395	5785	6175	6565	6955	7345	7735	8125
12		5041	5467	5893	6319	6745	7171	7597	8023	8449	8875
13			5929	6391	6853	7315	7777	8239	8701	9163	9625
14				6889	7387	7885	8383	8881	9379	9877	10375
15					7921	8455	8989	9523	10057	10591	11125
16						9025	9595	10165	10735	11305	11875
17							10201	10807	11413	12019	12625
18								11449	12091	12733	13375
19									12769	13447	14125
20										14161	14875
21											15625
22											

Figure 2: Composite numbers with the form $(6m-1)(6n-1) = (6p-1)(6q-1)$, the pink numbers are those which are repeated

Theorem 4.2. *(Danilo Chávez, January 22, 2022)*
Let be m, n, p, q, r, t, k integers. Let be $1 \leq m, 1 \leq n, 1 \leq p, 1 \leq q, 0 \leq r, 1 \leq t, 1 \leq k$.
When

$$(6m-1)(6n-1) = (6p-1)(6q-1)$$

If

$$p = 5r + 1$$

with $0 \leq r$, then

$$q = 1, 2, 3, 4...$$

or by symmetry, if

$$q = 5r + 1$$

with $0 \leq r$, then

$$p = 1, 2, 3, 4...$$

Proof. Let bem, n, p, q, r, t, k integers. Let be $1 \leq m, 1 \leq n, 1 \leq p, 1 \leq q, 0 \leq r, 1 \leq t, 1 \leq k$. We take the line $m = 1$ and we have

$$5(6n-1) = (6p-1)(6q-1)$$

or $(6p-1) = 5t$ or $(6q-1) = 5t$. By symmetry we can take $(6p-1) = 5t$ and the same will happen with $(6q-1) = 5t$ with inverted values.

Lets take $(6p-1)=5t$ and we have

$$p=\frac{5t+1}{6}$$

We are looking for the values where p is integer and we have that t is of the form $6r+1$ with $r=0,1,2,3...$

$$p=\frac{5(6r+1)+1}{6}=5r+1$$

p takes the values $p=1,6,11,16...$
Now we are looking for the values of q given p. We start with the equation

$$5(6n-1)=(6p-1)(6q-1)$$

We can see that

$$q=\frac{5n+p-1}{6p-1}$$

As we are looking for the integer values given p, we suppose that for $k=1,2,3,4...$

$$5n+p-1=k(6p-1)$$

giving

$$5n=6kp-k-p+1$$

Substituting we have

$$q=\frac{5n+p-1}{6p-1}=\frac{6kp-k-p+1+p-1}{6p-1}=k\left(\frac{6p-1}{6p-1}\right)=k$$

that shows that q takes all the values $k=1,2,3,4...$, for any value of p, remember that $p=5r+1$.
We conclude that when

$$p=5r+1$$

with $0\leq r$

$$q=1,2,3,4...$$

or by symmetry if

$$q=5r+1$$

then

$$p = 1, 2, 3, 4...$$

Quod erat demonstrandum (Q.E.D).

$\square$

To find the total quantity of composite numbers with the form $(6m-1)(6n-1) = (6p-1)(6q-1)$, we calculate the total composite numbers with that form lesser or equal to x, we subtract the numbers in the columns (it has repetitions), we subtract the numbers in the rows (it has repetition), we need to add the numbers between intersections (it was subtracted twice), we add the numbers in the first row (it was subtracted before) and finally we subtract the composite numbers that has repetition between the composite numbers with the form $(6m + 1)(6n + 1)$ and the composite numbers with the form $(6m - 1)(6n - 1)$.

C.0) To find the total quantity of composite numbers with the form $(6m - 1)(6n - 1) = (6p - 1)(6q - 1)$ lesser or equal to N.

If $(6m - 1)(6n - 1) \leq x$ then

$$n \leq \frac{x + (6m - 1)}{6(6m - 1)}$$

Because we want integers we define

$$C_{20}(x, m) = \left\lfloor \frac{x + (6m - 1)}{6(6m - 1)} \right\rfloor - m + 1$$

where the last term eliminates the number of composite numbers under the main diagonal. Thus, the total quantity of composite numbers with the form $(6m + 1)(6n + 1)$, lesser or equal to x is

$$\sum_{\substack{m \geq 1 \\ C_{20}(x,m) > 0}} C_{20}(x, m)$$

C.1) To find the total quantity of composite numbers with the form $(6m - 1)(6(5t + 1) - 1)$ lesser or equal to x in the columns.

$$(6m - 1)(6(5t + 1) - 1) = 5(6m - 1)(6t + 1)$$

If $5(6m-1)(6t+1) \leq x$ then

$$t \leq \frac{x - 5(6m-1)}{30(6m-1)}$$

but $t \geq 0$ and we want to count the number of columns, so

$$t + 1 \leq \frac{x - 5(6m-1)}{30(6m-1)} + 1$$

Because we want integers we define

$$C_{21}(x, m) = \left\lfloor \frac{x - 5(6m-1)}{30(6m-1)} + 1 \right\rfloor - \left\lceil \frac{m-1}{5} \right\rceil$$

where the last term eliminates the number of extra columns under the main diagonal. Thus, the total quantity of composite numbers with the form $(6m-1)(6(5t+1)-1)$, lesser or equal to x is

$$\sum_{\substack{m \geq 1 \\ C_{21}(x,m) > 0}} C_{21}(x, m)$$

C.2) To find the total quantity of composite numbers with the form $(6(5s+1)-1)(6n-1)$ lesser or equal to x in the rows.

