Unconscious Foundations of Mathematics

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Abstract

Mathematics accompanies throughout history the development of critical thinking, of philosophy, of physics and of the wide majority of modern research and technologies: in short, mathematics accompanies civilization and progress, for humanity acquired critical ideas, developing a peculiar language. Yet, epistemology overlooks a fundamental aspect of mathematics, in spite of all the literature speculating about its nature and its functions: being a language, mathematics grounds itself into human mind and culture, therefor mathematics reflects the structure of human mind, rather than being an intrinsic property (or domain) of nature[1]. Psyche expresses its structure in whatever human activity: every human activity expresses the structure of human mind. That circular statement summarizes the radical assumption in the theory about *symmetrical nature of unconscious* by Matte Blanco (1975), which displaced psychology into the domain of physics (Rossi 2019-2020): every discourse I develop on some subject expresses some inner characteristic of myself and of the culture a live in, or it expresses the way my society and I frame the world, rather than expressing some characteristic intrinsic to that subject itself[2]. That way, the two topics from Freud (1899; 1923) reveal themselves to be seminal tools in order to understand mathematics and its intrinsic unconscious structure, because various processes of *Symmetrization* pertain unconscious as much as mathematics.

Keywords: mathematics, psychology, subconscious, unconscious, semiotics, symmetry, equations.

1 Heintz (2005) noted the importance of challenging our culture in speculating about that topic, ever oscillating between the cognitive determinations of mathematics, on one hand, and the objective truth of mathematics, on the other hand.

2 Classical Hellenic and Indian philosophers knew that principle long before psychologists discovered its groundings (e.g., Berne 1953; Rosenberg 1998).

Introduction

Lakoff/Nuñez (2000) framed mathematics into cognitive psychology and linguistics, mathematics arranging a complex system of signs and syntactic rules. Scholars revealed the impossibility to frame mathematics out of human brain, recognizing the patterns (viz. the metaphors) that mind follows in order to acquire and to develop mathematics. I think that another step might be taken, for unconscious reveals a surprising and disregarded similitude with mathematical structures. Therefor, this paper frames mathematics in the Freudian function of ***subconscious association***, as discussed by Matte Blanco (1975), recalling and referring to their seminal concepts:

- *Condensation* of pluralities (of qualities, classes, ideas, etc.) into one single item or sign.
- *Displacement*, substituting one item (or sign or idea) with another item (or sign or idea).
- *Transference*, applying some function of one item (or sign or idea) to another item (or sign or idea).
- *Projection* of some characteristic of one item (or sign or idea) onto another item (or sign or idea).
- *Absence of negation*, including a non-existing item (or sign or idea) into a specific set, thus stating that the non-existing item exists (in some way).
- *Generalization* of items (or signs or ideas), absorbing their differences into items (or signs or ideas) apt to reflect some (even insignificant) similarity.
- *Symmetrization* or *symmetry principle*, assimilating differences in some ideal (virtual) undifferentiated continuum (viz. the domain of unconscious).

Matte Blanco (1975) provided the key feature for that framework: unconscious processes deploy *Symmetric* data (viz. an undifferentiated continuum where every data represent every other data), while conscious processes manage *Asymmetric* data, spotting differences between items, selecting and picking parts of that continuum, and ordering data through space-time; but conscious mind also “transmits the *appearance* of symmetry”, in order to get in touch with unconscious or “to become conscious of subconscious processes”. That is what I am trying to do in this paper: representing unconscious mathematical associations mostly via semiotic equations, viz. via statements reduced in the mold of an intuitive symbolic language.

1. Semiotics

People trust in ideas expressed by mathematics because mathematics reflect the structure of unconscious. People believe mathematical results to be true (viz. people accept mathematical results) because mathematics sprout from unconscious[1]: one goes along with the other, for they share the same structure. Boole (1847) and Russell (1903) reversed that assumption when they discussed **symbolic logic** as an intrinsic property of mathematics: discussing logic via symbols, they assumed logic as an *a priori* domain essentially intrinsic to the world, with its proper rules to be found and discussed; but, that way, they missed a substantial matter, for every logic is a way of managing symbols through unconscious associations; given that symbols themselves sprout from intuitive associations of ideas. Carroll (1886) showed it off, representing statements (based on false or truth propositions) as objects positioned into *loci* of a virtual space (viz. a metaphorical container), defining statements on the basis of *Inside*, *Outside*, and *Over* cognitive experiences[2].

Arithmetical signs convey symbolic "embodied" meanings: the signs convey concepts translated[3] from actual (cognitive) experience to internal (mind) representations, in terms of a sensory code[4], which is the natural language of unconscious, based on symbolic (and open) associations of ideas. Therefore, in this chapter I present the essential process of symbolic signification (viz. *semiosis*), in order to share a basic framework (both social or inter-personal, and individual or intra-personal), then I discuss the semiotics of the essential arithmetical signs.

1.1. Signification

Delving into the properties of numbers and functions, treating them like real items, scholars[5] overlooked the psychological nature of mathematics and the profound rooting of mathematics into human mind. Indeed, Penrose (1994: 411-420) remarked that mind relies on mathematics, considering mathematics an (intrinsic) *a priori* domain of nature, the "Platonic world of mathematical forms", hence considering mathematics a true world itself, preceding the mind: "The natural numbers where there before there were human beings… and they will remain after all life has perished" (*id.*: 413)[6]. But that statement discounts the paradoxical limit of Platonic thinking itself, for **mathematics is a system of signs**[7] (Black 1993), it is a language, on the basis of linguistic

1 The noun *mathematics* came from the Greek adjective *μαθηματικός* ("inclined to learning"), representing a set of symbolic knowledge, built both on logic and intuition, apt to acquire "accurate reckoning for inquiring into things, and the knowledge of all things, mysteries… all secrets" (*Ahmes papyrus*, XVII-XVI cent. BC, copied from a previous papyrus, XX-XVII cent. BC).

2 Lakoff/Nuñez (2000) recurred widely to that idea, despite they did not mention Carroll (1886).

3 The verb *to translate* blends the Latin preposition *trans* ("over", "beyond", "on the other side") and the adjective *lātus* ("borne", "carried", past participle of the verb *fĕrre*), thus *translating* means "to *Displace* something from one place to another".

4 Bodenhamer/Hall (1999) summarized how mind codifies experiences via *internal representations* organized on the basis of sets of sensory information: visual, auditory, olfactory, gustatory, haptic.

5 E.g., Peano (1890), Whitehead/Russell (1910-1913), Dirac (1930) and Gödel (1931), among the most appreciated.

6 Chap. 2 of this paper highlights the reasons why thinking about mathematics as an *a priori* world is a matter of (cognitive) faith, as Lakoff/Nuñez (2000) pointed out.

7 That is the insight common to Boole (1847), Carroll (1886) and Russell (1903).

theories, from Saussure (1916) on: I think of mathematics as of a cultural construct. That is why mathematics faces incompleteness limits stated by Gödel (1931): human constructs face intrinsic limits because of the limited capacity of human mind to decode universal complexity, following the seminal statement of Kant (1781). And that is why even Whitehead/Russell (1910-1913: 91-97) had to base their work on "primitive ideas" (or "undefined") and on "primitive propositions" (or "undemonstrated"), on *a priori* statements, taken for granted and un-discussed or indisputable: they created a system of signs (viz. a language) in order to provide a complete and non-circular description of mathematics, but they had to explain their (artificial) language via another (natural) language (viz. English), in order to describe mathematical language itself[8]. The simple fact that mathematics requires a natural language to be discussed (and scholars discussed it in terms of natural languages until XV cent.[9]) demonstrates why I must face the structure of language when I face the structure of mathematics.

A discussion about mathematics as a system of signs implies discussing significance and semiotics (viz. *semiosis*) in their basics: a *Sign* (a sound, a graphic, a gesture, etc.) "stands for" something else (a thought, an idea, an experience) via a complex relation, resulting from both from cognitive (natural) functions and social (artificial) conventions. A sign connects ideas on the basis of cognitive experiences.

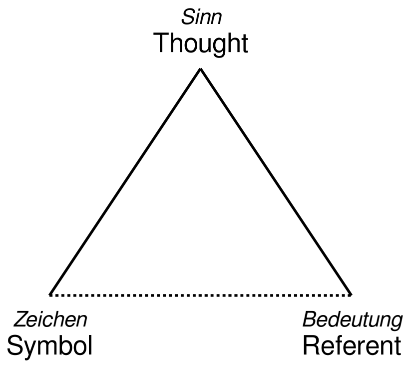

Figure 1: Semiotic Triangle

Frege (1892) and Ogden/Richard (1923) summarized the structure of significance via **semiotic triangles** (summed up in fig. 1): on one hand, a *Symbol* (viz. an auditory or visual or kinaesthetic element emerging spontaneously from social behavior or established in the domain of culture) represents and conveys some *Thought* (viz. a complex relation between ideas, cognition and experiences)[10]; on the other hand, a Thought relies on some *Referent* (viz. a cognitive experience or a set of cognitive experiences, occurring in nature or codified in culture); thus, on another hand, individuals infer the association between Symbols and Referents, which is a symbolic association or, in other terms, it is a subconscious association between the Symbol and its Referent, an indirect and unconscious process developed as a framing effect (viz. an intuitive associative process) in the mind of single individuals[11]. The (f)actual differences between Symbols and Referents collapse through subconscious Thoughts[12]: Symbols *generalize* the complexity of Referents, schematizing their structures, deleting some of their parts (focusing on other parts), and distorting them (e.g., stretching or shrinking their shapes)[13]. *Semiosis* operates through unconscious *Generalization* and *Symmetrization*, replacing (viz. *Displacing*) experiences (Referent-Thought)

8 The lexicographic paradox stated by Richard (1905) seems to really fit the case.

9 Cajori (1928) delved into that archaeology of mathematical language.

10 Saussure (1916) discussed the Symbol/Thought relation as a relation between *Signifiant* (French for "Signifier", the Symbol) and *Signifié* (French for "Signified", the Thought).

11 Individual experiences could lead to different interpretations of one same symbol, therefore culture procures standard criteria of significance (viz. meaning), apt to disambiguate social complexity.

12 Peirce (1907) discussed the Thought as an *Interpretant*: an entity interpreting the semiotic rapport between actual objects and the signifiers representing the objects.

13 Bodenhamer/Hall (1999: 135-157) discussed how those three cognitive processes operate: *Generalization*, *Deletion* and *Distortion*.

with items (Symbol-Though), thus establishing both a cognitive and an unconscious equivalence between different things (chap. 1.6 and 2.3).

Therefor, the expression “symbolic logic” stands for a set of rules associating multiple elements on the basis of **conscious *and* unconscious inferences**: Elias (1991) explained how that process of significance relies on the internal representations of experiences, culturally determined and shared by members of groups and societies; while Bodenhamer/Hall (1999) described how individuals generate and manage sensory internal representations (viz. the whole process depicted in fig. 1).

Just for the sake of speculation, I (venture to) define the semiotic triangle in mathematical terms: R defining the set of Referents, S defining the set of Symbols, T defining the set of Thoughts; **semiosis** (viz. a process of signification) operates whenever one element (t) from the set of Thoughts ($t \in T$) acts as a referential *image* ($r \in R$) of the symbolic function ($s \in S$); i.e. s is a *representation* of R on T. In other terms, a process of signification operates for:

[1] $$\{t = S(r)\} \leftrightarrow \{s = T(R)\}$$

E.g., $\mathbf{4}_{\text{Symbol}}$ (or $\mathbf{4}_{\text{S}}$) conveys 4_{Thought} (or 4_{T}), which in turn recalls what I experienced about 4_{Referent} (or 4_{R}), which I can infer directly from $\mathbf{4}_{\text{S}}$ on the basis of my personal conjunctive and intuitive associations of ideas, relying on my “internal representation of the world” or my “map[14]” or my “frame”. After all, $\mathbf{4}_{\text{S}}$ is just a conventional sign, for I can mean 4_{T} via different symbols (e.g., **IV** in Latin, 四 in Chinese, ४ in Devanagari); every conventional Symbol can convey 4_{T} (associated to some particular 4_{R})[15] on the basis of the underlying cultural codes and experiences.

Cajori (1928) proved that Western culture started adopting a system of mathematical signs around XV cent. (and stabilized the adoption from XVIII cent.): until that era scholars where used to discuss mathematics via their natural languages[16]. That meaning: the adoption of current mathematical language proves the development of a true semiotic process, for mathematics is an artificial language, established in culture by mutual agreements and conventions, throughout the last six centuries. Therefor, (in the terms of fig. 1) every mathematical Symbol conveys some meaningful Thought on the basis of some cognitive Referent, codified in culture and through education[17].

On one hand, language requires both auditory and visual semiosis, binding sounds and ideas as well as graphics and ideas, i.e. Symbols and Thoughts: every semiosis is a metaphor because something always stands for something else. On another hand, metaphors are signs revealing at least two meanings: a surface content (Symbol = Referent) and deep content (Symbol = Thought). Scholars applied those speculations on various topic, but I reckon they missed a few remarkable links that reveal a key property of language: cross-referencing Matte Blanco (1975) and Jaynes (1976), I think of

14 Korzybski (1933/1994: 750) stated that “the map is not the territory”: a seminal idea, meaning that mental representations of items *must* generalize, distort and erase information, in order for individuals to manage information. Bodenhamer/Hall (1999) delved into the dynamics of that topic.

15 That example clarifies the *linguistic nature* of mathematics, rather than its *a priori* essence, because every time I elicit some mathematical concept (e.g., “4_{T}”) I must employ *its* linguistic symbol ($\mathbf{4}_{\text{S}}$).

16 E.g., they would have written the expression “$x+y = 3$” like “three results from the sum of two different numbers”.

17 Education levels out the knowledge shared by society, with a risk of shared misunderstandings: e.g., Skemp (1987) pointed out the differences between *learning math* and *understanding mathematics*, showing how education could lead to mere notionism. On another hand, Polya (1945) showed the possibility to simplify hard problems, reducing them (or parts of them) to more simple problems, through the acquisition of an essential knowledge.

consciousness as a metaphorical representation of unconscious, apt to grasp subconscious processes (i.e. apt to develop introspection), and I can think of unconscious as a semiotic cognitive process apt to develop an internal “map of the world”[18], *generalizing* cognitive data; while cross-referencing Jaynes (1976) and Lakoff/Nuñez (2000), I can think of metaphors as of cognitive tools apt to acquire or to construct conceptual entities, *transferring* real items into symbolic items, mathematics being one product of that *Projection*; while cross-referencing Matte Blanco (1976) and Lakoff/Nuñez (2000), I can think of mathematics both as a result and a tool for introspection, for semiosis develops internal representations (apt to grasp and to construct inner properties of real entities) and semiosis transforms and *displaces* signs and meanings, (con)fusing one with another, revealing or constructing hidden properties of the mind.

Psychic, associative and symbolic processes are core abilities required for acquiring and processing data: Bodenhamer/Hall (1999) clarified that “acquiring” means codifying sensory-neural data, building in neural patterns; and “processing” means associating acquired data via neural patterns; Lakoff/Nuñez (2000) and Nuñez (2008) suggested how mathematical concepts and mathematical cognition rely on sensory-motor experiences, via specific motor activities (like counting numbers through fingers) embodying metaphorical representations; Bender/Beller (2012) demonstrated how “representational effects” (viz. systems of signs) affect cognition of numbers and processing of numbers (viz. operations on numbers). Indeed, functional magnetic resonance imaging scans locate brain regions associated with the ability to control fingers, and show the activation of that same regions in people performing numerical tasks: sensory-motor experience links specific bodily movements to “digital” activities, concerning both numbers and fingers. That remark identifies a peculiar class of symbols: ***the cognitive symbols*** that, in the perspective of the semiotic triangle (fig. 1), bind embodied sensory-motor activities (Symbols) with subconscious sensory-motor experience (Referent), both on one single endpoint, linked to some intellectual representation (Thought), on another endpoint (fig. 2). Lakoff/Nuñez (2000) accounted for cognitive symbols to be directly involved in mathematical thinking.

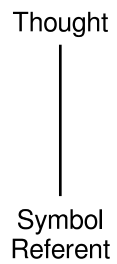

Figure 2: Cognitive Symbol

Mathematical language being a semiotic process (thus a cognitive process), the meaning of elementary mathematical signs derives from symbolic associations, involving Freudian topics: that process involves mainly the *Symmetry principle* (as discussed in chap. 2.3) and the *Projection* function (a way for conceptually associating inner representations of experiences). The arithmetical signs of the elementary operations (+, –, ×, ⁄, =)[19] reflect the semiotic process underlying a mutual convention: the adoption of a specific *signs*$_{\mathrm{S}}$ (in order to convey the meaning of a specific *operation*$_{\mathrm{T}}$) recalls some *experience*$_{\mathrm{R}}$ underlying the arithmetical process referred by the symbols. The arithmetical signification is a (now worldwide) *social experience*, based on **communal subconscious associations**. Indeed, different scholars opted for different signs, in order to represent same operations through ages[20], but scientific community reckoned (implicitly) the symbolic relevance of specific signs (thus their intrinsic meaning) when those signs were widely adopted as unique indices for specific arithmetical operations. That is a crucial point in understanding unconscious foundations of mathematics, because mathematical language is a universal language (a system of Symbols), common to all

18 Korzybski (1933/1994) and Bandler/Grinder (1975) should be included in the cross-references.

19 The noun *arithmetic* comes from the Greek noun *ἀριθμητική* (“number”, “the art of counting”): thus I refer to arithmetic (as a subset of mathematics) every time I treat numbers and operations.

