Relativity in Function Spaces

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Abstract

After proposing the Principle of Minimum Gravitational Potential, in a pursuit to find the explanation behind the correction to Newton's gravitational potential that accounts for Mercury's orbit, by finding all the higher-order corrections it is shown that the consequences of the existence of speed of light for gravity are not yet fully explored.

Keywords— function space, minimum gravitational potential, Mercury's orbit, energy-mass function space, functional analysis

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1 Analogies

1.1 Mass–Entropy

In an earlier work[1] I used the analogy

$$S \equiv m, \quad T \equiv \phi \tag{1}$$

where m is mass and ϕ is the gravitational potential, to develop a complete theory of heat in which T becomes a fundamental field propagating at the speed of light in vacuum, sourced by entropy. In this communication we look at the other side of the analogy to apply the methods of statistical mechanics to Newtonian gravity. Recall that in statistical mechanics entropy is defined by

$$S = k_B \log W$$

and temperature via

$$T = \frac{dE}{dS}.$$

If the analogy (1) is a true harmony of nature, we must have the following expressions

$$m = m_P \log W \tag{2}$$

where m_P is the Planck mass, and

$$\boxed{\phi = \frac{dV}{dm}} \tag{3}$$

where V is the gravitational potential energy. Note that (3) is perfectly compatible with the classical definition, as it coincides with

$$\phi = \frac{V}{m}$$

for a linear[1] function $V = V(m)$. We will utilize this important equation later.

1.2 Temperature–Gravitational potential

We also showed in [1] that mathematical implementation of the Principle of Maximum Temperature is done via the theta factor

$$\theta = \frac{1}{\sqrt{1 - \frac{T}{T_P}}},$$

which is justified by yielding the expected theorem of equipartition of energy (and correcting it in the realm of quantum gravity). By the aforementioned analogy we propose the following

Principle. Minimum Gravitational Potential[2]
In flat spacetime, Newtonian gravitational potential ϕ has the absolute minimum $-c^2/4$.

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[1] But not affine.

[2] This principle is in fact a *sub-principle* of the *Ontological Finiteness Principle*, which will be stated and justified in another paper to follow from the *Epistemological* Finiteness Principle.

Mathematical implementation of the Principle of Maximum Gravitational Potential leads us to introduce the Φ factor

$$\boxed{\Phi = \frac{1}{\sqrt{1+\frac{4\phi}{c^2}}}} \tag{4}$$

The pesky numerical 2 factor difference between general relativity and Newtonian gravitational (of ϕ/c^2) continues to trouble us, as usual: the coefficient 4 is chosen so that the resulting correction to the Newtonian gravitational potential be compatible with the new term arising from the famous correction[2]

$$V = -\frac{GMm}{r}\left(1+\frac{3GM}{c^2 r}\right), \tag{5}$$

which accounts for the precession of the perihelion of Mercury without any need to consider general relativity, but, to make connection with the metric of general relativity via

$$g_{00} = 1 + \frac{2\phi}{c^2},$$

we have to choose the numerical factor to be 2, viz.

$$\Phi = \frac{1}{\sqrt{1+\frac{2\phi}{c^2}}}. \tag{6}$$

A decisive answer cannot yet be given the question of *the true* numerical factor.

2 Corrections to Newtonian gravitational potential

As a consequence of the Φ factor, gravitational potential energy of a particle in a gravitational field ϕ is given by

$$\boxed{V = -\frac{mc^2}{2}\left(\frac{1}{\sqrt{1+\frac{4\phi}{c^2}}}-1\right)} \tag{7}$$

$$= m\phi - \frac{3}{c^2}m\phi^2 + \frac{10}{c^4}m\phi^3 + \mathcal{O}(\phi^4)$$

Expression (7) is well justified as it provides a *firm physical ground* for (5). To see this, in (7), let

$$\phi = -\frac{GM}{r},$$

thus

$$V = -\frac{mc^2}{2}\left(\frac{1}{\sqrt{1+\frac{4\phi}{c^2}}}-1\right) \approx -m\frac{GM}{r} - \frac{3}{c^2}m(\frac{GM}{r})^2 - \frac{10}{c^4}m(\frac{GM}{r})^3,$$

up to terms third order in ϕ.
So far there has been no *physical* explanation as to why this new term assumes this particular form (for example the coefficient 3); the new term in (5) is derived via mathematical comparison with the result of general relativity. I do not shy away from admitting that the number 4 in (4) is still not possible to be derived from anything other than comparison with the result of general relativity[3], but it is easy to see that the *form* of the Principle of Minimum Gravitational Potential is unaltered whether one is aware of general relativity or not: there is a unique fundamental constant of nature that has the same dimensions with the gravitational potential, therefore someone in 1906 can arrive at this principle up to a numerical factor.
Physically it is as if Mercury is not physically allowed locally (due to the *Principle of Minimum Gravitational Potential*) to reach a gravitational potential energy smaller than $-c^2/4$, and this acts as a 'cut-off' and explains the orbit without any need for general relativity[4].