$$(6(5s+1)-1)(6n-1) = 5(6s+1)(6n-1)$$

If $5(6s+1)(6n-1) \leq x$ then

$$n \leq \frac{x + 5(6s+1)}{30(6s+1)}$$

Because we want integers we define

$$C_{22}(x,s) = \left\lfloor \frac{x + 5(6s+1)}{30(6s+1)} \right\rfloor - 5s$$

where the last term eliminates the number of composite numbers before the columns that touches the main diagonal. Thus, the total quantity of composite numbers with the form $(6(5s+1)-1)(6n-1)$, lesser or equal to x and above the main diagonal is

$$\sum_{\substack{s \geq 0 \\ C_{22}(x,s)>0}} C_{22}(x,s)$$

C.3) To find the total quantity of composite numbers with the form $(6(5s+1)-1)(6(5t+1)-1)$ lesser or equal to x that are intersections between columns and rows.

$$(6(5s+1)-1)(6(5t+1)-1) = 25(6s+1)(6t+1)$$

If $25(6s+1)(6t+1) \leq x$ then

$$t \leq \frac{x - 25(6s+1)}{150(6s+1)}$$

Because we want integers we define

$$C_{23}(x,s) = \left\lfloor \frac{x - 25(6s+1)}{150(6s+1)} \right\rfloor - s + 1$$

where the last term eliminates the number of composite numbers under the main diagonal. Thus, the total quantity of composite numbers with the form $(6(5s+1)-1)(6(5t+1)-1)$, lesser or equal to x and above the main diagonal is

$$\sum_{\substack{s \geq 0 \\ C_{23}(x,s)>0}} C_{23}(x,s)$$

C.4) To find the total quantity of composite numbers with the form $5(6n-1)$ lesser or equal to x that are numbers in the first row.

If $5(6n-1) \leq x$ then

$$n \leq \frac{x+5}{30}$$

Because we want integers we define

$$C_{24}(x) = \left\lfloor \frac{x+5}{30} \right\rfloor$$

Thus, the total quantity of composite numbers with the form $5(6n-1)$, lesser or equal to x in the first row is

$$C_{24}(x)$$

C.Final) The total quantity of composite numbers with the form $(6m-1)(6n-1)$, without repetition is

$$C_{(6m-1)(6n-1)}(x) = \sum_{\substack{m \geq 1 \\ C_{20}(x,m)>0}} C_{20}(x,m) - \sum_{\substack{m \geq 1 \\ C_{21}(x,m)>0}} C_{21}(x,m) - \sum_{\substack{s \geq 0 \\ C_{22}(x,s)>0}} C_{22}(x,s) + \sum_{\substack{s \geq 0 \\ C_{23}(x,s)>0}} C_{23}(x,s) + C_{24}(x)$$

D) QUANTITY OF COMPOSITE NUMBERS IN COMMON WITH $(6m-1)(6n-1) = (6m+1)(6n+1)$

The next theorem shows the structure of the composites with $(6m-1)(6n-1) = (6p+1)(6q+1)$.

Theorem 4.3. *(Danilo Chávez, January 22, 2022)*
Let be m, n, p, q, r, t, k integers. Let be $1 \leq m, 1 \leq n, 1 \leq p, 1 \leq q, 1 \leq r, 1 \leq t, 1 \leq k$.
When