20 Cajori (1928; 1929) compiled an accurate history of mathematical notations worldwide.

humanity, built, acquired and shared throughout history: Cajori (1928; 1929) recorded how scholars strove to arrange a universal grammar (settling signs) and a syntax (settling rules for combining the signs). Hence mathematics conveys experience (Referents) and meanings (Thoughts), useful to understand inner traits of human thinking, along with a few other communal universal systems of symbols, like (conventional) musical notation and (innate) expression of emotions.

Elementary arithmetical signs express different levels of *Symmetry*, for they convey symmetric processes via their shapes: from the absolute (both vertical and horizontal) *Symmetry* of +, × and ·, to the relative or partial *Symmetry* of –, : and ÷. And they convey the specular duality of that *Symmetry*: from the singular stroke signs (–, /, ·) to the dual signs (+, ×, :, ÷). Those classes expand the two polarities of the fundamental dyads[21]: Durand (1963), Matte Blanco (1975) and Rossi (2019-2020) suggested that human imaging and knowledge developed on the basis of elementary archetypes.

	Projection	Transference	Displacement	Confusion
Single	–	/	·	xy no sign
Dual	+	×	:	÷ 3 strokes

Table 1: Matrix of Operational Signs

The tab. 1 recapitulates the discussion explained in the next four chapters: how basic operations *project* their meanings into the shapes (+ and –), and how complex experiences *transfer* their meanings manipulating the basic signs (× and /), how different meanings can be *displaced* (how multiplication can divide and how division can multiply), and how the application of processes is visualized via absence and redundance.

Semiotics shows an underlying process of signification, developing mathematics: the unconscious *Symmetry principle* translates (viz. it *transfers*) the cognitive experiences into a virtual continuum or an internal representation, processing (numerical) data and (logical) information in order to abolish their differences; then the conscious intellect renders *Asymmetry* into that continuum, deploying differences to be discussed in their relation with symmetry domain, for ideas binding *Symmetry* and *Asymmetry* (thus unconscious associations) survived in the very shapes of arithmetical signs. That way, the unconscious foundations of mathematics reveal themselves.

1.2. Addition

The addition can be considered as *the* fundamental arithmetical operation, for it goes along with the concept of *natural numbers* (chap. 2.1.1), which is bound to economical (viz. social and actual) experiences[22]: the noun *arithmetic* comes from the Greek noun *ἀριθμός* (“number”) and from the verb *ἀριθμέιν* (“to count” and “to pay”). Arithmetic *generalizes* the ability to observe or to examine (viz. to acquire) items (chap. 2.1.1) and the social activity to balance (viz. to distribute) debts and credits (chap. 2.1.2),

21 *Symmetry*/*Asymmetry*, as well as order/confusion, continuity/interruption, composition/fragmentation, etc.

22 The noun *economics* blends the Greek nouns *οἶκος* (“house”, “assets”) and *νόμος* (“habit”, “regulation”, “use”), thus it stands for “assets management”. Graeber (2011) showed (how much and how) many ancient cultures worldwide accounted debts and credits in formal documents, revealing that economics went along with the foundation and development of societies, as a set of rules for coexistence and as a set of rapports of power; and it went along with numerical thinking, as a symbolic metaphor of experience, as a tool necessary for accounting credits and debts (chap. 2.1.2).

paying something in return for something else[23]. Scholars from different disciplines, from Mauss (1924) to Cialdini (1984), recorded how *reciprocity* regulates coexistence on the basis of social *Symmetry* between individuals (viz. economics). Therefor it appears quite obvious that Cajori (1928: 128-130) recorded the first appearance of the current additional sign (**+**) in Widmann (1489), an essay on mercantile arithmetic, which was essentially an essay about counting and managing values, debts and credits: there the sign **+** referred to a practice found in previous manuscripts (1486), where Latin conjunction *et* ("and") got simplified into **+**, for it evokes the letter *t*[24]; it suggested how the addition means a repetition or a series of items (viz. $x+y+z = x$ **et** y **et** z) in order to identify the *extension* or the *size* of a group.

That radical assumption sends the semiotics of **+** out of the arithmetical domain and into the linguistic domain; but, that being the case, the Latin cross (✝) or a junction sign (⊤) or a dagger (†, ☨) would recall the letter ***t*** in a more proper way, rather than **+**. Yet **+** conveys peculiar symbolic meanings, sending **+** into the domain of radical signs, a domain of unconscious associations: Kandinsky (1926) and Frutiger (1978) explained that a t-shaped junction sign (⊤) conveys the idea of constructing something, for an horizontal item (–) gets positioned atop of a vertical item (|), that conveying the idea of equilibrium, for the horizontal item keeps a balanced position (–), rather than tilting down like a slant would do (/). That remarks validate the idea of an artifact as a summation of elements. On another hand, asymmetric crossings (✝, †, ☨) recall the shape of human body, arms spread open, that also validate the idea of a body as a sum of parts (viz. organs and limbs). Yet **+** conveys the idea of absolute symmetry, on the basis of the **ideas of *similitude* and *union***: a focal point (in the center of the cross) *unites* two similar strokes (| and –), it *connects* four identical arms, it *joins* the vertical and the horizontal dimensions, two essential cognitive experiences that Durand (1963) discussed as the foundations of imaging and that Bodenhamer/Hall (1999) discussed as the foundation of mental representations of experiences. The sign **+** (rather than ⊤ or † or the others) conveys a *generalized* idea of *equal joining* or, in other terms, the idea of putting together similar items: | and –. Which is a concept directly bound to the idea of *natural numbers*, a concept "embodied" in our experience about acquiring, managing and trading items[25] (chap. 2.1.1): | and **–** represents the experience of my fingers (viz. digits) summed up in my hands (or rods in a bunch).

The *Generalization* process carries out a seminal function in acquiring the concept of addition, for I can sum up similar items only ***generalizing* different items** into an identical class or into a unique conceptual container: "$3_{\text{Apples}} = 1_{\text{Apple}}+1_{\text{Apple}}+1_{\text{Apple}}$" means a group of 3 different items (each apple is a different apple, with respect to the other two apples) of the same class (apples differ from stones, from rivers, etc.). Understanding the expression "3 = 1+1+1" means to *transfer* the group "**1+1+1**" (a complex Symbol-Referent) into the entity "|||" (a Thought) via the Symbol "**3**". That way, the *Generalization* of individuals into a uniform class (viz. ignoring or abolishing differences) allows me to operate additions, so that I can state even apparently absurd ideas like (e.g.) $\{3_{\text{Apples}}+2_{\text{Pens}}+4_{\text{Stones}} = 9_{\text{Items}}\} \leftrightarrow \{w+x+y = z\}$. Etymology confirms that process: in the late XIV cent. the word *item* meant "moreover", "in addition", for it acquired the Latin adverb *ĭtĕm* ("likewise", "moreover") and pronoun *īdem* ("the same").

23 Romans talked about "give and take" framework stating "*do ut des*" ("I give [you something] so that you give [me something else in return]").

24 That remark explains the typical employment of ampersand (*&* abbreviating *et*) in business and commercial activities.

25 The experience of *giving* (**–**) is *Symmetric* to the experience of *taking* (**+**): that introduces the *Symmetry* reflecting **+** into **–** and vice versa.

When I think of an item, I think of a generic *thing* (from Old Norse noun *þing*, "assembly", and from Old German *ding*, "public assembly", both recalling the idea of a group or a multitude of individuals, belonging to the same conceptual class, people): Latin noun *rēs* ("thing", "fact", "event") became the Italian *cosa*, later on acquired in German and England languages, for *Coss* and *cossic art* meant "algebra" in the XVI cent. (see chap. 2.1.1 on that topic).

Before **+** spread worldwide, Cajori (1928: 229-236) recorded letters or natural linguistic abbreviations referring to addition: mainly the letter **p̃** shortening Latin nouns *plūs* ("more", "expensive") and *plēnus* ("full", "filled", "complete", just like the Greek noun *πλέως*), all of them recurring from the Sanskrit adjective *puṛṇàs* ("full")[26] and looking like references to economics, for the actual social experience of trading and saving, by the means of filling empty containers, and completing the "collection" of debts and credits. Moreover, the Greek adjective *πολύς* ("copious", "numerous", "many") was referred directly to the cognitive concept of *many*, which in turn is necessary in order to acquire the idea of numbers (chap. 2.1.1). Therefore, shifting from the employment of letters to the adoption of **+** (along the last five centuries) meant to recognize the symbolic (viz. associative) value of the addition, thus it meant to recognize the *Condensation* process at work, as a way to join items together, to form a unique bunch, as well as the possibility to order items into a full stack (chap. 2.1.1 delves into that topic), gathering (viz. developing) some *Symmetry* out of asymmetric entities, for a plurality of disordered items gets packed into a compact stack or directly into containers which (usually) are symmetric-shaped, hiding the irregularity of their contents.

The shape of the sign **+** conveys the *Symmetry* of a container built out of its asymmetric contents: two similar items (viz. vertical **|** and horizontal **–**)[27] are bunched (viz. *condensed*) together into a new item (**+**), summing up the two, hence conveying its very meaning (chap. 1.3 explores that assumption).

The addition of items (viz. employing **+**) is a symbolic cognitive experience, translating Referents into Symbols: hence I can draw a series of signs representing my competence in counting items (chap. 2.1.1). But I have available a limited set of Symbols-Referents (fig. 2): $||||||||||_{\text{Fingers}} = ||_{\text{Hands}}$ (viz. $10_{\text{Fingers}} = 2_{\text{Hands}}$). Hands and fingers are powerful tools because of their *Symmetry*: my unconscious perceives each hand as a mirror image of the other (chap. 2.1.5), so that I can think of $p = q$, even if $\boldsymbol{p \neq q}$ (chap. 1.6), which is a complex **symbolic association** implying *Generalization* (for $p_{\text{Item}} = Item_q$), *Condensation* (for $(\boldsymbol{p}, \boldsymbol{q}) = Items$), *Displacement* (for $\{p = q\} \leftrightarrow \{q = p\}$), and *Transference* (for $p_{\textbf{Item}} = q_{\textbf{Item}}$). That way (stating $10_{\text{Fingers}} = 2_{\text{Hands}}$), I symbolize my hands via my fingers: $\{1_{\text{Hand}} = 1_{\text{Finger}}\} \leftrightarrow \{Finger_{\text{Symbol-Referent}} = Hand_{\text{Thought}}\}$.

I have available only 10 Symbols-Referents or digits (or any other quantity of items I take as a reference, e.g., arms, legs, etc.), in order to count items (viz. to order items; chap. 2.1.1): I need more fingers, in order to count over |||||||||| (10 items), but I learned from the cognitive-symbolic experience ($10_{\text{Fingers}} = 2_{\text{Hands}}$) that I can operate a recursive semiosis: that way, I can *project* my fingers out of myself, collecting rods (viz. images of my fingers), then I can intentionally improve that *Projection*, drawing images of the fingers (that way linking consciousness to unconscious). And, as long as I have available 10 fingers only, I need more images of my fingers or more images of myself, as long as I have to count over 10: the limit of |||||||||| (10) items closes the range of my Referents, but it opens the possibility of another instance, then another one, then an-

26 www.etimo.it/?term=piu.
27 Chap. 1.6 explains why equality sign (**=**) groups two identical (viz. *Symmetric*) items.

other one, etc., viz. it represents **a cycle**[28]. The "first" finger opens a cycle and the "last" finger closes that cycle. Semiosis allows me to represent that cycle:

[2] $$0 = \bigcirc$$

Every time[29] I close a counting cycle (0), I *project* my hands into my fingers, so that I can count (viz. I can order) the cycles, *projecting* the count into the digits (**D**):

[3] $$\{\mathbf{D0} = \mathbf{D} \times \mathbf{0}\} \leftrightarrow \{\mathbf{D} \times \mathbf{0} = \Sigma\mathbf{0}\}$$

The statement [3] refers to the multiplication because any arithmetical sign conveys its meaning along with the others (chap. 1.3). Nevertheless, it introduces a crucial idea: semiosis allows me to conceptualize great numbers as **a series of small additions** (Σ) because I can fill the cycle **0** putting any referential quantity in front of **0**, viz. any digit (1 ≤ **D** ≤ 9) can signal how many cycles (**0**) I counted passing through my hands. That is an essential tool for the multiplicative process (chap. 1.4).

```
       8
     +70
    +500
  +3,000
 +30,000
=33,578
```

1.3. Subtraction

The operation of subtraction introduced ideal or abstract arithmetical concepts, absent in the (f)actual experience of additive series: the experience of removing (viz. subtracting) some item from a collection (e.g., picking an apple from a tree, as well as withdrawing money from an account) identifies **a negative set**, intended as the collection of absent items (e.g., money goes into my wallet, as well as picked apples go into a basket or into my stomach); so that (negative) debts balance (positive) credits (chap. 2.1.2), in terms of economic and social experience[30]. On the other hand, the experience of addition implies the experience of subtraction and vice versa, for stacking items into a container requires to remove those same items from another (even ideal) container. That way, subtraction goes along with addition and vice versa, fort they represent opposite polarities of one same dyadic entity (viz. *Symmetry principle*).

Cajori (1928: 128-130) recorded the first appearance of the minus sign (**–**) in Widmann (1489), along with **+** (chap. 1.2) and its economic characterizations. Yet scholars seem to know nothing certain of its origin, speculating about a simple bar used by merchants to point out the tare, called *minus*, from merchandise; or speculating about the Greek *ὀβελός*, a small horizontal line put in the place of spurious or unknown literary references (Cajori 1904: 232). Anyhow, the linguistic sign **–** marked an **absence of something**. Indeed, before the adoption of **–** spread worldwide, Cajori (1928: 229-236) recorded natural linguistic abbreviations referring to subtraction: mainly **m̃**, shortening the Latin adverb *mĭnus* ("less", "not") and the adjective *mĭnūtus* ("small", "futile"), from the Greek adjective *μεῖων* ("smaller than"); the latter implying a comparative relation between opposite polarities, like **–** and **+** (chap. 2.1.3 delves into that topic).

Dyadic relations (viz. **+** implying **–** and vice versa, or hotter implying colder, or *Ego* implying *Alter*, etc.) impose recursive or reflexive resolutions (chap. 2.1.5 delves into that topic): unconscious semiosis operates **a recursive process**, for symbols (on one hand, linguistic signs; on the other hand, internal representations) refer to other

28 Humanity developed cultures along with the interpretation of cycles: time and music (Rossi 2010/2018) represent the clearest examples.

29 Time represents the cyclic nature of arithmetical thinking: e.g., "$p \times q$" can be read as "*p* times *q*".

30 Graeber (2011) explained how (much) that way of thinking regulated most of the ancient societies.

symbols through an infinite series of references[31]. That idea explains the meaning of the minus sign (–) in terms of strict semiotics (referring to the concepts depicted in fig. 1 and 2), expressing the idea via the following symbolic formulation:

[4] $$- = + - \mathsf{I}$$

The definition of the minus sign (–) relies on the plus sign (+), *minus* (–) the vertical symbol (I): that is a *recursive symbolic process*, for $-_S$ conveys its meaning ($-_T$) through itself ($-_{SR}$) or by its dyadic relation with $+_S$. One must acquire $+_S$ in order to acquire – as the *absence* (–) of I_S. Given the same reason, the proposition [4] implies the following: + = – + I, meaning $+_{SR}$ conveys $+_T$ *adding* (+) I_S to $-_S$ (that proposition integrates chap. 1.2).

The proposition [4] shows how arithmetical signs convey visual clues or *evidences* about their meaning because they are part of a language. I *see* the elementary arithmetical signs conveying their meanings as long as I *see* the meaning of the arithmetical signs in their shapes. Each arithmetical sign incorporates itself into a **system of symbolic rapports**: subconscious association process (viz. semiosis) operates or *signifies* putting all the arithmetical signs on a same level, arranging them via *Symmetric* rapports, for every sign conveys its meaning (along) with the others.

1.4. Multiplication

The reiteration of additions (e.g., {3×4 = 4+4+4} ↔ {3+3+3+3 = 3×4}) represents just one meaning of the multiplication: in actual experience, a multiplication ($p \times q = r$) conveys the general idea of applying (×) some process (p) to some item (q) or, in other terms, it means to process ($p\times$ or "via" p) an item (q) in order **to transform** it (=) into another item (r). The repetition of additions (viz. serializing additions) represents just one possible process ($p\times$): the multiplicand and the multiplier (p and q) being numbers, they represent things or items (see chap. 2.1.1), while the operation (×) represents a relation (between numbers) conveyed by actual experiences. The algebraic expression $y = f(x)$ (viz. "y is a *function* of x") evidences why multiplication (implied between f and x) means to apply some process (viz. a function f) to something (x) (see chap. 2.2): the noun *function* (introduced by Leibniz 1692) derives from the Latin verb *fungi* ("to execute", "to perform"), meaning that a variable (x) goes under a performance (viz. it executes some transformation) resulting into y. That process (f) can be any elementary operation (+, –, ×, ∕) or any chain of operations, but the *application* of the function is given by the implied multiplication only.