2.1 Relativistic Lagrangian of a Particle

Obviously

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\dot{\mathbf{x}}^2}{c^2}} + \frac{mc^2}{2}\frac{1}{\sqrt{1+\frac{4}{c^2}\phi(\mathbf{x})}}. \tag{8}$$

This Lagrangian gives the *exact* orbit of planets, in particular Mercury's, to which general relativity's Rosette is but an approximation.

3 Energy-Mass *function space*

3.1 The arising metric and its consequences

Using (3), the theta factor (4) leads us to the following metric for *the energy-mass space*

$$\boxed{d\epsilon^2 = c^2 dm^2 + 4dE_g dm} \tag{9}$$

where $E_g = V$ is the gravitational potential energy. Accordingly

$$(\frac{2}{c}\frac{d\epsilon}{dm})^2 = 1 + \frac{4}{c^2}\frac{dE_g}{dm}$$

using (3)

$$\boxed{\frac{1}{c}\frac{d\epsilon}{dm} = \sqrt{1+\frac{4\phi}{c^2}}} \tag{10}$$

We must not forget the *transformative* nature and consequences of the principle of minimum gravitational potential. The principle is of the same spirit as the second principle of special relativity (constancy of velocity of

[3] Ideally, from a extreme rationalist point of view, the numerical factor too must be derivable from purely theoretical considerations. I have not however yet been able to do this.

[4] Of course it is plausible to suppose that general relativity must have somehow assimilated this principle; in a way which is still unknown to me

light for all inertial frames): *All inertial observers must agree on a certain minimum for the gravitational potential.* The theta factor (4) is in fact a transformation rule, *suggesting*

$$\tilde{\phi} = \frac{c^2}{\sqrt{1 + \frac{2\phi}{c^2}}}$$

or

$$\tilde{\phi} = \frac{c^2}{2} \frac{1}{\sqrt{1 + \frac{4\phi}{c^2}}},$$

but both of these transformation rules contradict the principle of minimum gravitational potential, as the transformed potential $\tilde{\phi}$ has a minimum different from the original potential ϕ. To resolve this, one must take

$$\tilde{\phi} = \frac{c^2}{4} \frac{1}{\sqrt{1 + \frac{4\phi}{c^2}}} \tag{11}$$

or

$$\tilde{\phi} = \frac{c^2}{2} \frac{1}{\sqrt{1 + \frac{2\phi}{c^2}}}.$$

This transformation rule (11) however, cannot be the result of a *coordinate* transformation (Lorentz transformations), for ϕ is a scalar function and must not change with Lorentz transformations, viz. we can write (11) as

$$\tilde{\phi} = \tilde{\phi}(\phi);$$

consequently

$$d\tilde{\phi} = \frac{d\tilde{\phi}}{d\phi} d\phi \Rightarrow d\tilde{\phi} \neq d\phi.$$

Therefore **the metric of energy-mass space is that of a *function space***, hence the title of this section. Accordingly the right way to look at (11) is to take it as a transformation in the *function space of gravitational potential energies* that leaves the inner product arising from the metric (9) invariant.

3.2 Coordinates

The natural coordinates that follow from (9) is

$$V^a = (cm, V), \quad a = 1, 2. \tag{12}$$

It can be shown that the transformations of this *two-vector* that leave the corrsponding inner product of (9) invariant are given by

$$\boxed{V'(m') = \frac{V(m) - \phi m}{\sqrt{1 + \frac{4\phi}{c^2}}}} \tag{13}$$

The full theory of these new transformations and the geometry of (9) will be explored in detail in a consequent paper.

References

[1] Alireza Jamali. On fundamentality of heat (preprint). *Preprints.org*, 2021.

[2] James D. Wells. When effective theories predict: the inevitability of mercury's anomalous perihelion precession (preprint). *arXiv:1106.1568 [physics.hist-ph]*, 2011.