$$(6m-1)(6n-1) = (6p+1)(6q+1)$$

If

$$p = 5r - 1$$

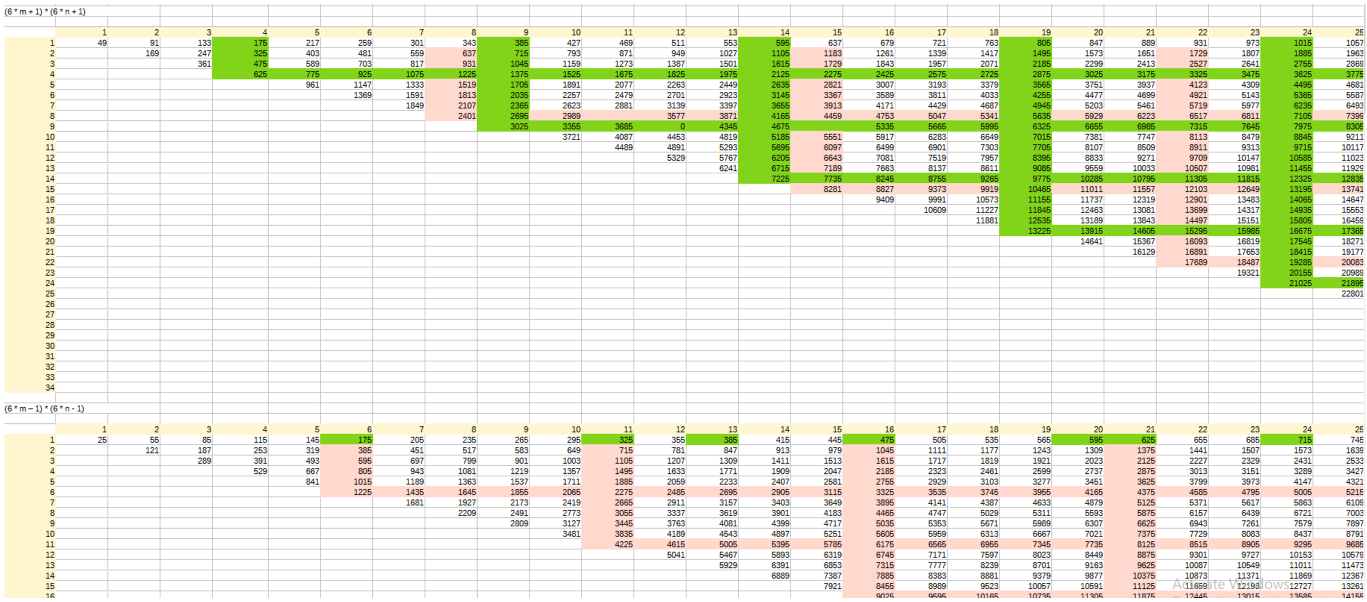

Figure 3: Composite numbers which has repetition between the composite numbers with $(6m-1)(6n-1) = (6m+1)(6n+1)$, the green numbers are those which are repeated

with $1 \leq r$, *then*

$$q = 1, 2, 3, 4...$$

or by symmetry, if

$$q = 5r - 1$$

with $1 \leq r$, *then*

$$p = 1, 2, 3, 4...$$

Proof. Let $be m, n, p, q, r, t, k$ integers. Let be $1 \leq m, 1 \leq n, 1 \leq p, 1 \leq q, 1 \leq r, 1 \leq t, 1 \leq k$. We take the line $m = 1$ and we have

$$5(6n-1) = (6p+1)(6q+1)$$

or $(6p+1) = 5t$ or $(6q+1) = 5t$. By symmetry we can take $(6p+1) = 5t$ and the same will happen with $(6q+1) = 5t$ with inverted values.
Lets take $(6p+1) = 5t$ and we have

$$p = \frac{5t-1}{6}$$

We are looking for the values where p is integer and we have that t is of the form $6r-1$ with $r=1,2,3,4...$

$$p=\frac{5(6r-1)-1}{6}=5r-1$$

p takes the values $p=4,9,14,19...$
Now we are looking for the values of q given p. We start with the equation

$$5(6n-1)=(6p+1)(6q+1)$$

We can see that

$$q=\frac{5n-p-1}{6p+1}$$

As we are looking for the integer values given p, we suppose that for $k=1,2,3,4...$

$$5n-p-1=k(6p+1)$$

giving

$$5n=6kp+k+p+1$$

Substituting we have

$$q=\frac{5n-p-1}{6p+1}=\frac{6kp+k+p+1-p-1}{6p+1}=k\left(\frac{6p+1}{6p+1}\right)=k$$

that shows that q takes all the values $k=1,2,3,4...$, for any value of p, remember that $p=5r-1$.
We conclude that when

$$p=5r-1$$

with $1\leq r$

$$q=1,2,3,4...$$

or by symmetry if

$$q=5r-1$$

then

$$p=1,2,3,4...$$

Quod erat demonstrandum (Q.E.D).

$\square$

To find the total quantity of composite numbers that are in common between the composites with $(6m-1)(6n-1) = (6m+1)(6n+1)$, we calculate the numbers in the green columns, we add the numbers in the green rows and finally we subtract the numbers between intersections.

D.1) To find the total quantity of composite numbers with the form $(6m+1)(6(5t-1)+1)$ lesser or equal to x in the columns.

$$(6m+1)(6(5t-1)+1) = 5(6m+1)(6t-1)$$

If $5(6m+1)(6t-1) \leq x$ then

$$t \leq \frac{x+5(6m+1)}{30(6m+1)}$$

Because we want integers we define

$$C_{121}(x,m) = \left\lfloor \frac{x+5(6m+1)}{30(6m+1)} \right\rfloor - \left\lfloor \frac{m}{5} \right\rfloor$$

where the last term eliminates the number of extra columns under the main diagonal. Thus, the total quantity of composite numbers with the form $(6m+1)(6(5t-1)+1)$, lesser or equal to x is

$$\sum_{\substack{m\geq 1\\ C_{121}(x,m)>0}} C_{121}(x,m)$$

D.2) To find the total quantity of composite numbers with the form $(6(5s-1)+1)(6n+1)$ lesser or equal to x in the rows.

$$(6(5s-1)+1)(6n+1) = 5(6s-1)(6n+1)$$

If $5(6s-1)(6n+1) \leq x$ then

$$n \leq \frac{x-5(6s-1)}{30(6s-1)}$$

Because we want integers we define

$$C_{122}(x,s) = \left\lfloor \frac{x - 5(6s-1)}{30(6s-1)} \right\rfloor - 5s + 2$$

where the last term eliminates the number of composite numbers before the columns that touches the main diagonal. Thus, the total quantity of composite numbers with the form $(6(5s-1)+1)(6n+1)$, lesser or equal to x and above the main diagonal is

$$\sum_{\substack{s\geq 1\\ C_{122}(x,s)>0}} C_{122}(x,s)$$

D.3) To find the total quantity of composite numbers with the form $(6(5s-1)+1)(6(5t-1)+1)$ lesser or equal to x that are intersections between columns and rows.