Etymology confirms that insight: the noun *multiplication* blends the Latin adverb *multa* ("many", "many times", "a lot", "often") and the verb *plīcāre* ("to fold", "to wrap", "to curl"), thus it means a "repeated folding" or a "repeated curling". The adverb *multa* refers to a serialization, a sequence, thus it refers to the process of addition; while the verb *plīcāre* conveys an experience implying the manipulation of *one* object (e.g., I take a fabric and I fold it, growing it in size, or I take a sheet of paper and I fold it into an *origami*), rather than a summation of multiple objects (e.g., piling up sheets of paper, building a ream). Deleuze (1988) explained the conceptual relevance of the fold, for it transforms a straight line into a space: hence, the "fold" of a number *trans-*

31 The paradox identified by Richard (1905) can be extended to semiosis: every sign employed to denote another sign is a sign itself that, in turn, has to be defined employing other signs; every sign standing for other signs. Otherwise, language couldn't define a sign: but subconscious intuition can.

fers it from the one-dimensional idea or series to the two-dimensional idea of complexity (chap. 2.1.5).

The manipulation (*plīcāre*) of many objects (*multa*) stands for the possibility to correlate or to combine or to *condensate* different things into another item. In the same way, the multiplication *condenses* different processes into one symbol in three different ways, then **the intellect reads**[32] the appropriate meaning (Thought-Referent axis in fig. 1) out of the Symbol: (1) repeating an additive process, in order to compute the size of a series, speeding up seriation tasks (viz. the ability to arrange items[33]), e.g., $\{2\times3 = 2+2+2\} \leftrightarrow \{3+3 = 2\times3\}$; (2) comparing a same quantity between different classes, in order to compute the amount of resources needed to accomplish a task, e.g., $\{2_{Pens}\times3_{Kids} = (1_{Pen}+1_{Pen})\times(1_{Kid}+1_{Kid}+1_{Kid})\} \leftrightarrow \{2_{Pens}\times3_{Kids} = (1_{Pen}\times1_{Kid})+(1_{Pen}\times1_{Kid})+(1_{Pen}\times1_{Kid})\}$; (3) computing combinations of different items belonging to different classes, in order to compute the number of possible outcomes in problem solving, e.g., $(1^{Red}+1^{Blue})\times(1_{Red}+1_{Blue}+1_{Green}) = (1^{Red}+1_{Red})+(1^{Red}+1_{Blue})+(1^{Red}+1_{Green})+(1^{Blue}+1_{Red})+(1^{Blue}+1_{Blue})+(1^{Blue}+1_{Green})$[34].

Those different ways to manipulate or to combine items really mean the idea of a *Condensation*: I can (con)fuse one item into another (or a class into another class, or quantities into qualities), if I insert one item into the other or if I intersect or I *interpolate*[35] them. Again, my fingers (depicted like | and – rods or strokes) represent and visualize that experience: e.g., $\{3\times3 = 9\} \leftrightarrow$ {≡×||| = ⋕}[36]. The interpolation returns the product as **the number of intersections** between the rods: the count (viz. the sum) of the intersections in the grid returns the product (e.g., ⋕ = 9). The symbolization of that experience (viz. substituting fingers with strokes and visualizing them like items arranged in a recursive semiosis, as discussed in chap. 1.2) allows me to multiply greater numbers via interpolation, separating multiples (via rule [3]), counting intersections, and juxtaposing the sums (fig. 3 represents **0** like a wide null space between the rods). But the cognitive symbol of interpolation does not work when dealing with great numbers (e.g., intersecting my fingers I cannot multiply 27×49): yet, semiosis does.

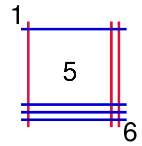

Figure 3: ***12***×***13***

The process of interpolation establishes (viz. it symbolizes) a correlation between two items (left and right hands, or left and right fingers, or blue and red strokes in fig. 3) that I can *see* and check through my fingers: my (opposable) thumb and my index finger identify the horizontal and vertical axes of the area represented within the interpolation. That two finger-axes represents Symbols-Referents-numbers[37] as well as symbolic numerals identify 10 digits (viz. 10 fingers). That way, visualizing a (virtual) area within my thumbs and my index fingers, I can identify two referential axes (viz. fin-

32 The noun *intellect* comes from the Latin verb *intellīgēre* ("to understand" "to notice", "to discern"), blending the adverb *intŭs* ("into") and the verb *lēgēre* ("to grasp", "to read"). Hence (from here on) I will write *intellect* meaning conscious processes (like stating a sentence), rather than subconscious activities (like subitizing two items).

33 Inhelder/Piaget (1964) explained *seriation* as the ability to group similar items (e.g., triangles) into different classes (e.g., red, blue and green), thus I can group (e.g.) $\{12_{Triangles} = 4_{Triangles}\times3_{Colors}\} \leftrightarrow \{12_{Triangles} = 2_{Red}+5_{Blue}+5_{Green}\}$.

34 In that example, *Up* and *Down* categories has been visualized as ^{Up} and $_{Down}$.

35 The verb *to interpolate* came from Latin verb *interpōlāre* ("to repair"), blending the adverb *intĕr* ("amid") and the verb *pōlīre* ("to smooth", "to complete"), hence "operating in order to define".

36 Unconscious associates fingers with elementary items like rods (sticks, branches, snakes, etc.) and vice versa, for *Symmetry* principle *Generalizes* qualities of items, *Deleting* their differences.

37 That could be the source of the intuition in Oresme (1350) and named after Descartes (1637: 297-314) as the *Cartesian coordinate system*.

gers) as the two signifiers of the objects that I want to interpolate; hence I can develop a semiosis in order to correlate (viz. to interpolate) digits, rather than fingers[38]:

	1,000	**400**	**30**	**2**	
50	50,000	20,000	1,500	100	71,600
7	7,000	2,800	210	14	10,024
					81,624

Table 2: A Multiplicative Process (1,432×57 = 81,624)

The example in tab. 2 illustrates both a conceptual representation of numerical interpolation and a practical tool, helpful in multiplying numbers: translating *huge numbers* into sums via rule [3] (e.g., 1,432 = 2+30+400+1,000 in the top headings of tab. 2 and 57 = 7+50 in the left headings), I multiply *small digits* (e.g., 5×1 rather than 50×1,000) via cognitive interpolation (⋕), then I sum up the results (other small digits, like 5+2, rather than 50,000+20,000), filling the voids (viz. summing digits into zeros via rule [3]).

1	**2**	**3**	**4**	**5**
2	4	**6**	**8**	**10**
3	6	9	**12**	**15**
4	8	12	16	**20**
5	10	15	20	**25**

Table 3: Multiplication Table

The multiplication table (or *times table*) studied in schools derives from an interpolation that visualizes a recursive *Symmetry principle*, for the tab. 3 depicts a whole area built out of two mirror images (◪ = ◺+◥, recurring in the proposition [12]): I need to memorize only one half of the table, in order to memorize the elementary products that I could see through the interpolation of my fingers (⋕) and that I can employ for multiplying huge numbers via a symbolic interpolation (tab. 2).

That way, the sign **×** evidences its meaning (i.e. *interpolation process*), for the Symbol-Referent **×** signifies its Thought (×) via recursive semiosis, viz. it represents the application of a process that produces the sign **×** itself:

[5] $$\times = | \times -$$

The (recursive) definition of the multiplication sign (**×**) relies on elementary items, like fingers or rods (viz. **|** and **–** shaping **+**), undergoing a process of interpolation (×): the very shape of the sign **×** represents a *diagonal intersection*, thus a complex[39] way of employing **+** or a complex way of counting numbers.

Yet I must provide an evidence for the obvious visual difference between **+** and **×**.

Cajori (1928: 190) recorded the first appearance of **×** (the *St. Andrew's Cross* employed as a multiplication operator) in Oughtred (1631), even if Cajori (1928: 251-260) recorded previous scholars employing **×** in multiplying numerators and denominators between couples of fractions, in order to solve proportions[40]: a fact that validates the insight of an *interpolation* visualized in the shape of **×**. While Cajori (1928: 267) recorded the first appearance of the *middle dot* or *dot operator* (·) in Harriot (1631) and in Leibniz (1698), then in scholars quoting Leibniz on their turn, that way spreading worldwide the reference. Before those evidences, Cajori (1928: 250) found the capital letter **M** (for German *Multiplikation*) in Stifel (1545), while the Bakhshali manuscript (III-X cent.) juxtaposed multiplicands and multipliers, like in the current ex-

38 Chap. 2.1.1 speculates about the Latin noun *dĭgĭtus*.
39 Chap. 2.1.5 delves into that topic.
40 E.g., { $\frac{3}{2} \times \frac{n}{4}$ } ↔ {3×4 = 2×n}.

pression *xy* or *x*(*y*); then Fibonacci (1202) introduced that same juxtaposition, also known as *implied multiplication*.

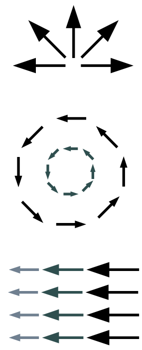

Figure 4: Divergence, Curl, Gradient

Scholars, from Heaviside (1891) on, bound the *nabla* operator (∇)[41] to those three instances of the multiplication sign, in order to represent three different meanings (viz. processes) of the multiplication concept in higher mathematics and physics: the dot operator (·) stands for the ***divergence*** (viz. diffusion) of a vector field along two or three dimensions or, in other terms, it stands for the spatial orientation of a flux with respect to some source or to some destination point in space (fig. 4 top); while × stands for the ***curl*** (viz. circulation) of a vector field rotating around some point in space (fig. 4 middle); while the implied multiplication (*xy*) stands for the ***gradient*** (viz. density or intensity) of a vector field increasing or decreasing through space (fig. 4 bottom).

Heaviside (1891) grasped the semiosis underlying the cross multiplication sign (×), when he employed it as a **rotational operator**: × rotates +, it *curls* (viz. it multi-plicates) a number (recalling the meaning of the Latin verb *plīcāre*), like the imaginary unit (*i*) of complex numbers manipulates entities (chap. 2.1.5). Putting it in semiotics (recalling the proposition [4]):

[6] $$\times = \circlearrowleft_{45°} \times +$$

The multiplication sign (×) results as a 45° rotation ($\circlearrowleft_{45°}$) of the sign +[42]: that conveying the idea of the multiplication as a transformation of the addition, rather than a mere repetition (that idea introduces the semiotics of division too: chap. 1.5). Hence $\times_S$ conveys $\circlearrowleft_T$ on the basis of $\#_R$: a process of interpolation or intersection transforms a concept (viz. a number) into another concept (viz. a greater number), via the cognitive properties of addition, just like the **rotation of an item** (viz. rotating a frame) makes me see a different *image* of that same item (viz. it reframes an idea, changing the visual perspective).

That last remark sends the function [6] out of the domain of semiotics and into the domain of arithmetic, viz. it *projects* the symbolic properties of × into the syntactic rules of mathematics. Indeed, the possibility to apply a symbolic process to + allows me to apply an arithmetic process (e.g., 2×) to more numbers (e.g., 3 and 4), hence I can calculate 2×3 = 6 *and* 2×4 = 8 (viz. I can interpolate || and |||, as well as || and ||||): that “and” means that I can sum up (2×3)+(2×4) = 6+8. The referential interpolation (viz. playing with rods or strokes) evidences that {2×(3+4) = 14} ↔ {14 = (2×3)+(2×4)}: semiosis builds a cognitive experience that identifies the **distributive property**[43] of multiplication (a fundamental rule of mathematics):

[7] $$x(a+b) = xa+xb$$

That law represents a crucial experience because its right side (*xa*+*xb*) multiplies the *x* on the left side by the items gathered inside the brackets on the left side: that is another recursive semiotic process, for *x*(*a*+*b*+*c*) = *xa*+*xb*+*xc*, etc., evidencing (viz. vi-

41 Hamilton (1853: 610) introduced ∇ as a differential operator decomposing a vector in space.

42 Similar rotations (135°, 225°, 315°) convey the same visual effect on +, just like 90°, 180°, 270° rotations leave + unaltered, because of its *Symmetric* properties (chap. 1.2).

43 The verb *to distribute* comes from Latin verb *distrībūēre* (“to distribute”, “to divide”), blending the prefix *dis* (for separation or disjuncture or a negation) and *trībūēre* (“to assign”, “to appreciate”): the perfect infinite, employed as an adjective, *trĭbūtus* (“tribute”, “tax”), has a functional bound with the noun *trĭbŭs* (“tribe”, “community”): another social reference for mathematical etymology.

sualizing) the meaning of the multiplication in its very conceptual property, for **it multiplies the presence** of an item inside an expression, repeating its occurrence. That semiotic process gains its meaning on the basis of the seriation ability: I can compute a multiplication (viz. I can manipulate signs) just because I can identify similarities and differences in items, I can *generalize* them, and I can arrange groups of items on the basis of some property (Inhelder/Piaget 1964). That way, the multiplication *condenses* different qualities or classes. E.g., I understand the statement "3×5 = 5+5+5" because I focus on items, in order to catch differences (**3** ≠ **5** and **×** ≠ **+**), then I *displace* the meaning of those differences and I replace that meaning with some *Symbolic association* fitting the similarities: $\{(\mathbf{5}, \mathbf{5}, \mathbf{5})_{\text{on the right side}} = \mathbf{3}_{\text{identical items}}\}$ and $\{\mathbf{\times 5}_{\text{multiplicand}} = \mathbf{5}_{\text{addend}}\}$ mean the reiterated addition of 3 instances of **5**, or 5 instances of **3**.

That way, the multiplication introduces the possibility to (con)fuse or to compose categories, extending that characteristic property of the addition: if, on one hand, I can sum up only similar items or items taken from the same category (e.g., $2_{\text{Apples}}+6_{\text{Apples}} = 8_{\text{Apples}}$ or $2_{\text{Apples}}+6_{\text{Kids}} = 8_{\text{Items}}$), on the other hand, the multiplication allows me to combine items taken from different categories into a specific category (e.g., $2_{\text{Pens}}\times 6_{\text{Kids}} = 12_{\text{Pens}}$ or $2_{\text{Meters}}\times 6_{\text{Meters}} = 12_{\text{Square meters}}$). The result (viz. the product) *condenses* quantities (e.g., 2 and 6) of two or more different categories (e.g., pens and kids) into one single category (e.g., pens) or it transforms one single entity (e.g., meters) into another entity (e.g., square meters).

That last remark highlights a crucial consideration: generic multiplication *operands* commute ($p\times q = q\times p$ because both sides of the equation give the same *numerical* result), but specific multiplication ***categories*** **do not commute** (e.g., $12_{\text{Pens}} \neq 12_{\text{Kids}}$). The arithmetical operation "2×6 = 12" conveys a correct (abstract) solution ($2_{\text{Digit}}\times 6_{\text{Digit}} = 12_{\text{Digit}}$), but the logical statement "$2_{\text{Pens}}\times 6_{\text{Kids}} = 12_{?}$" lacks of information (about the category assigned to 12): the meaning of that solution depends on the experiences (viz. Thoughts-Referents in fig. 1) of who operates the multiplication. A multiplication framed in reality constraints (viz. referring to specific objects or categories) changes the meaning of the operation itself, for I get $2_{\text{Pens}}\times 6_{\text{Kids}} = 12_{\text{Pens}}$, having Pens as a target (e.g., I feel it is easier to pop up pens out of the blue, rather then popping out kids); but, having Kids as a target, I could get 3_{Kids} after processing 2_{Pens} for 6_{Kids}: the result of a division ($3_{\text{Kids}} = 6_{\text{Kids}}\div 2_{\text{Pens}}$) fits better than the result of the multiplication, on the basis of my experiences with the real possibilities of increasing the number of kids and pens in a specific time-frame or given small numbers. In other terms, writing "$6_{\text{Kids}}\times 2_{\text{Pens}} = 12_{?}$" makes me think of the (obvious) result of 12_{Pens}. That meaning that multiplication *projects* my actual experience in the abstract process of transformation (viz. I conceive the multiplication as a symbolic process, for the multiplication complicates, it intertwines or it entangles categories and sets): that is why $p\times q$ reiterates p times q (e.g., 2×3 = 3+3) or it reiterates q times p (e.g., 2×3 =2+2+2), for it *projects* p into q and vice versa, interpolating the items.

That is a reason why $p\times 0 = 0$: cognitive null experience (×0) nullifies phenomena, as Berkeley (1710) suggested. Moreover, $\{p\times 0 = p\times(q-q)\} \leftrightarrow \{(p\times q)-(q\times p) = 0\}$, as it will be discussed via the propositions [8] and [9] and in chap. 3.1: null or void experience (×0) collapses in arithmetic via the *Absence of negation* principle, for zero can be computed as a *difference* (hence, 0 is a conceptual construct *and* it is a real number, rather than being no-thing): I can translate $p\times 0 = p\times(q-q)$ to the expression $1\times 0 = 1\times(1-1)$. And the distributive property (law [7]) reframes that idea:

[8] $$\{1\times(1-1) = (1\times 1)+(-1\times 1)\} \leftrightarrow \{1+(-1\times 1) = 0\}$$

A general property of equations, based on the *Symmetry principle* (see chap. 2.3), reframes the proposition [8]:

[9] {1+(–1×1) = 0} ↔ {–1×1 = –1}

That means: {**–** × **–** = **+**}. And, in the same way, processing the proposition [8] via its *Symmetric* formulation (viz. multiplying items in brackets by –1, rather than by 1), the proposition [9] *projects* its *Symmetric* image:

[10] {–1×(1–1) = (–1×1)+(–1×–1)} ↔ {–1×–1 = 1}

Then the propositions [9] and [10] can be *generalized* as one of the fundamental rules of arithmetic:

[11] {**+** × **+** = **+**} ↔ {**+** × **–** = **–**} ↔ {**–** × **–** = **+**}

Here lies the *Symmetric* property of arithmetic: natural numbers (viz. positive numbers, **+**) derive from unconscious *Symmetry*, i.e. **+×+** and **–×–** (chap. 2.1.1), while negative numbers derive from intellectual *Asymmetry*, i.e. **+×–** (chap. 2.1.2).