$$(6(5s-1)+1)(6(5t-1)+1) = 25(6s-1)(6t-1)$$

If $25(6s-1)(6t-1) \leq x$ then

$$t \leq \frac{x + 25(6s-1)}{150(6s-1)}$$

Because we want integers we define

$$C_{123}(x,s) = \left\lfloor \frac{x + 25(6s-1)}{150(6s-1)} \right\rfloor - s + 1$$

where the last term eliminates the number of composite numbers under the main diagonal. Thus, the total quantity of composite numbers with the form $(6(5s-1)+1)(6(5t-1)+1)$, lesser or equal to x and above the main diagonal is

$$\sum_{\substack{s\geq 1\\ C_{123}(x,s)>0}} C_{123}(x,s)$$

D.Final) The total quantity of composite numbers with repetition between composite numbers with the form $(6m+1)(6n+1)$ and $(6m-1)(6n-1)$, is

$$C_{common}(x) = \sum_{\substack{m \geq 1 \\ C_{121}(x,m) > 0}} C_{121}(x,m) + \sum_{\substack{s \geq 1 \\ C_{122}(x,s) > 0}} C_{122}(x,s) - \sum_{\substack{s \geq 1 \\ C_{123}(x,s) > 0}} C_{123}(x,s)$$

E) QUANTITY OF COMPOSITE NUMBERS WITH THE FORM $(6m-1)(6n+1) = (6p-1)(6q+1)$

(6m-1)(6n+1) = (6p-1)(6q+1)

(6m-1)(6n+1)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
1	35	65	95	125	155	185	215	245	275	305	335	365	395	425	455	485	515	545	575	605	635
2	77	143	209	275	341	407	473	539	605	671	737	803	869	935	1001	1067	1133	1199	1265	1331	1397
3	119	221	323	425	527	629	731	833	935	1037	1139	1241	1343	1445	1547	1649	1751	1853	1955	2057	2159
4	161	299	437	575	713	851	989	1127	1265	1403	1541	1679	1817	1955	2093	2231	2369	2507	2645	2783	2921
5	203	377	551	725	899	1073	1247	1421	1595	1769	1943	2117	2291	2465	2639	2813	2987	3161	3335	3509	3683
6	245	455	665	875	1085	1295	1505	1715	1925	2135	2345	2555	2765	2975	3185	3395	3605	3815	4025	4235	4445
7	287	533	779	1025	1271	1517	1763	2009	2255	2501	2747	2993	3239	3485	3731	3977	4223	4469	4715	4961	5207
8	329	611	893	1175	1457	1739	2021	2303	2585	2867	3149	3431	3713	3995	4277	4559	4841	5123	5405	5687	5969
9	371	689	1007	1325	1643	1961	2279	2597	2915	3233	3551	3869	4187	4505	4823	5141	5459	5777	6095	6413	6731
10	413	767	1121	1475	1829	2183	2537	2891	3245	3599	3953	4307	4661	5015	5369	5723	6077	6431	6785	7139	7493
11	455	845	1235	1625	2015	2405	2795	3185	3575	3965	4355	4745	5135	5525	5915	6305	6695	7085	7475	7865	8255
12	497	923	1349	1775	2201	2627	3053	3479	3905	4331	4757	5183	5609	6035	6461	6887	7313	7739	8165	8591	9017
13	539	1001	1463	1925	2387	2849	3311	3773	4235	4697	5159	5621	6083	6545	7007	7469	7931	8393	8855	9317	9779
14	581	1079	1577	2075	2573	3071	3569	4067	4565	5063	5561	6059	6557	7055	7553	8051	8549	9047	9545	10043	10541
15	623	1157	1691	2225	2759	3293	3827	4361	4895	5429	5963	6497	7031	7565	8099	8633	9167	9701	10235	10769	11303
16	665	1235	1805	2375	2945	3515	4085	4655	5225	5795	6365	6935	7505	8075	8645	9215	9785	10355	10925	11495	12065
17	707	1313	1919	2525	3131	3737	4343	4949	5555	6161	6767	7373	7979	8585	9191	9797	10403	11009	11615	12221	12827
18	749	1391	2033	2675	3317	3959	4601	5243	5885	6527	7169	7811	8453	9095	9737	10379	11021	11663	12305	12947	13589
19	791	1469	2147	2825	3503	4181	4859	5537	6215	6893	7571	8249	8927	9605	10283	10961	11639	12317	12995	13673	14351
20	833	1547	2261	2975	3689	4403	5117	5831	6545	7259	7973	8687	9401	10115	10829	11543	12257	12971	13685	14399	15113
21	875	1625	2375	3125	3875	4625	5375	6125	6875	7625	8375	9125	9875	10625	11375	12125	12875	13625	14375	15125	15875
22	917	1703	2489	3275	4061	4847	5633	6419	7205	7991	8777	9563	10349	11135	11921	12707	13493	14279	15065	15851	16637
23	959	1781	2603	3425	4247	5069	5891	6713	7535	8357	9179	10001	10823	11645	12467	13289	14111	14933	15755	16577	17399
24	1001	1859	2717	3575	4433	5291	6149	7007	7865	8723	9581	10439	11297	12155	13013	13871	14729	15587	16445	17303	18161
25	1043	1937	2831	3725	4619	5513	6407	7301	8195	9089	9983	10877	11771	12665	13559	14453	15347	16241	17135	18029	18923
26	1085	2015	2945	3875	4805	5735	6665	7595	8525	9455	10385	11315	12245	13175	14105	15035	15965	16895	17825	18755	19685
27	1127	2093	3059	4025	4991	5957	6923	7889	8855	9821	10787	11753	12719	13685	14651	15617	16583	17549	18515	19481	20447
28	1169	2171	3173	4175	5177	6179	7181	8183	9185	10187	11189	12191	13193	14195	15197	16199	17201	18203	19205	20207	21209
29	1211	2249	3287	4325	5363	6401	7439	8477	9515	10553	11591	12629	13667	14705	15743	16781	17819	18857	19895	20933	21971
30	1253	2327	3401	4475	5549	6623	7697	8771	9845	10919	11993	13067	14141	15215	16289	17363	18437	19511	20585	21659	22733
31	1295	2405	3515	4625	5735	6845	7955	9065	10175	11285	12395	13505	14615	15725	16835	17945	19055	20165	21275	22385	23495
32	1337	2483	3629	4775	5921	7067	8213	9359	10505	11651	12797	13943	15089	16235	17381	18527	19673	20819	21965	23111	24257
33	1379	2561	3743	4925	6107	7289	8471	9653	10835	12017	13199	14381	15563	16745	17927	19109	20291	21473	22655	23837	25019
34	1421	2639	3857	5075	6293	7511	8729	9947	11165	12383	13601	14819	16037	17255	18473	19691	20909	22127	23345	24563	25781
35	1463	2717	3971	5225	6479	7733	8987	10241	11495	12749	14003	15257	16511	17765	19019	20273	21527	22781	24035	25289	26543
36	1505	2795	4085	5375	6665	7955	9245	10535	11825	13115	14405	15695	16985	18275	19565	20855	22145	23435	24725	26015	27305
37	1547	2873	4199	5525	6851	8177	9503	10829	12155	13481	14807	16133	17459	18785	20111	21437	22763	24089	25415	26741	28067
38	1589	2951	4313	5675	7037	8399	9761	11123	12485	13847	15209	16571	17933	19295	20657	22019	23381	24743	26105	27467	28829
39	1631	3029	4427	5825	7223	8621	10019	11417	12815	14213	15611	17009	18407	19805	21203	22601	23999	25397	26795	28193	29591
40	1673	3107	4541	5975	7409	8843	10277	11711	13145	14579	16013	17447	18881	20315	21749	23183	24617	26051	27485	28919	30353
41	1715	3185	4655	6125	7595	9065	10535	12005	13475	14945	16415	17885	19355	20825	22295	23765	25235	26705	28175	29645	31115