Getting back to fig. 3 and to tab. 2: the multiplicative interpolation process introduces a peculiar dimension in the visualization (viz. the metaphors) of arithmetic, for it *transfers* the one-dimensional domain of integers, obtained by addition (chap. 2.1.1), into a two-dimensional domain, returned by multiplication. The interpolation (#) identifies both a perimeter (□ as the outer rods intersected in the process) and **an area** (■ as a *Gestalt Condensation* resulting from the intersection points ▦ returned by the interpolation), a grid or a lattice; while the addition and the subtraction identify just a line (chap. 2.1.2). The area identified in the visualizations of interpolations evidences the very meaning of the square, or the elementary exponentiation process: squaring an item (viz. a number) means multiplying (viz. interpolating) it by itself ($n^2 = n \times n$), hence it means one same number means both an item (n^2) and a process ($n \times n$). Therefor, **every square results from 2 halves**[44]:

[12] {◪ = ◸+◢} ↔ {⬔ = ◺+◥}

Condensing the two sides of statement [12], every square results from 4 quarters:

[13] {⊠ = ◀+▷+▲+▽} ↔ {⊠ = ◆+⬘}

Every square (□) results from the summation of 2 smaller squares (◆ = ◀+▷, and ⬘ = ▲+▽). **The Pythagorean Theorem** ($r^2 = p^2+q^2$) results from semiosis and it delves into areas (more precisely, into multiplications), rather than into triangles: the theorem validates even avoiding triangles (◧ = ▮+▯)[45] and, indeed, scholars apply it to *every* square (viz. to solve every quadratic equation), even if (traditionally) it should fit right-angled triangles only.

Exponentiation represents a chain (viz. a series) of multiplications: the exponent provides the total amount of the links of the chain (e.g., $n^3 = n \times n \times n$), just like the multiplication provides a chain of additions (e.g., $3 \times n = n+n+n$), for both **+** and **×** operate a recursive semiosis. Exponentiation ($\boldsymbol{n^m}$) *transfers* the cardinal function of numerals (viz. exponents m) to the ordinal function of numbers (viz. how many links build the chain $n \times n \times$…):

44 That remark introduces the division discussed in chap. 1.5.
45 That version validates, substituting the concept of *square* with the concept of *area*.

[14] $$n^r = \boldsymbol{r} \times \{\boldsymbol{n}\boldsymbol{\times}\}$$

And recursive semiosis indeed explains why $n^0 = 1$ (rather than $n^0 = 0$) and why multiplication differs from addition ($0 \times n = 0$ while $n^0 = 1$): the *Absence of negation* principle states that $\{0 = n - n\} \leftrightarrow \{n^0 = n^{n-n}\}$, an operation which will be discussed in chap. 1.5 (for negative exponentiation conveys the opposite of a chain of multiplications, thus it conveys a chain of divisions).

Leibniz (1698) preferred to represent the multiplication via the dot operator (·), in order to avoid any confusion between the operation **×** and the variable $\boldsymbol{x}$. As well as **×** conveys the multiplication as an interpolation, · conveys a compressed (viz. *condensed* and *generalized*) aspect of the multiplication, which is an aspect very close to the juxtaposition of multiplicand and multiplier:

[15] · = (**+** × ⇠) + (**+** × ⇡) + (**+** × ⇢) + (**+** × ⇣)

Heaviside (1891), again, grasped the inner meaning of the semiosis of dot operator, for · conveys the idea of a transformation, visualizing a force *diverging* from some source (fig. 4 top) into a recipient: the middle dot collapses the branches of the cross on their intersection point (for ⇠, ⇡, ⇢ and ⇣ are applied to **+**), that way retrieving again the meaning of multi-plication as a folding process that compresses items. Leibniz (1698) opted specifically for a *middle* dot (·), rather than an acclimated (thus a more comfortable and a more basic) typographical sign like an *Asymmetric* full stop (.)[46]: he aspired to *Symmetry*. That idea replicates the *Symmetry principle*, as the absence of the branches of the cross stands for what made them vanish: the **principle of conservation** (which is a variation of the *Symmetry principle* that imprinted human imagery, from ancient China[47] to Greece[48] to modern Europe[49])[50] operates so that the lack of something (viz. the lack of the branches converging into the dot) signals the process operated in order to remove (viz. to fold) that something.

Moreover, the *Condensation* of **+** into · *transfers* the addition into the domain of subtraction (viz. from the "dual" sign **+** to the "monistic" **–**, as it is showed in tab. 1), that way introducing another *Symmetric* idea (that will be discussed in chap. 1.5): a multiplication can decrease values (in addition to increasing them), as well as a division can increase items (in addition to decreasing them). That is another effect of the *fold*, revealing duple faces of ideas conceptualized through signs and language, as Deleuze (1988) pointed out.

The multiplication (as well as the equality; chap. 1.6) reveals how duplicity (embodied in human neural system) manages mathematics, for every chain of processes should be operated in couples: two items must be processed before the result is processed by another item, and so on. E.g., the calculation "3+4×3+4" can be reframed as "3+(4×3)+4 = 19" (prioritizing the multiplication), or it can be reframed as "(3+4)×(3+4) = 49" (prioritizing the additions), or it can be reframed as "[(3+4)×3]+4 = 25" (following the positional order of operations): anyway, I must decide the computational order to impose to *couples* of items, and every possible order of calculation displays couples of items. That dual nature of arithmetic rules every equation and every function (chap. 1.6 and 2.3).

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46 Leibniz (1684) opted for the acclimated colon (**:**), in order to represent divisions in a simpler and more elegant way (chap. 1.5).

47 E.g., Taoist text *Zhuāngzǐ* (IV-III cent. BC).

48 E.g., Empedocles (IV cent. BC).

49 E.g., Lavoisier (1774).

50 Rossi (2019-2020) delved into the relevance of that topic in epistemology.

1.5. Division

Just like the multiplication signs (chap. 1.4), the division signs *condense* multiple ideas: reiterating subtractions, fair sharing resources, breaking something apart, comparing scales of quantities and, more generally, applying an inverse process with regard to the multiplication. Thus, the most notable idea conveyed by division is that of *Asymmetry* or the inverse of the *Symmetry principle*, that generally regulates mathematics.

The idea of **reiterating subtractions** elicits a peculiar trait of the division as an inverse process with regard to the multiplication:

[16] $$\{s \div r = t\} \leftrightarrow \{s-(t \times r) = 0\}$$

Calculating a division means *to solve a problem* about multiplication[51], viz. to find how many times (t) the divisor (r) must be subtracted to the dividend (s) in order to get 0; while the multiplication simply means *to execute* a certain number of times an addition. E.g., the operation "8×2 = 16" provides me with the instructions for summing up 2 instances of **8**, or 8 instances of **2** (chap. 1.4), whereas the operation "8÷2 = 4" implies my conceptual acquisition of the multiplication process, that I should apply (as a problem to be solved) to the process of subtraction, reiterating that subtraction until I find the number (viz. factor) of reiterations that answers the question. Hence I can think of "8÷2 = 4" as a problem to be solved via multiplication, {**8**÷2 = 4} ↔ {4×2 = **8**} ↔ {**8**–(4×2) = 0}, or a problem to be solved via addition, **8**÷2 = {**–2–2–2–2**} ↔ {4+4 = **8**}. Hence, the division implies an intellectual interpretation of arithmetic expressions and an *intellectual semiosis* (while other operations imply unconscious semiosis and intellectual interpretation of expressions): I must count (viz. sum up) the number of operands (***t***) involved in a series of subtractions, *displacing* the numeral and the cardinal property of addition (viz. I must count the operands ordered in the series of subtractions), *displacing* addition and subtraction (viz. counting subtractions implies summing up operands, and the quantity of the subtracted operands returns the quotient, so that $t \times r = s$), and *projecting* the Symbol (e.g., **2**) into the Referents (e.g., 4+4).

On another hand, the ideas of the division as a fair way to **share resources** and as a way to break something apart come from economics, as an intrinsic property of society. The economic ideas of *welfare state*, *equal opportunities* and *equitable distribution of wealth* exemplify how current policymakers strive to regulate well known experiences from ancient societies: Staid (2015) recorded how the leaders of the "societies without a State" strove to eliminate inequalities in order to put their people in a position apt to refuse or to ignore their instinct of dominance. And Rossi (2009; 2019) showed how even primitive blades made humans able to materially divide resources (cutting and slicing them via thin edges) and to symbolically gather societies up (threatening and intimidating deviant individuals): the criteria of sharing, via slicing, and the enforcement of rules, via violence, established laws and order. Then organized societies divided space, via geometry, and divided time, via calendars (Zerzan 1988).

The process of sharing material resources elicits an inner property of division: having n items and n individuals, a group can assign one item to each individual ($^n/_n = 1$), that way establishing **a bijective relation** between different categories (viz. people and items, or more generally X and Y), which has a crucial role in developing algebra (chap. 1.6 and 2.3) because that relation develops the possibility to differentiate enti-

51 Campbell (1999). Moreover, the multiplication "implied in the division" could be the reason why readers tend to recall the Gospels' episode about Jesus *dividing* loaves and fishes (Matthew 14: 13-21; Mark 6: 30-44; Luke 9: 10-17; John 6: 1-14) as the "*multiplication* of loaves and fishes".

ties, to analyze their components and to think of infinitesimal calculus, whereas multiplication only develops series. On the contrary, economics had a relevant impact on arithmetic because univocal relations or greedy ratios ($^n/_1 = n$) represent the very instinct of dominance that societies strive to abolish. Moreover, *Asymmetric* sharing processes reveal problems to be solved via economics and policies: on one hand, $\frac{n}{p>n} < 1$, e.g., $\frac{4}{8} = 0.5$ (viz. too many people, p, can get very little amounts of resources) and, on the other hand, $\frac{n}{p<n} > 1$, e.g., $\frac{8}{4} = 2$ (viz. surplus, $2 > 1$, must be shared by a few people). Both ratios reveal how insufficient and abundant resources present social problems, to be solved via rational solutions, viz. via criteria based on divisions.

The latter ratio can be extended to a more general limit: $\frac{n}{0<p<1} > n$ (e.g., $\frac{5}{0.2} = 25$) means that dividing something (n) by a very little something ($0 < p < 1$, given $p = \frac{s<r}{r>s}$) returns an entity bigger than the dividend (n), so that a division returns the same result of a multiplication, restating the conclusion of the proposition [16]. That way, a multiplication can operate or compute a division and vice versa: firstly, $\frac{n}{0<p<1} = \frac{10}{10} \times \frac{n}{0<p<1}$ (e.g., $\{ \frac{3}{0.6} = \frac{10}{10} \times \frac{3}{0.6} \} \leftrightarrow \{ \frac{30}{6} = 5 \}$; and, conversely, $n \times \{0 < p < 1\} < n$ (e.g., $\{5 \times 0.6 = 5 \times \frac{6}{10} \} \leftrightarrow \{ \frac{30}{10} = 3 \}$).

That is a *real* introduction to macroeconomics: if, on one hand, I can addition and subtract numbers in order to account debts and credits and in order to manage money and power, on the other hand I need some rate ($\frac{n}{p>n}$) in order to calculate taxation and in order to exert power, imposing taxes[52]. Many societies indeed (from Babylonian to Hebrew, to Catholic Church until XVIII cent., among many others) computed and named their tributes as *tithes* ($^1/_{10} = 0.1$).

That *dual* aspect of the dividing tools (viz. the blade separating and grouping at once, just like the division reducing and growing entities) recurs in the essence of the arithmetical operation of division, which always **breaks something in 2** parts: unconscious *condenses* the two parts like reflecting segments of a *Symmetric* continuum or like specular elements of a unified whole[53], while the conscious intellect ponders the parts like different items, different sizes, different places, etc., or different quantities or numbers, each one to be managed in its proper way. E.g., breaking $1 \div 3 = t$ requires at least 2 "cuts": the first cut returning 2 parts, and the second cut returning the required 3 parts; but I must exert an *Asymmetric* first cut (—|———), in order to get a final fair sharing (—|—|—), rather than exert a fair first cut (——|——), that will get me an unfair final sharing (——|–|–). That intellectual operation (viz. an *Asymmetric* process eliciting differences in similar items) returns the very meaning of the division.

Comparing scales of quantities, **establishing ratios**, operating with fractions: all of those processes imply visualizing numbers like containers[54]. The division ($^s/_r = t$) implies a comparison between containers (s and r) processing their contents in order to filter out an entity (t): the division operates a disjunctive process (reverting an interpolation), whereas the multiplication operates a conjunctive process (chap. 1.4). The noun *proportion* blends Latin preposition *pro* ("in front of") and noun *portĭo* ("portion", "part"), meaning "given a portion": $\{ \frac{s}{r} = t \} \leftrightarrow \{ \frac{s}{r} = \frac{t}{1} \}$ means that the dividend

52 Graeber (2011) recorded how societies accounted debts and credits in the origins of civilization and how taxation meant the exercise of power and dominance.

53 Matte Blanco (1975) and Bohm (1962; 1977) seemed to agree on that point.

54 Lakoff/Nuñez (2000) insisted on the metaphor of containers and contents employed in order to develop the concept of number: unfortunately they did not parse Jaynes (1976) on consciousness as a spatial metaphor.

(*s*) gets divided by the divisor (*r*) until it results as a "part" (*t*) "in front of" 1 individual (viz. per capita or *pro capite*). Portions or parts can be compared via scales, that way *condensing* different items (*s* and *r*) into one individual entity or ratio (e.g., $\frac{8}{4}=\frac{4}{2}$ means that the ratio on the left side of the equation can be managed as a singular item, with respect to the ratio on the right side; both ratios being items themselves), thus they can be operated (e.g., $\{ \frac{8}{4} \div \frac{4}{2} = \frac{8}{4} \times \frac{2}{4} \} \leftrightarrow \{ \frac{8}{4} \div \frac{4}{2} = \frac{4}{8} \times \frac{4}{2} \}$ means that singular items allow me to invert a division by a multiplication, that way the *Asymmetry* renders the appearance of *Symmetry*, for operands of multiplication commute, whereas operands of division do not) and they can be decomposed (e.g., $\frac{8}{7} = \frac{7}{7} + \frac{1}{7} = 1 + \frac{1}{7}$ means that singular scales can be reduced to singular items or actually "1", just like every natural number can be reduced to a series of additions of single items, like 3 = 1+1+1).

Those ideas (reiteration of subtractions, sharing of resources, breaking of items and ratios) show *Asymmetry* as an intrinsic property of the division: **the operands do not commute** ($s \div r \neq r \div s$)[55]; e.g., the expression "8÷2 = 4" formulates a different problem rather than "2÷8 = 0.25", for the two expressions employ different numerals under different circumstances. The order of the operand changes the result of the division (viz. the quotient) because it determines the function of the operands (i.e. *dividend÷divisor* and $^{numerator}/_{denominator}$). That property reveals itself in the very shapes of division signs.

The symbolic relevance of *Asymmetry* with regard to the division showed up after conventional signs spread over the world (**:**, ÷, /). Cajori (1928: 250) recorded the capital letter **D** (for German *Division*) in Stifel (1545) and in other previous scholars, showing how semiosis operated on a phonetic level: scholars *projected* the qualities of the operation into its lexical abbreviation (the same applies to addition, subtraction and multiplication), until Cajori (1928: 269-273) recorded a widespread adoption of the fractional line, writing numerators above and denominators below the line ($\frac{s}{r}$). Rahn (1659) suggested the adoption of ÷, the two small dots representing generic numerators (*s*) and denominators (*r*), while Leibniz (1684) preferred to employ the colon (**:**), getting rid of the redundant fractional line between the dots; then Senillosa (1818) introduced the slant (/), explicitly assuming it as an analogy with ×.