Figure 4: Composite numbers with the form $(6m-1)(6n+1) = (6p-1)(6q+1)$, the pink numbers are those which are repeated

The next theorem shows the structure of the composites with $(6m-1)(6n+1) = (6p-1)(6q+1)$.

Theorem 4.4. *(Danilo Chávez, January 22, 2022)*
Let be m, n, p, q, r, t, k *integers. Let be* $1 \leq m, 1 \leq n, 1 \leq p, 1 \leq q, 0 \leq r, 1 \leq t, 1 \leq k$.
When

$$(6m-1)(6n+1) = (6p-1)(6q+1)$$

We have two cases:
CASE A:
If

$$p = 5r + 1$$

with $0 \leq r$, *then*

$$q = 1, 2, 3, 4...$$

or if

$$q = 5r - 1$$

with $1 \leq r$*, then*

$$p = 1, 2, 3, 4...$$

CASE B:
If

$$p = 7r - 1$$

with $1 \leq r$*, then*

$$q = 1, 2, 3, 4...$$

or if

$$q = 7r + 1$$

with $0 \leq r$*, then*

$$p = 1, 2, 3, 4...$$

Proof. Let bem, n, p, q, r, t, k integers. Let be $1 \leq m, 1 \leq n, 1 \leq p, 1 \leq q, 0 \leq r, 1 \leq t, 1 \leq k$.
CASE A:
We take the line $m = 1$ and we have

$$5(6n + 1) = (6p - 1)(6q + 1)$$

or $(6p - 1) = 5t$ or $(6q + 1) = 5t$.
CASE A1
Lets take $(6p - 1) = 5t$ and we have

$$p = \frac{5t + 1}{6}$$

We are looking for the values where p is integer and we have that t is of the form $6r + 1$ with