Semiotics of those signs represent the *Asymmetry* as the peculiar trait of the division, for every sign has been developed through time along with its own genesis (whereas the multiplication signs share common ideas):

[17] $\{/ = \times - \backslash\} \leftrightarrow \{/ = \circlearrowleft_{45^\circ} -\}$

[18] $\{\div = \frac{s}{r}\} \leftrightarrow \{\div = (\uparrow s \times -) + - + (\downarrow r \times -)\}$

[19] $\{\cdot < :\} \leftrightarrow \{: > \cdot\}$

Senillosa (1818) introduced the sign / as an analogy with ×: his intuition relies on the absence of one slant in the sign × and on a 45° rotation of the sign – (prop. [17]), both of that remarks relying on the already discussed link between **multiplication and subtraction**, for I can think of the division as a series of subtractions; the rotation (viz. a continuous motion) signifies the series, just like the multiplication does with re-

55 While $s \times r = r \times s$, and $s+r = r+s$. And subtractions commute on a symbolic level: e.g., $\{8-5 = 3\} \leftrightarrow \{-3 = 5-8\}$ for the same identical numerals *reflect* the positive and the negative poles of the operands (chap. 2.1.2 and 2.1.5).

gard to the addition (chap. 1.4), and the very sign of the slant (viz. a minus) signifies the subtractions. Moreover, **/** is the only asymmetric sign of elementary operations, suggesting imbalance as a reference to the weighing rod of scales: the sign **/** conveys the one and only operation resulting asymmetric values or non-integer numbers or pure ratios (e.g., $7 \div 2 = 3+1$ and $^7/_2 = 3.5$), thus they are called *rational numbers* (chap. 2.1.3). It should seems quite obvious that fractions (viz. asymmetric operations signifying *Asymmetric* rapports between numbers) defined the study of higher mathematics and the rational thinking or the "way of thinking ratios" as inner traits of the Age of Reason: asymmetric thinking, as an intellectual effort, made scholars discover and deploy the symmetric rules underlying logical dipoles and even (through ages) described the structure of true matter[56]. The division is a conscious process, compared to the other three operations that are based on unconscious representations: dividing or sharing resources is an intentional and *rational* (thus a strategic and political) behavior, rather than an instinctive identification of items or a semiosis resulting from cognitive experiences, like the other elementary operations show.

Semiotics of **/** reveal how the concept of rotation, recurring in division and multiplication (prop. [6]), can be linked to numbers themselves: 1 is the quotient of the slope for a 45° inclination of a line, represented by **/**. E.g., $\{x = y\} \leftrightarrow \{1(x) = 1(y)\}$: that meaning a straight line 45° sloped passing through the intersection of the two axes in a plane (given a Cartesian coordinate system). That way, the number 1 represents the idea of a rotation (chap. 2.1.5), that implies a force (i.e. a process) operating on an item, in order to transform it or to manipulate it. The overwhelming interest of scholars throughout history for the rotation and for the revolution of planets and particles (viz. studying their angular momentum and the curl of the vector fields) could derive from the circularity bringing along that symbolic struggle: that could be a reason why the operational signs for multiplication and division acclimated in the mathematical lexicon after the development of trigonometry and infinitesimal calculus, from XVIII cent. on (Cajori 1929: 336-340).

The proposition [18] (validating even for **:**, thus without the sign **–**) recurred in Hindus (III-X cent.), who "simply wrote the divisor beneath the dividend" (Cajori 1928: 269), and in Arabs (XII cent.), who wrote divisions like $\frac{s}{r}$, influencing Fibonacci (1202), who introduced that representation in Europe: everywhere the division established a relation between **metaphorical containers**[57] that are visualized as an *upper source* (s) and a *lower recipient* (r), for quantities contained in the upper s flow (viz. get distributed) into the lower r (via natural or real gravity) and get filtered by the **–**, returning the result $\{t = s \div r\} \leftrightarrow \{\frac{s}{r} = t\}$. Durand (1963) explained how the relation between upper and lower polarities developed human imagery, so that universal polarities (like good/bad, hot/cold, male/female, etc.) always need to be mediated by a third factor (the **–** within **÷**, with regard to the division), that is the human unconscious semiotic ability (chap. 1.1). That way, the horizontal slant (**–**) introduced *Asymmetry* in arithmetic, introducing *oddity* via that third Symbol (**–**) in the very sign of division, whereas the other elementary signs imply only even (e.g., dual and quaternary) polarities. And the process of *filtering* really seems to validate the case stated with proposition [18], for the division is the only elementary operation resulting in a quotient and maybe in a *remainder*, which is exactly "what remains" or "what is left" after filtering some substance.

56 Rossi (2019-2020).
57 Lakoff/Nuñez (2000).

Leibniz (1684) swiftly commented the ease of employing **:** in the place of other signs of division, while his contribution framed a curious interest in duality[58]: the left side member of proposition [19] shows how the sign **:** *projects* or splits (**<**) one item (·) into two items (**:**), *transferring* the properties of · (i.e. the ability to establish relations between objects) into a semi-symmetric sign, for **:** reflects · along the vertical dimension only (while, e.g., ∵ would display complete symmetry); on the other hand, the right side member of proposition [19] shows how the sign · *condenses* or joins or interpolates (**>**) two items (**:**) into one (that proposition integrates chap. 1.4). The *Transference* from · to **:** (and vice versa) recurs even in the semiotics of the signs for *less than* (**<**) and *greater than* (**>**): one item (·) is less (<) than two items (**:**), even if splitting (viz. dividing) one item in two (· < **:**) returns two smaller items; vice versa, two items (**:**) are more (>) than one item (·), even if the interpolation (viz. multiplication) returns one bigger item. The *Asymmetry* (viz. the intellectual process) typical of the division *displaces* the cardinal property in the place of the numeral property of numbers.

One last remark about *Asymmetry* must be discussed: the division by zero conveys completely different ideas, compared to the multiplication by zero, because the division introduces a very specific framework (viz. the conscious intellectual manipulation of items) into mathematics. On one hand, calculating $n{\times}0 = 0$ applies to n the idea of zero as a process of interpolation (chap. 1.4), so that interpolating something (n) by 0 returns 0 because **0** represents a quality, rather than a quantity (n), i.e. the quality of concluding or closing a cycle of ten additions (chap. 1.2). Applying that quality (0) to any quantity (n) returns the quality itself, collapsing quantification because an arithmetical process (viz. a numerical process) deals only with numbers, rather than with qualities[59]: law [7] states that $\{n{\times}0 = n{\times}(m–m)\} \leftrightarrow \{(n{\times}m)–(n{\times}m) = 0\}$. Arithmetic states $n{\times}0 = 0$ just like $n{\times}1 = n$ because multiplying any n by any ratio smaller than 1 (i.e. $0 < r < 1$) returns a product smaller than zero (e.g., $n{\times}0.5 = {}^{n}/_{2}$): the product tends to 0 as long as the factor tends to 0, until the limit $n{\times}0 = 0$.

On another hand, a "mirror argumentation" (with respect to the latter, i.e. switching $\{0\} \leftrightarrow \{\infty\}$, and $\{\times\} \leftrightarrow \{/\}$) does not validate the "illegal" division ${}^{n}/_{0} = \infty$, even if ${}^{n}/_{n} = 1$ and if $\frac{n}{0<r<1} > n$, and even if the quotients of divisions tend to grow as long as the divisors tend to 0. In that case, scholars tend to accept stating "trends" in divisions (i.e. $\lim_{r\to 0} {}^{n}/_{r} = \infty$, meaning that the ratio ${}^{n}/_{r}$ tends to calculate 0, as long as r tends to the limit 0), but they tend to refuse the statement ${}^{n}/_{0} \approx \infty$ and they consider ${}^{n}/_{0}$ a *meaningless* operation, returning an undefined quotient, because consciousness cannot think of the division like a *Symmetric* formulation of the multiplication, while consciousness thinks of the subtraction like a *Symmetric* formulation of the addition (chap. 1.3). That is because of the *Symmetric* nature of the addition, subtraction and multiplication, and because of the *Asymmetric* nature of the division. In other terms, the *Symmetry principle* implies $\{{}^{n}/_{0} = t\} \leftrightarrow \{n = t{\times}0\}$, the latter validating *only* $0{\times}0 = 0$, but not $0 = {}^{0}/_{0}$ because cognitive experiences prove that any ${}^{n}/_{n} = 1$. Moreover, the *Asymmetric* nature of division can be grasped noting that the expression $\frac{n}{0} = \frac{n}{r-r}$ cannot be reformulated as $\frac{n}{0} = \frac{n}{r} - \frac{n}{r}$ (a *Symmetrization* of law [7]) because $\frac{n}{r-r} \neq \frac{n}{r} - \frac{n}{r}$ (cognitive experiences validate that expression), while $n{\times}0 = (n{\times}r)–(n{\times}r)$. That way, the division elicits *Asymmetric* cognitive functions as well as the *Symmetric* nature implied in the semiosis of **0**. In other terms, the division elicits the arithmetical paradox implied in the conscious effort (an idle effort) to manage *Symmetry* (**0**) via *Asymmetry* (**/**). Every

58 Confirmed in Leibniz (1703) and in the discussion of Deleuze (1988).

59 Symbolical applications of the sign **0** (e.g., 1×**0** = 10) are different process, compared to arithmetical applications of the number 0 (e.g., 1×0 = 0).

calculation implies differentiating[60] items via *Asymmetric* cognition, "representing" items via *Symmetric* symbolic processes (viz. semiosis; chap. 1.1), developing the effort to regulate incompatible activities: Matte Blanco (1975) pointed out that effort as the way consciousness grasps unconscious mind. And that is a crucial remark, because mathematicians strove to acquire a conceptualization of numbers validating the division ${}^{n}/_{0} = \infty$, considered "illegal" for arithmetic, but valid for semiosis. And they did it via the *projectively extended real line* that translates the real line (<——>) to a circle (◯), which is a symbolic *Symmetrization* of an *Asymmetric* entity (chap. 2.1.4).

At last, the decomposition of $0 = r-r$ (via *Absence of negation*) deploys the exponentiation $n^0 = n^{r-r}$, implying the acquisition of the concept of *negative exponentiation*: $\{n^r = \boldsymbol{r} \times (\boldsymbol{n}\times)\} \leftrightarrow \{n^{-r} = \frac{1}{n^r}\}$, that means that negative exponentiation conveys recursive divisions, just like positive exponentiation conveys recursive multiplications, which is the case of *Symmetric* semiosis. On that basis, $n^0 = 1$ because $\{n^{n-n} = \frac{n^n}{n^n}\} \leftrightarrow \{\frac{n^n}{n^n} = 1\}$, as well as any $\frac{n}{n} = 1$. Again, the need for *Symmetry* regulates conscious processes, like the last one: $n^{\frac{1}{r}} = \sqrt[r]{n}$ means the possibility to invert the square root as an exponentiation (e.g., $\sqrt{n} = n^{\frac{1}{2}}$), that way simplifying calculus via semiosis, because fractional exponentiation *projects* the calculation of square roots into the manipulation of multiplications.

1.6. Equality

Equality (as a sign and as a concept) has a key role in mathematics, because it deploys algebra[61]: every equation states an equality between different entities (chap. 2.3), developing the unconscious process of *Symmetrization* and the conscious process of differentiation that characterize human knowledge. That two processes rely on social experiences: from an ancestral attitude in deploying and managing radical conceptual dipoles, to a sophisticate competence in elaborating economic systems. Lévi-Strauss (1949) recorded how even elementary structures of kinship (a radical experience in human existence) rely on ideal oppositions, balanced by rules and traditions (like the prohibition of incest) that deployed economic systems (like exchange economies). And economics developed many (mathematical) models of *equilibrium*, in order to manage differences between debts and credits, demand and supply, etc., *symmetrizing* (viz. balancing) differences. The idea and the sign of equality *condense* all of those ideas, deploying a unique and universal language, based on subconscious symbolic processes.

The sign **=** conveys many ideas. First, **=** specifies the result of an arithmetic operation or of a series of operations (e.g., $8 = {}^{40}/_{5}$): here the *Symmetrization principle* conveys the possibility *to read* the equalities from right to left (e.g., "40 divided by 5 returns 8") as well as from left to right (e.g., "8 can be written as ${}^{40}/_{5}$ rather than 2×4 or 5+3"). That way, **=** conveys the internal ***Symmetry* of algebraic statements** (*left* = *right*), even if, in the same time, the meaning of a statement depends on the *Asymmetric* reading order opted by an intellectual intention: the reading order switches different meaning *condensed* into a single expression. Moreover and more generally, the sign **=**

60 To compute via differentiation here means to grasp and state cognitive differences between similar items, manipulating and processing them in order to identify specific groups (chap. 1.2 and 2.1.1). Nevertheless, infinitesimal calculus really is the operation of "differentiating" values.

61 The noun *algebra* comes from the Arabic noun *al-jabr* ("reunion of broken parts", "restoration"). That way, the sign = restore the unity (viz. the *Symmetry*) of meanings fragmented or differentiated into different pieces or signs, conveying the symbolic idea of a connection between different items.

establishes a correspondence or **a correlation between different items** located in different position (e.g., $w+x = y-z$), making evident differences collapse under an intellectual intention (i.e. different symbols, like x and z, are written on different sides of equality): conscious *Asymmetry* strives to find similarities (or identities) in different items, *generalizing* them and, that way, revealing the underlying *Symmetric* nature of mathematics, which elicits the need to manipulate and to transform formal statements, revealing the very essence of equations (chap. 2.3). Moreover, the sign = simply defines variables and entities (e.g., $v = {}^{s}/_{t}$), that way stating and acquiring information via processing data: every **equality operates a semiosis** (chap. 1.1), for equality represents something (v) via something else (${}^{s}/_{t}$); equality represents complex Thoughts, *condensing* chains of inferences and *projecting* the meaning of language into potentially unspoken or unspeakable Symbols, conceived as universal texts, as long as they speak the unconscious language of associative intuition. Finally, = introduces abstract thinking into cognitive (actual) experiences: e.g., I can convert "7 = 4+3" into "x = 4+3", the latter expression showing that equality translates (viz. *transfers*) a real entity (like the sum 7) to an abstract entity (x) standing for every possible Thought; as well as it can translate any abstract Thought (x) to a real or tangible (viz. perceptible) thing (4+3 meters, pounds, etc.). Then I can convert the latter equality into the expression "$x = y+z$", showing that differences ($\boldsymbol{x \neq y \neq z}$ just like $7 \neq 4 \neq 3$) can be *transferred* into similitudes given by the sign = that defines the equation.

Symmetry (shown in the sign =) and *Asymmetry* (shown in the sign ≠, {≠ = = + /}) regulate the uses of =, *condensing* unconscious intuitions and conscious thoughts via semiosis: the discussions above here (about =) imply the rapport between the sign = and its opposite, the inequality (≠), that paradoxically implies an identity because every rapport (even inequality) establishes a bond between two things (Matte Blanco 1975) or, in other terms, every rapport (even inequality) *projects* one item into another. E.g., a statement "$x \neq y$" conveys a rapport (a connection) between x and y or, in other terms, x and y take parts of a unique Thought that reciprocally *condenses* one item into the other; my conscious mind considers x and y as detached entities, but my unconscious grasps the inequality (≠) as a process binding and *condensing* x and y, allowing me to think of them as (differentiated) parts of a same (undifferentiated) continuum. The inequality (≠) allows me to infer differences because it allows me to identify some common part or some similar element (e.g., I can think of two different people because they have different eyes, different hands, different hairs, etc., but they both have eyes, hands, hairs, etc.). Every inequality implies some sort of community: a rock is quite different from a cat, but even rocks and cats share common elements (e.g., *quanta* for particle physics, *mana* for magic thinking, *Dào* for ancient Chinese philosophers, etc.). The equality develops the possibility for conscious mind to reconcile conflicts or **to synthesize the differences** that the same conscious mind historically operated on unconscious unity of animal instincts, determining human conflicts, as Brown (1959/1985: 85-86) pointed out. Hence mathematics reveals an instance of conscious mind: to compose differences, reconciling the *Asymmetry* with *Symmetry* or differences with unity. Monetary economics reveal that same instance, that is why economics and mathematics share a common property in consciously revealing unconscious unity: money is the *sign* of a debt (in the terms of fig. 1), thus it is the *sign* of a credit; and that sign can be passed on, it can be *transferred* and *displaced* between people, and it can be balanced or equalized or reset (viz. putting *debt* = *credit*), as Graeber (2011) evidenced.

The very sign = shows the *Symmetryzation* property of equality:

[20] {**=** = **:** × ⇢} ↔ {**=** = **:** × ⇠} ↔ {**=** = **:** ⇠⇢ **:**}

Two distinct items can be translated to a new position (**:** × ⇢ as well as **:** × ⇠). That *Displacement*[62] identifies two different locations (one on the left and one on the right side of space, and one on the "*before* side" and on the "*after* side" of time); and those locations identify two couples of items (**:**$_{\text{Left}}$ and **:**$_{\text{Right}}$), both resulting in the same entity (**=** = **:** ⇠⇢ **:**) because one couple is the image of the other, as well as today I am a different image of myself from yesterday: Matte Blanco (1975) explained how unconscious treats actual different entities like one same symbolic entity.

About semiotics of equality, Cajori (1928: 15) recorded the equality symbol in *Ahmes papyrus*, and he (id.: 297-305) recorded how ancient scholars expressed equality "rhetorically" by such words as *ἴσοι*, *aequales*, *esgale*, *ghelijck*, etc., or by **ι**$^{\sigma}$ (an abbreviation of the Greek word *ἴσοι*, "equal") in Diophantus (III cent. BC); the latter resembling a sign used by Al-Qalasâdî (XV cent. AD) (Cajori 1928: 93). Later on, in Western Europe, Pacioli (1494) and Ghaligai (1521) represented equality via a long single lower dash (___). Then Cajori (1928: 298) recorded the first appearance of **=** in Recorde (1557), in Oughtred (1631) and in many other scholars in XVII cent., even if others competed in spreading alternative signs in that era, like Glorioso (1613) and Ricci (1668), until Parent (1713) represented equality by two vertical lines, ‖, rotating **=** (i.e. ‖ = ⟲$_{90°}$ × **=**) or, presumably, vertically *displacing* two items (i.e. ‖ = **..** × ⇡ as well as ‖ = ¨ × ⇣), validating the proposition [20]. Then Cajori (1928: 300-301) noted as "the greatest oddity" the sign **2|2** adopted by Hérigone (1634) for equality, that way neglecting the inner meaning of equality, that in **2|2** *symmetrizes* two numeral items (**2**$_{\text{Left}}$ and **2**$_{\text{Right}}$) as if they were mirror images of one another (the mirror evidenced in the vertical slant): the sign **2|2** conveys the same symbolic meaning of ‖, where the central white space stands for a mirror and the two vertical lines stand for the two items. Nevertheless, Newton and Leibniz widely employed the sign **=** in their works (opting for vertical reflection), and scholars spread it widely, citing their works.

As a last remark, another evidence for *Displacement* underlying the idea of equality comes from the sign **≈** (i.e. "almost equal"), which conveys the idea of approximation or *Generalization* via wavy lines: an interference in motion given as **≈** = **=** × ⇡ × ⇣. Lakoff/Nuñez (2000) explained how the cognitive metaphor of *motion* develops the mathematical concept of *function*: a starting point represents source numbers, a direction over a path represents processes and operations, and a destination point represents results or images of the function. Unconscious metaphors drive the understanding of mathematics.

62 Kandinsky (1926) explained that every line represents the movement of a dot.

2. Psycholinguistics

Mathematical languages sprouted all around the world from ancient civilizations, but modern (and current) mathematical language grounds itself in classical Hellenic thinking, that Fibonacci (1202) translated in Arabian numerical system, on which basis Western culture developed scientific thinking (e.g., Galilei 1638 and Newton 1687 developed a systematic method for analysis of phenomena) and all its subsequent branches, from binary logic (e.g., Leibniz 1703 and Boole 1847 evidenced how duality and polarities suffice for express every computation) to entropy theory (e.g., Clausius 1864 applied calculus to describe one of the fundamental property of matter), to quantum physics (e.g., Dirac 1930 assembled a specific algebra for describing new conceptions of matter states), to economics (e.g., Von Neumann/Morgenstern 1944 formulated an economic theory and a specific formalization) and so on[1]. The scientific models for describing phenomena ground themselves in mathematics, which in turn grounds itself on essentials entities: natural numbers (chap. 2.1.1) and elementary operations (chap. 1) that are embodied in unconscious human mind[2], which sprouts and organizes all the subsequent higher mathematics, as conscious manipulations and permutations of elementary processes[3].

Mathematical language condenses complex thoughts in a small set of signs[4] via an essential syntax. Transforming mathematical expressions via specific rules of algebra, people acquire new and critical ideas[5]: algebraic rules (re)edit unconscious phenomena like *Condensation*[6] and *Displacement*[7]. Along civilization humanity developed the ability to express complex thoughts via simple expressions, in order to elicit new ideas, which is a method typical of scientific thinking[8]: humanity boosted conceptions in physics, economics, etc., when it acquired the ability to manipulate formal algebraic expressions and equations, around XVI cent. (Cajori 1928: 227-229), starting up a cultural process based on subconscious semiosis. Since then, humans *entrusted* mathematical language with the goal to provide functional meanings, *trusting* in its semiotic processes, because the processes themselves rely on the embodied human ability to de-

1 In Rossi (2019-2020) I delved into the seminal groundings common to all of those branches.

2 Sklar et al. (2012) reported the ability of processing arithmetic outside conscious awareness.

3 Moscovici (1972) pointed out the paradox intrinsic to the classical opposition *nature/culture*, for human nature *has* to develop culture. So the development of higher mathematics could be considered a natural product, just like trying to consciously introspect unconscious is a natural process. I discussed that topic about epistemology in Rossi (2019-2020: 141-151).

4 Ten digits (0-9), eight operational signs (chap. 1), three couples of specular parentheses (first order, **(** and **)**, second order, **[** and **]**, and third order, **{** and **}**), two inequality signs (**<** and **>**), and the square root (**√**), just for the basics.

5 E.g., a seminal (here simplified) economical model states that $Y = a{+}bY{+}I$: national income (Y) collects (=) autonomous consumption (a), marginal propensity (b) to consume income itself (Y), and investments (I). Grouping Y on one hand of the equation, it becomes: $Y{-}bY = a{+}I$. Then, factoring out Y, the equation becomes: $Y(1{-}b) = a{+}I$. Then, solving for $Y = (a{+}I)\cdot{}^{1}/_{1-b}$, the initial expression conveys a new concept: the ratio ${}^{1}/_{1-b}$ (viz. the simplified *Keynesian multiplier*) affects Y on the basis of b. Therefor economists trust in the power of consumption, and they named ${}^{1}/_{1-b} = k$ in the name of Keynes (1936), who systematized the multiplier in macroeconomics.

6 E.g., a variable (x) condenses all of the possible real numbers: $x = 0, {}^{1}/_{3}, -1, {}^{-7}/_{5}\ldots$

7 E.g., applying an operation on both sides of equations implies the possibility to relocate the terms of the equation: $\{ax = y\} \leftrightarrow \{x = {}^{y}/_{a}\}$.

8 I delved into that topic in Rossi (2019-2020).

velop and to manipulate language, viz. the processes rely on the structure of human mind itself.

There lies the essence of mathematical thinking: unconscious (symbolic) **associations manage numbers, operators and algebraic signs**, on which bases we construct mathematical expressions coherent with our experiences and with our mind-sets: then algebraic rules (based on the *Symmetry principle*[9]) allow us to modify mathematical expressions (e.g., factoring out, operating on both sides of equations, etc.), revealing insights and defining meanings underlying the initial formulation of expressions. Therefor we *trust* in the "new" meaning of the expression (resulting from formal manipulations) because we *trust* in the initial formulation (given by empirical cognition) and we *trust* in the process of manipulation itself (given by the structure of human psyche): we (must) *trust* in our senses, because our psyche processes our senses.

Suppes (1967) discussed about "abstraction" and "imagery" in mathematics, and Lakoff/Nuñez (2000) discussed about "conceptual metaphors" of mathematical ideas, both accounting for those processes as focal operations and focal operators concerning the acquisition and the manipulation of mathematical concepts and of symbolic logic. Suppes explained those operations in terms of "intuition", "association", "representation" and "transfer": all seminal concepts in the topics of Freud (1899; 1923), even though Suppes did not mention or hint that similitude with Freudian topics. Even Polya (1945) insisted on mathematical thinking as a way of structuring "chains of equivalent problems" or, in other terms, of finding similarities (viz. *Generalizations*, *Projections* and *Transference*) between different problems, or (again) of looking for elements *associating* (apparently) different concepts, on the basis of the idea that one concept can be a metaphor expressing another concept (chap. 1.1), which can be traced back to geometrical experiences, viz. to sensory experiences, for geometry "measures reality" (blending Greek noun *γή*, "earth", "world", and *μετρία*, "measure"), translating items in numbers (viz. *displacing* and *transferring* qualities in quantities, and items in symbols) and vice versa.

2.1. Numbers

The etymology of the English noun *number* conveys some critical insight about symbolic topics of arithmetic: Latin noun *nŭmĕrus* ("quantity", "multitude", "mass") and Greek verb *νέμειν* ("to divide", "to split", "to distribute") recall an idea intrinsic to the division (chap. 1.5), summarized also by the Sanskrit radical *nàmas* ("assigned portion"). All of those ideas recall economic and social experiences: from harvesting and collecting food, saving commodities (like hunters-gatherers used to); to marking fields via geometry and distributing resources within a group of people (like farmers did); and also to recording debts and credits (like Babylonians and Egyptians did). Ancient cultures seem to have deployed mathematical tools in order to manage their social activities and the stability of society[10], appealing to the embodied (natural) numerical abilities of humans: firstly the acquisition of natural numbers ($\mathbb{N}$) via actual cognition (chap. 2.1.1), like also other animals do[11]; then the acquisition of integers ($\mathbb{Z}$), developing unconscious associations regarding $\mathbb{N}$ (chap. 2.1.2); then the acquisition of rational numbers ($\mathbb{Q}$), developing both cognitive and cultural processes (chap. 2.1.3); then

9 Chap. 2.3 and Matte Blanco (1975) delve into that topic.
10 Zerzan (2009) and Graeber (2011) delved into that topic.
11 Capelewicz (2021) recorded seminal studies on the ability of animals in counting and managing numbers.

the identification of real numbers ($\mathbb{R}$) as a discussion about the properties of $\mathbb{Q}$ (chap. 2.1.4); then constructing the complex numbers ($\mathbb{C}$) as an apparent intellectual set, even if it integrates surprising unconscious embodied patterns (chap. 2.1.5). *Generalization* process *projects* $\mathbb{N} \in \mathbb{Z} \in \mathbb{Q} \in \mathbb{R} \in \mathbb{C}$, and each infinite set of numbers deploys higher levels of infinity, given as a set of permutations and repetitions of the ten numerals.

2.1.1. Natural numbers

The seminal idea of *number* arises from the experience of *natural numbers* ($\mathbb{N}$). The locution *natural numbers* evidences how numbers different from $\mathbb{N}$ rely on "artificial" processes (viz. intellectual constructs), rather than on actual perceptions. Lakoff/Nuñez (2000) insisted on the idea that the adjective *natural* identifies entities embodied in human mind and peculiar to human perceptions, thus peculiar to human abilities, just like the expression *natural language* identifies an innate competence in talking[12]: in the same way, counting items and ordering items rely on actual perceptions, following Piaget/Szeminska (1965) about qualitative seriations abilities. Perceptions and unconscious core abilities allow me to identify small groups of items, distinguishing items from a homogeneous background (or continuum); then I can extend that cognitive process, grouping small groups of items, then grouping small groups of small groups of items, and so on, idealizing an infinite series of limited grouping activities as metaphors of my seminal (small scale) actual experience. That (infinite) process is a recursive *Projection* of (the characteristic of) a number into a group, so that a group of groups is thought of as a group of numbers, and so on: *number* being a recursive semiotic concept (viz. a label or Symbol) recalling a *Generalization* (Thought) of cognitive experiences (Referent), just like (e.g.) *Tree* = {*Leaves*+*Branches*+*Roots*…} is a mental construct of many different actual experiences gathered together into one image[13].

Lakoff/Nuñez (2000) showed how the concept of *number* relies on the **perception of items**. That, in turn, relies on the cognitive differentiation and *Symmetrization* of a continuum. E.g., when I see a "tree", my unconscious (viz. a set of cognitive processes) groups a bunch of different entities into one single item ("leaves", "branches", "fruits" and even "particles", if I think of my visual system as of a particle detector or a sensor of light photons): Wertheimer (1922; 1923) and other *Gestalt* psychologists discussed that unconscious cognitive ability of perceiving single items as a whole *and* as individual items in the same time[14]. That way, I can conceive the set of natural numbers, $\mathbb{N} = \{1, 2, 3\ldots \infty\}$, as a series of individuals or distinct elements or items, *generalizing* the whole continuum (both of items and Symbols) as a group (viz. a set), characterized by some property belonging to every single item: I can measure the size of the set (counting infinite cardinal items) or I can order a sequence, establishing different levels of relevance (aligning infinite ordinal items, for different levels of relevance or utility[15]). In any case, I think of numbers managing items: the set $\mathbb{N}$ being the container of items, but in the end, the set being a Symbol for a complex Referent, summarized (viz. embodied) in some subconscious Thought.

That *Generalization* process relies on perceptions and cognition, both partaking to an innate semiotic system that I experience from the very first moments of my life, for I came into society acquiring a *name*, that is a Symbol identifying myself as a Thought bound to some Referents: my body (with every single part), my actions, my voice, my

12 Chomsky (1968) argued about human innate competences in acquiring and developing language.
13 There the word *image* conveys both a "mental representation" of a Referent and the "result of a function" that assembles Symbols and items.
14 Pritchard (1961) provided evidences of the embodied nature of *Gestalt* psychology.
15 Again, economics seems to be a core medium in developing mathematics.

choices, my emotions, etc., they identify myself as a person (*generalized* as a member of a society or a citizen of the world or a human being passing through history) as well as they identify single limbs or organs (*generalized* as biologic components or functional tools or molecular complexes). Names and numbers (as well as every other word and Symbol) operate a *Symmetric* function, for they identify items and groups of items in a continuum. Lakoff/Nuñez (2000) explained how cognitive abilities base the concept of *set* on the actual experience of containers and contents: the container ℕ contains infinite different elements (Symbols **0**, **1**, **2**… **∞** as well as Thoughts 0, 1, 2… ∞)[16] belonging to the same container (viz. expressing the same quality or *generalizing* a common property), for cognitive abilities identify them as "items" or "specific items" (like *trees*, *apples*, *stones*, etc., with *branches* as subsets of *trees*, or *hands* as subsets of *body*, etc.). That way, a collection or a set of natural number is a subconscious cognitive experience given by mere embodied perception.

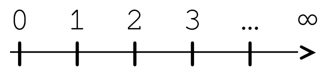

Figure 5:* ℕ *Spatial Metaphor

Oriented lines (like → in fig. 5) usually represent a collection, ℕ being a virtual sequence or a repetition or a reiteration of a symbolic association *generalized* around some property. That line *condenses* ordering and counting cognitive properties: the sequence (1, 2… *n*) depends on the individual differences between items, thus the order measures the relevance of the items with regard to the subject of the sequence; while the sequence returns a quantity, given by the *Generalization* of items, collapsing their differences (viz. relevance) in a same conceptual class.

ℕ in fig. 5 identifies an additivity series as a sequence of infinite **nuclear additions** (1+1+1+1…), where ellipsis (…) stands for infinite reiteration of the additivity process (tending to ∞), and the adjective *nuclear* means that ℕ is a *discrete* or *digital* series: nothing stands between two digits or elements of ℕ (viz. a gap ⊢⊣ is just a gap or a null void), just like nothing stands between two *fingers* (viz. items) in one hand. The noun *digit* comes from Latin noun *dĭgĭtus* ("finger" *and* "unit of measurement"): Lakoff/Nuñez (2000) argued that the idea of *number* comes from the embodied experience of counting elements (like the fingers) from a complex object (like the hand), or picking contents from containers (viz. differentiating items in a continuum); just like Matte Blanco (1975) argued that the nature of unconscious (viz. the opposite process) is a recursive *Generalization* giving rise to a series of *infinite sets* or containers or, in other words, a *Symmetric* continuum collapsing and homologating differences. That is a two-way semiotic process operating at once: numbers symbolize points or segments identified in the continuous line, differentiating and breaking it; while the line *generalizes* and *Symmetrizes* actual differences into a continuous ideal property.

In order to define ℕ, our mind needs to acquire two basic concepts:

- *One* (item)[17], distinct from other (different) items, focusing itself with respect to the homogeneous background. The *one* item being the nuclear element needed to start and develop any series.

16 Lakoff/Nuñez (2000) explained that ℕ includes 0 because numbers are represented via the spatial metaphor of a line (fig. 5), 0 being the (point of) origin of the line. While representing the numbers via the embodied tools (like hands, fingers, etc.), ℕ excludes 0.

17 Frutiger (1979) noticed hand-written numbers evidence no ligatures because numerals represent individual entities, viz. things, whereas letters are bind one another in each word by ligatures.

- *Many*[18] copies of that referential (one) item, or the sum or juxtaposition of identical or similar items, needed to idealize the series and to close it.

The two concepts (*one* and *many*) rely on actual experiences or Referents (fig. 1): the mind processes sensory information and identifies and isolates something (viz. some physical entity) from other physical entities. The mind differentiates the continuum of physical reality (viz. a background), identifying individual[19] bodies (viz. items in the foreground): I can identify complex items (built out of singular parts) as unique items, and their parts as other unique items too, and so on. That is a cognitive competence, without which it would be impossible for me to manage and think about one single item[20], while I can chunk up or chunk down the scaling[21] of that process of identification.

Language develops that process, constructing peculiar syntagmata, like definite and indefinite articles, that specify the different meanings of the *one* concept with regard to the *many* concept: e.g., English definite article *the* identifies one unique item in a discourse, isolating one individual item in the background *from* the many others; while indefinite article *a* (*an*) identifies one item within a group of similar items, isolating one *among* many (similar) items in the background. The experience of *one* and *many* sum up the meaning of $\mathbb{N} = \{0+1+1+1\ldots\}$ for we need to acquire 1_T (the ideal concept) from 1_R (the embodied experience) in order to acquire $many_T = \{1_T+1_T+1_T\ldots\}$ from the experience $many_R$[22]. Then I can adopt any conventional sign (e.g., $\mathbf{1}_S$, $\mathbf{I}_S$, $|_S$, $—_S$, etc.) in order to convey 1_T and its successors ($2_T = 1_T+1_T$, $3_T = 2_T+1_T$, etc.).