$r = 0, 1, 2, 3...$

$$p = \frac{5(6r+1)+1}{6} = 5r+1$$

p takes the values $p = 1, 6, 11, 16...$
Now we are looking for the values of q given p. We start with the equation

$$5(6n+1) = (6p-1)(6q+1)$$

We can see that

$$q = \frac{5n-p+1}{6p-1}$$

As we are looking for the integer values given p, we suppose that for $k = 1, 2, 3, 4...$

$$5n-p+1 = k(6p-1)$$

giving

$$5n = 6kp-k+p-1$$

Substituting we have

$$q = \frac{5n-p+1}{6p-1} = \frac{6kp-k+p-1-p+1}{6p-1} = k\left(\frac{6p-1}{6p-1}\right) = k$$

that shows that q takes all the values $k = 1, 2, 3, 4...$, for any value of p, remember that $p = 5r+1$.
We conclude that when

$$p = 5r+1$$

with $0 \leq r$

$$q = 1, 2, 3, 4...$$

CASE A2
Lets take $(6q+1) = 5t$ and we have

$$q = \frac{5t-1}{6}$$

We are looking for the values where q is integer and we have that t is of the form $6r-1$ with $r = 1, 2, 3, 4...$

$$q = \frac{5(6r-1)-1}{6} = 5r-1$$

q takes the values $q = 4, 9, 14, 19...$
Now we are looking for the values of p given q. We start with the equation

$$5(6n + 1) = (6p - 1)(6q + 1)$$

We can see that

$$p = \frac{5n + q + 1}{6q + 1}$$

As we are looking for the integer values given q, we suppose that for $k = 1, 2, 3, 4...$

$$5n + q + 1 = k(6q + 1)$$

giving

$$5n = 6kq + k - q - 1$$

Substituting we have

$$p = \frac{5n + q + 1}{6q + 1} = \frac{6kq + k - q - 1 + q + 1}{6q + 1} = k\left(\frac{6q + 1}{6q + 1}\right) = k$$

that shows that p takes all the values $k = 1, 2, 3, 4...$, for any value of q, remember that $q = 5r - 1$.
We conclude that when

$$q = 5r - 1$$

with $1 \leq r$

$$p = 1, 2, 3, 4...$$

CASE B:
We take the line $n = 1$ and we have

$$7(6m - 1) = (6p - 1)(6q + 1)$$

or $(6p - 1) = 7t$ or $(6q + 1) = 7t$.
CASE B1
Lets take $(6p - 1) = 7t$ and we have

$$p = \frac{7t + 1}{6}$$

We are looking for the values where p is integer and we have that t is of the form $6r - 1$ with

$r = 1, 2, 3, 4...$

$$p = \frac{7(6r-1)+1}{6} = 7r - 1$$

p takes the values $p = 6, 13, 20, 27...$
Now we are looking for the values of q given p. We start with the equation

$$7(6m-1) = (6p-1)(6q+1)$$

We can see that

$$q = \frac{7m-p-1}{6p-1}$$

As we are looking for the integer values given p, we suppose that for $k = 1, 2, 3, 4...$

$$7m - p - 1 = k(6p-1)$$

giving

$$7m = 6kp - k + p + 1$$

Substituting we have

$$q = \frac{7m-p-1}{6p-1} = \frac{6kp-k+p+1-p-1}{6p-1} = k\left(\frac{6p-1}{6p-1}\right) = k$$

that shows that q takes all the values $k = 1, 2, 3, 4...$, for any value of p, remember that $p = 7r - 1$.
We conclude that when

$$p = 7r - 1$$

with $1 \leq r$

$$q = 1, 2, 3, 4...$$

CASE B2
Lets take $(6q+1) = 7t$ and we have

$$q = \frac{7t-1}{6}$$

We are looking for the values where q is integer and we have that t is of the form $6r+1$ with $r = 0, 1, 2, 3...$

$$q = \frac{7(6r+1)-1}{6} = 7r + 1$$

q takes the values $q = 1, 8, 15, 22...$
Now we are looking for the values of p given q. We start with the equation

$$7(6m - 1) = (6p - 1)(6q + 1)$$

We can see that

$$p = \frac{7m + q - 1}{6q + 1}$$

As we are looking for the integer values given q, we suppose that for $k = 1, 2, 3, 4...$

$$7m + q - 1 = k(6q + 1)$$

giving

$$7m = 6kq + k - q + 1$$

Substituting we have

$$p = \frac{7m + q - 1}{6q + 1} = \frac{6kq + k - q + 1 + q - 1}{6q + 1} = k\left(\frac{6q + 1}{6q + 1}\right) = k$$

that shows that p takes all the values $k = 1, 2, 3, 4...$, for any value of q, remember that $q = 7r + 1$. We conclude that when

$$q = 7r + 1$$

with $0 \leq r$

$$p = 1, 2, 3, 4...$$

Quod erat demonstrandum (Q.E.D).