Referential (or actual) experiences about $\mathbb{N}$ reveal the conventional nature of numeral digits: common decimal numeral system (based on ten digits: 0-9) can be replaced by binary numeral system (based on two digits: 0, 1)[23]; as long as Symbol conveys specific Thoughts (value), every numeral system makes it possible to count an additivity series and to "imagine" infinite series, through ideal repetitions (i.e. recursive semiosis) of the same additive operation. Indeed, Suppes (1967) stated that it is possible to identify simple symbolic systems in order to define numbers, like substituting each *numeral*$_S$ with the equivalent *number*$_R$ of strokes implied in the count[24] (on the basis of cognitive symbols, fig. 2): $|_{SR} = 1_T$, $||_{SR} = 2_T\ldots$ $|||||_{SR} = 5_T$ and so on. Our innate competence in associating experiences to symbols is a process of signification (fig. 1) where (e.g.) $|||||_S$ works as well as 5_S in order to convey $|||||_T$ on the basis of $|||||_R$.

The strokes depicted in $|||||_R$ convey $|||||_T$ because of the *grouped juxtaposition* of the strokes[25]. Nevertheless, that process, explained by Suppes (1967), rather than explain-

18 See chap. 1.2 about Greek adjective *πολύς*.

19 The etymology of the noun *individual* helps to clarify that topic: Latin noun *indīvĭdŭus* blends the prefix *in-* (a privative particle) and the adjective *dīvĭdŭus* ("divided", "separated"); thus *individual* means "undivided".

20 Bohm (1980) pointed out that the distinctions between items disappear on quantum level of matter (which he called "implicate order"), identifying a *quantum field* where particles are superposed one another in the entangled background.

21 Bodenhamer/Hall (1999) delved into the dynamics of chunking up and down.

22 Subscripts R and T stand for Referent and Thought in fig. 1.

23 Leibniz (1703) demonstrated the mathematical possibilities of binary system, grounded in ancient Chinese philosophy.

24 Fingers fit as well as the strokes, for Lakoff/Nuñez (2000) explained how mathematical ideas come from sensory-motor experiences or "embodied" experiences.

25 Goldfarb (2011: 79-80) recorded an innate competence in animals to compare the *Gestalten* (German for "images") of items: counting competence is based on that "pattern recognition mechanism" that Lakoff/Nuñez (2000) accounted for as *subitization* or the ability to identify small groups of (around 3) similar items with a glimpse of an eye. *Gestalt* psychology accounted for a cognitive

ing how humans manage numbers, appeals to the human innate ability to count items, which is a psychological and objective competence. Besides, counting appears to be a specific process of signification, where $|_{\mathrm{R}}$ or one single *finger*$_{\mathrm{R}}$ (some standard physical phenomenon[26] taken as a Referent) impresses in the mind the relative $|_{\mathrm{T}}$ (a cognitive experience deploying an internal representation), which any individual can represent via any Symbol (like $\mathbf{1}_{\mathrm{S}}$, $\mathbf{I}_{\mathrm{S}}$, ⸮$_{\mathrm{S}}$, ecc.) that makes the individual recall $|_{\mathrm{T}}$ through repetition of associations[27].

Roman numerals develop ℕ as well as Arabic numerals do, for **I**, **II**, **III**, **V**, **X**, **L** highlight semiosis binding $|_{\mathrm{S}}$ to *finger*$_{\mathrm{R}}$: Chinese language share the same experience with the first three numerals (一, 二, 三), like Babylonian language did (𒁹, 𒈫, 𒐈), as well as Phoenician language did (*/*, *//*, *///*). Most of the numeral systems adopted the same semiosis (viz. juxtaposed strokes) for the first 3 numerals, changing semiosis from number 4 on: that confirms the *subitization theory* of Lakoff/Nuñez (2000), for the mind changes its framework over the number 3, representing groups greater than 3 items via metaphors, symbolizing quantities as a recursive process of signification (chap. 1.3). Cognitive processes acquire |, || and ||| as (f)actual experiences, and semiosis represents them via *cognitive symbols* (fig. 2), like Roman I, II and III; then subconscious semiosis builds up numerical concepts greater than 3, building up new *conceptual symbols* (fig. 1) like **4** = |||+| and so on.

Frutiger (1979) suggested a peculiar semiosis of Roman numerals: **V** resulted as a representation of the angle between the thumb and the index fingers of the hand ($\vee_{\mathrm{R}}$), so that **X** resulted from the summation of the two hands, for $\{\mathbf{X} = \mathbf{V} + \mathbf{\Lambda}\} \leftrightarrow \{\mathbf{X} = \mathbf{V} + (\circlearrowleft_{180°} \times \mathbf{V})\} \leftrightarrow \{\mathbf{X} = \mathbf{V} + (0+i^2) \times \mathbf{V})\}$[28]. *Projecting* that idea, **L** stood for a wider angle (thus a larger quantity) of the same kind of **V**. That way, Roman numerals for very large numbers came from a recursive semiotic process based on words, rather than on actual Referents: **C** stood for *centum* (“100”); **M** stood for *mille* (“1,000”), expressed also by **CIↃ**; hence **D** (“500”) abbreviated the “right side” half of 1,000, for {$^{\mathrm{CIↃ}}/_2$ = C|+|Ↄ} → {**IↃ** = **D**}.

Gestalt psychology revealed relevant properties of numerical conceptualization because counting relies on our ability to group (images of) items: Dedekind (1888) and Peano (1889), and the theories that followed[29], ground themselves on the seminal concept of *set*: **a collection of distinct elements**[30]. All scholars took (and still take) for granted the idea of a *collection*, which appears to be indefinable in mathematics because a collection implies gathering something, which in turn implies catching something with senses, and senses base all of our ideal constructs[31].

competence in organizing information like complete and ordered images, following the law of *Prägnanz* (German for “pithiness”, in Wertheimer 1922; 1923), and a cognitive competence in grouping items juxtaposed one another, following the law of proximity (Koffka 1935: 164-165).

26 Goldfarb (2011: 79-81) wrote about “*reference*” in that sense, apparently ignoring the concept of Referent in the semiotic triangle (fig. 1).

27 Goldfarb (2011: 80) wrote about the concept of number and about the “artificial nature of its reduction to a symbol”, which is an idea consistent with the process binding Symbols and Thoughts depicted in fig. 1.

28 Chap. 1.4 discusses the application (×) of rotational process ($\circlearrowleft$) and chap 2.1.5 discusses it in numerical terms.

29 Whitehead/Russell (1910-1913) above all.

30 Bolzano (1851); Cantor (1874); Zermelo (1908); Fraenkel (1922).

31 Bodenhamer/Hall (1999) explained how knowledge is built upon (and mediated by) sensory-neural information. Moreover, Lakoff/Nuñez (2000) explained how we speculate on abstract domains exploiting the inferential structure of tangible domains, associating a Symbol to a Thought on the basis of experiences (viz. Referents). Again, both references recall indirectly the semiotic triangle (fig.

Indeed ancient documents account for the idea of a collection to symbolize any generic number: Egyptian *Ahmes papyrus* represented unknown variables with the symbol  ("heap", "bunch", "pile", "stack") (Cajori 1928: 15, 379), meaning numbers (Thought in fig. 2) via the idea of things (Symbol and Referent in fig. 2); and Fibonacci (1202) represented unknown variables as ***res*** (Latin noun for "thing"), while Pacioli (1494) used the word ***cosa*** (Italian noun for "thing"), contracted in ***co.***, in order to represent unknown variables[32]. Cajori (1928: 379-380) pointed out how only modern scholars (from XV cent.) started to represent generic numbers via single letters (***a***, ***b***… *z*), substituting or *condensing* Referents (things) with Symbols (letters), while ancient documents recalled generic concept of $number_{\mathrm{T}}$ recalling indistinct $things_{\mathrm{R}}$: the semiosis of abstraction, codified in culture, founded the mathematical language.

Quantum theories propose the same idea: quantum level of matter identifies undefined or *superposed* states of quanta and physics can provide only statistical descriptions of matter on that level; while classical physics can provide extremely exact descriptions of molecular (viz. ordinary) state of matter. Conscious senses and psyche operate on ordinary level of matter or, in other terms, explicate actual cognition of an underlying continuum generating unconscious: actual reality and perceptions are just a specific *collapse* of superposed quantum states that Bohm (1980) distinguished as our sensible "explicate order", collapsed through the imperceptible "implicate order" of matter.

2.1.2. Integers

Cross-referencing Matte Blanco (1975) and Bohm (1980), Freudian topics represent the "implicate order" as a homogeneous continuum differentiated by conscious cognition, that "explicates" that continuum. The set of integers ($\mathbb{Z}$)[33] reveals that process of explication in mathematics, for $\mathbb{Z}$ deploys the set of natural numbers ($\mathbb{N}$) *and* its opposite, via the possibility to subtract natural numbers ($-\mathbb{N}$) from other natural numbers or, in other terms, to remove items from a collection, establishing the idea of the collection of "removed items": thus $\mathbb{Z}$ represents the essential *Symmetry* in mathematics, which is just an expression of the many (infinite) forms of the *Symmetry principle* (see chap. 2.3). Here *Symmetry* involves addition and subtraction concepts as opposite polarities to be introduced in the continuum.

Subtraction is the seminal operation developing arithmetic as an expression of the *Symmetry principle*: any number ($n \in \mathbb{N}$) has **its mirror counterpart** ($-n \in \mathbb{Z}$). That is an economic and social experience, for debts always balance credits because a surplus in debts stands for an equal credit to be accounted on its counterpart: the creditor has an amount of "social power" to be exerted against its debtor[34]. Lakoff/Nuñez (2000) identified the *mental rotational competence*[35] as the essential cognitive framework founding $\mathbb{Z}$: they suggested that any $-n$ results as a 180° rotation of n on the ideal ordinal axis (fig. 5), thus $-n = \circlearrowleft_{180°} \times n$. Whereas I think $-n$ as a *reflection* implying *Symmetry principle* (chap. 2.1.5 delves into that topic): along with rotational competence, I

1) as the essence of seminal Freudian concepts of *Association*, *Condensation* and *Transference*.

32 (Cajori, 1928: 107) noted that "from this word were derived in Germany and England the words *Coss* and "cossic art", which in the sixteenth and seventeenth centuries were synonymous with "algebra"".

33 The letter $\mathbb{Z}$ stands for the German word *Zahlen*, "numbers".

34 Graeber (2011) discussed the worldwide habit to compare and record activities and losses (viz. credits and debts) in order to manage power and to run ordered societies.

35 Shepard/Metzler (1971) and Shepard/Cooper (1981).

perceive visual, kinaesthetic and proprioceptive experience of my two hands (as well as other limbs of my body[36]), each one *reflecting* its counterpart.

Hence, on the basis of cognitive experiences, I organize an internal positional representation (viz. an embodied conceptualization) of one set of natural numbers ($\mathbb{N}$), on the right hand ($\text{IIIII}_{SR} = 5_T$) with five different individual fingers[37], as an internal representation fitting the set of five ordered digits (1-5, each I fitting each *finger*)[38], along with their mirror image, experienced on the left hand (5-1) and vice versa. Hence I develop the representation of an infinite continuum (<——> in fig. 6) as a positional metaphor for $\mathbb{N}_{\rightarrow}$ and ${}_{\leftarrow}\mathbb{N}$: a continuum split by a border (I) balanced on the edge of the two sides, like a mirror reflecting one side to the other. Zero point operates that reflection.

∞ … 3 2 1 0 1 2 3 … ∞

<————|————>

Figure 6: Reflected $\mathbb{N}$ Spatial Metaphor

That way, I have available two infinite series (${}_{\leftarrow}\mathbb{N}$ and $\mathbb{N}_{\rightarrow}$) joined together in a new infinite series (${}_{\leftarrow}\mathbb{N}_{\rightarrow} = {}_{\leftarrow}\mathbb{N}+\mathbb{N}_{\rightarrow}$)[39] mirroring items: following the *Symmetry principle*, unconscious grasps {*left* = *right*} ↔ {*right* = *left*} (chap. 2.3) or $\{{}_{\leftarrow}\mathbb{N} = \mathbb{N}_{\rightarrow}\}$. Thus I have available a numeral continuum (${}_{\leftarrow}\mathbb{N}$ or $\mathbb{N}_{\rightarrow}$) that I can employ to represent the subtraction, the other continuum representing the addition: *displacing* and *condensing* ${}_{\leftarrow}\mathbb{N}$ with $-\mathbb{N}$[40], I construct $\mathbb{Z}$ (fig. 7) via semiosis, *displacing* and *condensing* Referents (${}_{\leftarrow}\mathbb{N}$) with Symbols ($-\mathbb{N}$).

−∞ … −3 −2 −1 0 +1 +2 +3 … +∞

<————|————>

Figure 7: $\mathbb{Z}$ Spatial Metaphor

Developing $\mathbb{Z} = \{-\mathbb{N}+\mathbb{N}\}$ is a subconscious semiotic process based on *Symmetrization*, just like $\mathbb{N}$ is a semiotic process based on actual experience and cognitive signification. The experience of subtraction is an innate competence in recognizing “differences” within collections of *many*$_T$ items[41]: the subtraction makes it possible to think of negative numbers because a “subtraction set” (e.g., $x-y = z$) identifies a series of symbolic subsets (e.g., $\{x-y\}$, $\{-y =\}$, $\{= z\}$, etc.), where the subset $\{-y\}$[42] is the mirror image or the *Symmetric* image of $\{+y\}$ (for $y \in \mathbb{N}$). Having deployed $\mathbb{N}$ on the one *hand*, I deploy $-\mathbb{N}$ on the *other hand*, on the basis of what *Gestalt* psychology called the *law of symmetry*: every individual item becomes a portion of a whole[43] when I pair that item with its mirror image[44].

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36 Lakoff/Nuñez (2000) recorded that many populations extend their basic competence in counting over 10 (fingers), implementing limbs of human body. And even systems of measurement, like the imperial system, implemented *fingers*, *palms*, *hands*, *arms*, and *feet*.

37 Thumb, index/pointer, middle finger, ring finger, pinkie: each finger has its individual denotation.

38 The *infinite* set $\mathbb{N}$ deploys as the recursive (∞) mental representation of one hand (or other small groups of items). Lakoff/Nuñez (2000) explained that process as a cognitive competence in metaphorical recurrent representation of things: Symbols can operate as a virtual Referents (fig. 1).

39 The paradox given in $\{\infty+\infty = 2\infty\} \leftrightarrow \{2\infty = \infty\}$ is a problem just for conscious mind, because subconscious mind *generalizes* every *kind* of ∞ in a recursive metaphorical representation, according to the cognitive pattern that Lakoff/Nuñez (2000) named *Basic Metaphor of Infinity* (*BMI*).

40 The inverse ($-\mathbb{N}$ in the place of $\mathbb{N}_{\rightarrow}$) could be valid too, even though putting negative numbers on the left of the zero (and positives on the right) is just a symbolic convention. Yet Bodenhamer/Hall (1999) identify a subconscious tendency in arranging past events (← = –) on the left side of our (inner) symbolic representational space and future events (→ = +) on the right side.

41 Lakoff/Nuñez (2000) recorded children being aware of subtractions operated on small collections.

42 Every $\boldsymbol{-y}_S$ conveys $-y_T$ as well as *eating*-y-*grapes-from-a-bunch*$_{SR}$ does.

43 The noun *symbol* blends the Greek preposition *σύν* (“together”, “with”) and the verb *βάλλειν* (“to cast”): a *σύμβολον* was a “sign of identification” composed of two complementary parts to be joined together in order to compose a whole image.

Differently from ℕ, the *Symmetry* developing ℤ = {–ℕ+ℕ} via mirror images operates a radical symbolic disjuncture, for it multiplies the possibilities of *ordering* items on one hand, while it nullifies the results of *counting* items on the other hand: the two infinite series (–ℕ and ℕ) double up the logical possibility to order infinite items (viz. {–ℕ, +ℕ} = 2∞) because they represent two distinct infinite series; while the arithmetic summation of the two series returns nothing (viz. +ℕ–ℕ = 0), for every number (or value) in ℕ has its negative counterpart in –ℕ and the two annihilate one another. That remark (a property of 0 superposed to a property of ∞) suggests a *Transference* between the opposite poles of ℕ (0 and ∞ in fig. 5), as it can be recorded through the series of rational numbers (chap. 2.1.3).

2.1.3. Rational and irrational numbers

The experience of division (e.g., cutting an apple in two) introduces a peculiar perspective on the linear metaphors of numbers, breaking up the common-sense criterion of order. Rational numbers (ℚ) identify the (infinite) collection of *fractions* resulting from divisions: the common (and essential) social experience occurring when we split and distribute something (a whole) between a group of people. Etymology reveals some symbolic remark on that point: on one hand, the noun *fraction* comes from the Latin noun *fractĭo* (“breaking”) and from the verb *frangēre* (“to break”), both referred to the social experience of divisions and distribution of resources; while, on the other hand, the verb *to divide* may come from Sanskrit *vydh-ŷami* (“to pierce”, “to hit”), recurrent in the “breaking” experience of fractions, and it blends the Latin preposition *di* (“two”, “double”) with the verb *vīdēre* (“to see”), hence a *division* means a “double vision”. That duality recurs in the very semiotics of division (chap. 1.5) and it means the very function carried out by conscious mind, that is always correlating and parsing distinct entities or items (*Asymmetry* being the figure of that function), while the unconscious confuses and *generalizes* them (*Symmetry* being the figure of that other function): the function of comparing items can be carried out comparing couples of items or groups of two items per time.