$\square$

To find the total quantity of composite numbers with the form $(6m - 1)(6n + 1)$, we calculate the total composite numbers with that form lesser or equal to x. We will calculate ROW BY ROW, beginning from $m = 2$. We subtract the numbers with the form $(6m-1)(6(5t-1)+1)$. We subtract the numbers with the form $(6m - 1)(6(7t + 1) + 1)$. We add the intersections between the numbers with the forms $(6m-1)(6(5t-1)+1)$ AND $(6m-1)(6(7t+1)+1)$, we always avoid the rows with $m \equiv 1 mod(5)$ and $m \equiv -1 mod(7)$. We subtract every composite numbers lesser or equal to x with $m \equiv 1 mod(5)$ OR $m \equiv -1 mod(7)$. We add every composite numbers lesser or equal to x that are in the intersection where $m \equiv 1 mod(5)$ AND $m \equiv -1 mod(7)$. Finally we subtract the composite numbers in the column $n = 1$ that are in the row $m = 1$ (it has repetition) .

E.0) To find the total quantity of composite numbers with the form $(6m-1)(6n+1)$ lesser or equal to x.

If $(6m-1)(6n+1) \leq x$ then

$$n \leq \frac{x-(6m-1)}{6(6m-1)}$$

Because we want integers we define

$$C_{30}(x,m) = \left\lfloor \frac{x-(6m-1)}{6(6m-1)} \right\rfloor$$

Thus, the total quantity of composite numbers with the form $(6m-1)(6n+1)$, lesser or equal to x is

$$\sum_{\substack{m \geq 1 \\ C_{30}(x,m)>0}} C_{30}(x,m)$$

E.1) To find the total quantity of composite numbers with the form $(6m-1)(6(5t-1)+1)$ lesser or equal to x in the columns. Adding from $m=2$ and avoiding $m \equiv 1 mod(5)$ AND $m \equiv -1 mod(7)$.

$$(6m-1)(6(5t-1)+1) = 5(6m-1)(6t-1)$$

If $5(6m-1)(6t-1) \leq x$ then

$$t \leq \frac{x+5(6m-1)}{30(6m-1)}$$

Because we want integers we define

$$C_{31}(x,m) = \left\lfloor \frac{x+5(6m-1)}{30(6m-1)} \right\rfloor$$

Thus, the total quantity of composite numbers with the form $(6m-1)(6(5t-1)+1)$, lesser or equal to x, adding from $m=2$ and avoiding $m \equiv 1 mod(5)$ AND $m \equiv -1 mod(7)$ is

$$\sum_{\substack{m\geq 2\\ m\neq 1mod(5)\\ m\neq -1mod(7)\\ C_{31}(x,m)>0}} C_{31}(x,m)$$

E.2) To find the total quantity of composite numbers with the form $(6m-1)(6(7t+1)+1)$ lesser or equal to x in the columns. Adding from $m=2$ and avoiding $m \equiv 1mod(5)$ AND $m \equiv -1mod(7)$.

$$(6m-1)(6(7t+1)+1) = 7(6m-1)(6t+1)$$

If $7(6m-1)(6t+1) \leq x$ then

$$t \leq \frac{x-7(6m-1)}{42(6m-1)}$$

Because we want integers we define

$$C_{32}(x,m) = \left\lfloor \frac{x-7(6m-1)}{42(6m-1)} \right\rfloor$$

Thus, the total quantity of composite numbers with the form $(6m-1)(6(7t+1)+1)$, lesser or equal to x, adding from $m=2$ and avoiding $m \equiv 1mod(5)$ AND $m \equiv -1mod(7)$ is

$$\sum_{\substack{m\geq 2\\ m\neq 1mod(5)\\ m\neq -1mod(7)\\ C_{32}(x,m)>0}} C_{32}(x,m)$$

E.3) To find the total quantity of composite numbers with the form $(6m-1)(6(35t-6)+1)$, that are the intersections between the numbers with form $(6m-1)(6(5t-1)+1)$ and the form $(6m-1)(6(7t+1)+1)$, lesser or equal to x in the columns. Adding from $m=2$ and avoiding $m \equiv 1mod(5)$ AND $m \equiv -1mod(7)$.

$$(6m-1)(6(35t-6)+1) = 35(6m-1)(6t-1)$$

If $35(6m-1)(6t-1) \leq x$ then

$$t \leq \frac{x + 35(6m-1)}{210(6m-1)}$$

Because we want integers we define

$$C_{33}(x,m) = \left\lfloor \frac{x + 35(6m-1)}{210(6m-1)} \right\rfloor$$

Thus, the total quantity of composite numbers with the form $(6m-1)(6(35t-6)+1)$, lesser or equal to x, adding from $m = 2$ and avoiding $m \equiv 1mod(5)$ AND $m \equiv -1mod(7)$ is

$$\sum_{\substack{m \geq 2 \\ m \neq 1mod(5) \\ m \neq -1mod(7) \\ C_{33}(x,m) > 0}} C_{33}(x,m)$$

E.4) To find the total quantity of composite numbers with the form $(6m-1)(6n+1)$ lesser or equal to x, in the rows where $m \equiv 1mod(5)$ and not in the column $n = 1$, we have

If $(6m-1)(6n+1) \leq x$ then

$$n \leq \frac{x - (6m-1)}{6(6m-1)}$$

Because we want integers we define

$$C_{34}(x,m) = \left\lfloor \frac{x - (6m-1)}{6(6m-1)} - 1 \right\rfloor$$

Thus, the total quantity of composite numbers with the form $(6m-1)(6n+1)$, lesser or equal to x, in the rows where $m \equiv 1mod(5)$ and not in the column $n = 1$, is

$$\sum_{\substack{m \geq 2 \\ m \equiv 1mod(5) \\ C_{34}(x,m) > 0}} C_{34}(x,m)$$

E.5) To find the total quantity of composite numbers with the form $(6m-1)(6n+1)$ lesser or equal to x, in the rows where $m \equiv -1 mod(7)$ and not in the column $n = 1$, we have

If $(6m-1)(6n+1) \leq x$ then

$$n \leq \frac{x-(6m-1)}{6(6m-1)}$$

Because we want integers we define

$$C_{35}(x,m) = \left\lfloor \frac{x-(6m-1)}{6(6m-1)} - 1 \right\rfloor$$

Thus, the total quantity of composite numbers with the form $(6m-1)(6n+1)$, lesser or equal to x, in the rows where $m \equiv -1 mod(7)$ and not in the column $n = 1$, is

$$\sum_{\substack{m \geq 2 \\ m \equiv -1 mod(7) \\ C_{35}(x,m)>0}} C_{35}(x,m)$$

E.6) To find the total quantity of composite numbers with the form $(6m-1)(6n+1)$ lesser or equal to x, in the rows where $m \equiv 1 mod(5)$ AND $m \equiv -1 mod(7)$ and not in the column $n = 1$, we have

If $(6m-1)(6n+1) \leq x$ then

$$n \leq \frac{x-(6m-1)}{6(6m-1)}$$

Because we want integers we define

$$C_{36}(x,m) = \left\lfloor \frac{x-(6m-1)}{6(6m-1)} - 1 \right\rfloor$$

Thus, the total quantity of composite numbers with the form $(6m-1)(6n+1)$, lesser or equal to x, in the rows where $m \equiv 1 mod(5)$ AND $m \equiv -1 mod(7)$ and not in the column $n = 1$, is

$$\sum_{\substack{m\geq 2\\ m\equiv 1 mod(5)\\ m\equiv -1 mod(7)\\ C_{36}(x,m)>0}} C_{36}(x,m)$$

E.6) To find the total quantity of composite numbers which lies in the first row $m=1$ and in the columns $n=7s+1$, which has repetition in the rows with $m=5t+1$ and in the columns $n=1$, we have

$$7(6(5t+1)-1)=35(6t+1)$$

Because we want integers we define

$$C_{37}(s)=35(6s+1)$$

Thus, the total quantity of composite numbers which lies in the row $m=1$, which has repetition in the column $n=1$, lesser or equal to x is

$$\sum_{\substack{s\geq 1\\ C_{37}(s)<x}} 1$$

E.Final) The total quantity of composite numbers with the form $(6m-1)(6n+1)$, without repetition is

$$C_{(6m-1)(6n+1)}(x)=\sum_{\substack{m\geq 1\\ C_{30}(x,m)>0}} C_{30}(x,m)-\sum_{\substack{m\geq 2\\ m\neq 1 mod(5)\\ m\neq -1 mod(7)\\ C_{31}(x,m)>0}} C_{31}(x,m)-\sum_{\substack{m\geq 2\\ m\neq 1 mod(5)\\ m\neq -1 mod(7)\\ C_{32}(x,m)>0}} C_{32}(x,m)$$
$$+\sum_{\substack{m\geq 2\\ m\neq 1 mod(5)\\ m\neq -1 mod(7)\\ C_{33}(x,m)>0}} C_{33}(x,m)-\sum_{\substack{m\geq 2\\ m\equiv 1 mod(5)\\ C_{34}(x,m)>0}} C_{34}(x,m)-\sum_{\substack{m\geq 2\\ m\equiv -1 mod(7)\\ C_{35}(x,m)>0}} C_{35}(x,m)$$
$$+\sum_{\substack{m\geq 2\\ m\equiv 1 mod(5)\\ m\equiv -1 mod(7)\\ C_{36}(x,m)>0}} C_{36}(x,m)-\sum_{\substack{s\geq 1\\ C_{37}(s)<x}} 1$$

F) QUANTITY OF PRIME NUMBERS WITHOUT THE FORM $6k+1$ or $6k-1$

The prime numbers without the form $6k+1$ or $6k-1$ under x are the numbers 2 and 3, two of them, so we need to add a constant at the end to complete the task.

Now we define

$$C_2 = 2$$

G) THE FUNCTION $\pi(x)$

Finally we have all the elements to formulate the function $\pi(x)$ if $25 \leq x \leq 846$.

$$\pi(x) = C_{6k+1}(x)+C_{6k-1}(x)-C_{(6m+1)(6n+1)}(x)-C_{(6m-1)(6n-1)}(x)+C_{common}(x)-C_{(6m-1)(6n+1)}(x)+C_2$$

5 Python Code to Test the Formula

With the next python code you can test the formula of $\pi(x)$. You need to download python from their website https://www.python.org/ and then install it. Download the pi_N_846.py file in this address

https://www.mediafire.com/file/m7qaxi5vge3dc8s/pi_N_846.py/file

After installing the software you can run the file, just double clicking on it.

References

[1] Aurelio Baldor. *Aritmética. Teórico Práctica.* Pag. 201. ISBN:84-357-0079-8, 1985-1986.