The name for *rational* numbers comes from Latin noun *rătĭo* (“calculation”, “balance”, “comparison”, “rapport”), which represents economic thinking, for a positive integer (s) gets divided by another integer (r) in order to assign portions (${}^{s}/_{r}$) to each individual (r), calculating a fair rapport of the whole (s) with the total number of individuals (r), so that assigning a single portion (${}^{s}/_{r}$) to every individual (r) it returns the whole (s): $r \times {}^{s}/_{r} = s$. The letter ℚ stands for “quotient” (“rate”), from the Latin adverb *quŏtĭens* (“how much”, “how many times”). That way, the rapport ${}^{s}/_{r}$ represents the rapport between r individuals of a society, mediated by some medium (s), for that ratio represents the possibility to *really* split and share some common thing: a division bounds people[45], even if it results in some imperfect or unfair (viz. *Asymmetric*) allocation[46]. That experience relies on the *Generalization* process (viz. on the ability to collapse differences under some common quality *projected* into different items): e.g.,

44 E.g., an ordered series of 3 pairs of brackets, [ ] ( ) [ ], conveys an image absent in a random series of 6 brackets, [( ]] [ ), because of my subconscious grouping ability.

45 The Gospels’ episode about Jesus dividing loaves and fishes (Matthew 14: 13-21; Mark 6: 30-44; Luke 9: 10-17; John 6: 1-14) represented a cornerstone in community-building strategies, as it introduced an insight on divisions returning a product, rather than a quotient (e.g., ${}^{1}/_{0.5} = 2$).

46 Many families experience (quite ordinary) unfair distributions of resources: somebody gets much more cares than others, or more love, or more money, etc. Adults understand imbalance as a natural property of families, while children and parents (thinking of the categories in Berne 1964) strive for acquiring balanced allocations: the *Symmetry principle* operates as a subconscious drive in people focused on themselves, rather than on the community.

given 4 apples, I can get 2 *small* apples reckoning a fair settlement between me and my brother, even if my brother had 2 *medium* apples; or I can get $^1/_3$ of a cake, via a simple geometric settlement or via a visual check on the three slices, even if the net weight of my slice differs from the net weight of the other two slices.

Economic thinking manages ratios ($^s/_r$) like *per se* items or individual symbols apt to represent a rapport between two distinct items (s and r), defining the role of each subordinate ($s = 1$) under the power of the ruler (r, meaning the representation of the total amount of people in the community), in order to compute taxation: Graeber (2011) recorded money has been there from the beginning of human societies based on power and State[47], and money has been counted (and accounted) for taxation records (just like in present times), needed in order to manage power and leadership: "$t = {}^1/_r$" means a tax rate (t) calculated as the range (r) of the power applied by the ruler to every item (1) present in the community[48]; t having to be multiplied by the income (Y) in order to compute the total amount of taxes (T) to be collected by the State ($T = tY$). That way, rulers need to calculate ratios or, at least, they need to identify ratios (t) to be applied to incomes (Y), in order to calculate taxes on those incomes. That way, tax rates develop **the elementary series of ratios** given in fig. 8[49], where the fraction $^1/_\mathbb{N}$ means that one item can be divided by any $n \in \mathbb{N}$, returning smaller numbers as long as n gets bigger.

$^1/_1$	$^1/_2$	$^1/_3$	… →	$^1/_\infty$
1	0,50	0,33…	… →	0

Figure 8: Series of Elementary Rates ($^1/_\mathbb{N}$)

Representing $\mathbb{N}$ individuals along the positional axis (→ in fig. 5) and *projecting* that axis into the slash of division (/), thus *transferring* the ordinal property of numerals (viz. the location of a number on →) to their cardinal property (viz. the value of the divisor located under the slash)[50], it is possible to think of dividing a common whole resource (1) between a group ($\mathbb{N}$), for every body gets a stake of the bunch, so that the stake gets smaller as long as the group gets bigger; as well as it is possible to think of dividing every single item (1) by the power of the ruler ($\mathbb{N}$), the State collecting a fair contribution from every body, so that the collection gets bigger as long as the group gets bigger. The lower row in fig. 8 lists the values of each ratio[51], listed on the upper row as an infinite series of "first order" fractions (for the numerator $s = 1$): the symbolic ratio $^1/_\mathbb{N}$ is developed step by step, as the numerator (1) gets divided by the series $\mathbb{N}$ "sliding" at the denominator. Yet negative branch of $\mathbb{Z}$ reflects $\mathbb{N}$ (chap. 2.1.2), developing a ↔ oriented series, rather than the single → oriented series in fig. 8.

That elementary instance of $\mathbb{Q}$ introduces another instance of the *Symmetry principle*, inverting the growth given by left-to-right order metaphor: the series on the lower row (fig. 8) tends to 0 (it decreases) as long as the series on the upper row tends to ∞ (it grows). The symbolic division $^1/_\mathbb{N}$ establishes **a dual nature** of the upper and lower series, with the extreme limit ($^1/_\infty = 0$) restating the *Transference* between 0 and ∞ given by the properties of $\mathbb{Z}$ (chap. 2.1.2): the *Condensation* suggests that ∞ can trans-

47 While Staid (2015) recorded how ages developed also societies "without a State".

48 E.g., a *tithe* (chap. 1.5) is the tenth part collected by the state out of every unit of money circulating throughout the community: $t = {}^1/_{10}$.

49 The following representations of $\mathbb{Q}$ omit the negative hand of the set (for the sake of simplicity) because the negative hand mirrors the positive hand, just like it has been pointed out about $\mathbb{Z}$ in chap. 2.1.2.

50 The representation given in fig. 8 evidences the meaning of equality sign (=) as "a correlation between different items located in different positions" (chap. 1.6).

51 Chap. 2.1.4 discusses the result of last fraction represented in fig. 8 ($^1/_\infty = 0$).

form something into 0 (fig. 8) and, vice versa, 0 can transform something into ∞[52], as well as divisions and multiplications can be inverted employing any operand $0 < q < 1$ (chap. 1.5), that way the extreme poles of $\mathbb{N}$ (0 and ∞ in fig. 5) coincide through the *Symmetry principle*. And that idea can be extended also to the extreme poles of $\mathbb{Z}$ ($-\infty$ and $+\infty$ in fig. 7), returning the symbolic tool given in chap. 2.1.4 (fig. 9) in order to consciously explain the unconsciously obvious fractions ${}^1/_\infty = 0$ and ${}^1/_0 = \infty$.

The series of elementary rates (fig. 8) enters the possibility of every fraction given in $\mathbb{Q} = {}^{\mathbb{Z}}/_{\mathbb{Z}}$ (tab. 4), for the numerator (1) can be replaced by every other number ($\mathbb{N}$) via recursive semiosis, viz. via combining symbols below and under the slash. That way, the complete series of combinations of numerators and denominators ($\mathbb{Q}$) identifies extremes polarities, employing 0 and ∞, given as the *unconscious limits of conscious intellect*, while every other combination (contained in the range of those two limits) identify real items, viz. numbers and symbols that measure *Asymmetry* in real phenomena. The ratios employing extreme polarities always return *Symmetry* (viz. unconscious representations of continuum): the divisions ${}^s/_0 = \infty$ and ${}^s/_\infty = 0$ return **mirror images** of the polarity employed as a divisor; while ${}^0/_r = 0$ and ${}^\infty/_r = \infty$ return an identical image for quotient and dividend. The former hand of the symmetric limits has been discussed here, while the latter needs a little digression: no-thing divided by some-thing returns no-thing (${}^0/_r = 0$), while every-thing (viz. ever-y-thing)[53] divided by something will always return every-thing (${}^\infty/_r = \infty$), for the noun *infinity* blends Latin prefix *in-* (privative particle) and *fīnītus* (“certain”, “limited”, “completed”), thus it means “impossible to be limited”, “impossible to be measured” and, substantially, “impossible to be divided”.

[21] $$\{{}^0/_r = 0\} \leftrightarrow \{0 = 0{\times}r\}$$

[22] $$\{{}^\infty/_r = \infty\} \leftrightarrow \{\infty = \infty{\times}r\}$$

Propositions [21] and [22] integrate the discussions given in chap. 1.4 and 1.5: via the division, the conscious mind catches the unconscious *Transference* between 0 and ∞, for both limits always result in themselves when they get divided by something, flowing to some recipient (r). That meaning the unconscious *projects* 0 and ∞ from the category of *qualities* (entities, containers apt to be filled by outer meanings) into the category of *quantities* (numbers, contents with an intrinsic value, even though an uncountable value). That *Projection* resumes the meaning of infinite sets representing conscious mind grasping unconscious, as discussed by Matte Blanco (1975).

The *Symmetry principle* keeps reiterating the divisions throughout $\mathbb{Q}$: the unconscious *projects* the first elementary instance of $\mathbb{Q}$ (fig. 8) from 1 to the other natural numbers (taken from $\mathbb{N}$)[54], thus it *displaces* two infinite series along the line of fraction (${}^{\mathbb{N}}/_{\mathbb{N}}$). That way, the “second instance” of $\mathbb{Q}$ defines a **continuous** (analog) set, rather than a discrete set (like $\mathbb{N}$ and $\mathbb{Z}$), for the adjective *analog* comes from the Greek adjective *ἀνάλογος* (“proportional”, “commensurate”): $\mathbb{Q}$ identifies an infinite number of ratios between discrete integers ($n = {}^n/_1$), so that every “gap” in the natural series (${}^1/_1 \ldots {}^2/_1 \ldots {}^3/_1 \ldots$) can be “filled up” with a ratio (${}^1/_1 \ldots {}^2/_n \ldots {}^2/_1 \ldots$). That (symbolic) ratios open up a paradox analogous to that in chap. 2.1.2: the infinite set $\mathbb{Q} = {}^{\mathbb{Z}}/_{\mathbb{Z}}$ explodes the ordering possibilities of rational numbers because $\{\mathbb{Q} = \mathbb{Z}^2\} \leftrightarrow \{\mathbb{Z}^2 = \infty^2\}$ (i.e. $\mathbb{Q}$ deploys multiplied combinations of the infinite series $\mathbb{Z}$, rather than juxtaposed

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52 Ancient cultures like Taosim delved into that paradox (Rossi 2019-2020).

53 The adverb *ever* could come both from Proto-Indo-European root **aiwi* (“vital force”, “eternity”) an from Latin noun *aeternus* (“eternal”).

54 Again: $\mathbb{Q}$ defines its mirror image of negative ratios, operating $\mathbb{Z}$, rather than $\mathbb{N}$.

combinations like $\mathbb{Z} = {}_{\leftarrow}\mathbb{N}+\mathbb{N}_{\rightarrow}$); while the arithmetic computation of the set returns a recursive set $\{\mathbb{Q} = {}^{\mathbb{Z}}/_{\mathbb{Z}}\} \leftrightarrow \{{}^{\mathbb{Z}}/_{\mathbb{Z}} = {}^{0}/_{0}\}$, where $^{0}/_{0}$ represents the idea of infinite recursive possibilities.

Cantor (1891) stepped into that paradox when he ordered the ratios deployed in $\mathbb{Q}$, correlating or substituting every ratio ($q \in \mathbb{Q}$) with a natural number ($n \in \mathbb{N}$), viz. *displacing* $\mathbb{N}$ into $\mathbb{Q}$: that way he proved the existence of different orders or ranges of infinity ($\infty_{\mathbb{Q}} > \infty_{\mathbb{N}}$), organizing a table based on the same idea employed in tab. 3, where every integer[55] listed in the columns is divided by every integer listed in the rows (or vice versa). **The table identifies two infinite halves**, one mirroring the other along the diagonal unit fractional ($^{n}/_{n}$ = 1) *and* through the slash of ratios, that way integrating the meaning of the sign / discussed in chap. 1.5: integers taken as numerators on one half of the table are taken as denominators on the other half.

	0	**1**	**2**	**3**	**…**	**∞**
0	$^{0}/_{0}$	$^{0}/_{1}$	$^{0}/_{2}$	$^{0}/_{3}$	…	$^{0}/_{\infty}$
1	$^{1}/_{0}$	$^{1}/_{1}$	$^{1}/_{2}$	$^{1}/_{3}$	…	$^{1}/_{\infty}$
2	$^{2}/_{0}$	$^{2}/_{1}$	$^{2}/_{2}$	$^{2}/_{3}$	…	$^{2}/_{\infty}$
3	$^{3}/_{0}$	$^{3}/_{1}$	$^{3}/_{2}$	$^{3}/_{3}$	…	$^{3}/_{\infty}$
…	…	…	…	…	…	…
∞	$^{\infty}/_{0}$	$^{\infty}/_{1}$	$^{\infty}/_{2}$	$^{\infty}/_{3}$	…	$^{\infty}/_{\infty}$

Table 4: Elaboration based on Cantor (1891)

The *Symmetry* employed in representing $\mathbb{Q}$ gets even more evident, if I translate the fractions in terms of their quotients (tab. 5). The first and the last cells of the two tables ($^{0}/_{0} = \mathbb{Q}$ and $^{\infty}/_{\infty} = \mathbb{Q}$, thus entering the equivalence $^{0}/_{0} = {}^{\infty}/_{\infty}$) represent the extreme limits of the entire set ($\mathbb{Q}$) via the set itself: hence I can consider the ratio $^{0}/_{0}$ as a number (n) because the proposition $\{^{0}/_{0} = n\} \leftrightarrow \{0 = 0{\times}n\}$ validates for every $n \in \mathbb{Q}$, as well as $\{^{\infty}/_{\infty} = n\} \leftrightarrow \{\infty = \infty{\times}n\}$ does; $^{0}/_{0} = 1$ (and $^{\infty}/_{\infty} = 1$), from $^{n}/_{n} = 1$, being just one of the *infinite* possibilities given in $^{0}/_{0} = \mathbb{Q}$ (and in $^{\infty}/_{\infty} = \mathbb{Q}$). On another hand, the lower-left cell and the upper-right cell ($^{\infty}/_{0} = \updownarrow$ and $^{0}/_{\infty} = \updownarrow$, thus entering $^{\infty}/_{0} = {}^{0}/_{\infty}$) return the inner *Symmetrical* property of $\mathbb{Q}$: the set comes from inverting the positional order of symbols employed in the fractions or (in other words) every possible combination of dividends and divisors has its vertical mirrored counterpart (above and below the division slashes). That way I (consciously) understand that I have available an infinite set of possibilities to mirror numerals along the horizontal axis (viz. $\mathbb{Z} = {}_{\leftarrow}\mathbb{N}+\mathbb{N}_{\rightarrow}$) *and* along the vertical axis (viz. $\mathbb{Q} = {}_{\uparrow}\mathbb{Z}+\mathbb{Z}_{\downarrow}$). On another hand, the diagonal of tables 4 and 5 ($^{\mathbb{Q}}/_{\mathbb{Q}} = 1$) divides the set in two halves: one infinitesimal set (gathering every $0 < q < 1$) and one infinite set (gathering every $q \geq 1$).

$\mathbb{Q}$	**0**	**0**	**0**	**…**	$\updownarrow$
∞	1	0,5	0,33	…	**0**
∞	2	1	0,66	…	**0**
∞	3	1,5	1	…	**0**
…	…	…	…	1	**…**
$\updownarrow$	**∞**	**∞**	**∞**	**…**	$\mathbb{Q}$

Table 5: Translating $\mathbb{Q}$

55 Here I represent natural numbers only, in order to simplify the visualization.

Stating $\{\mathbb{Q} = {}^{\mathbb{Z}}/_{\mathbb{Z}}\} \leftrightarrow \{\mathbb{Q} = {}^{\mathbb{N}}/_{\mathbb{N}} + {}^{-\mathbb{N}}/_{\mathbb{N}} + {}^{\mathbb{N}}/_{-\mathbb{N}} + {}^{-\mathbb{N}}/_{-\mathbb{N}}\}$ means that the ordinal property of $\mathbb{Q}$ doubles the ordinal property of $\mathbb{Z}$ (chap. 2.1.2), viz. $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ identify different levels or ranges of infinite. Moreover, I cannot really order items in $\mathbb{Q}$ along a positional line (like the lines given in fig. 5 and 7) because every gap in the growing series (${}^{1}/_{1} \ldots {}^{2}/_{1} \ldots {}^{3}/_{1} \ldots$) should be filled with other (infinite) series of fractions, mixing increasing an decreasing quotients (${}^{1}/_{1} \ldots {}^{2}/_{p} \ldots {}^{2}/_{q} \ldots {}^{2}/_{1}$): the ordinal property of numbers collapses in $\mathbb{Q}$ because of the impossibility to respect any linear order. That way, $\mathbb{Q}$ reveals that linear order does not matter when conscious mind grasps the structure of unconscious: in other terms, conscious mind differentiates a (con)fused continuum, putting *Asymmetry* in the place of *Symmetry*, on a relative basis (viz. a basis relative to each subjective perspective) because, through semiosis, every Symbol (viz. every-thing) could stand for every partial Thought taken from a complex undifferentiated whole. That is why I can state $\{{}^{\infty}/_{0} = {}^{0}/_{\infty}\} \leftrightarrow \{{}^{0}/_{0} = {}^{\infty}/_{\infty}\}$ in order to confine (in tab. 5) the framework of $\mathbb{Q}$ (from tab. 4): conscious linear order in $\mathbb{Q}$ must be replaced by a patchwork (or an arbitrary pattern through a patchwork), in order "to transmit the appearance of unconscious symmetry" (Matte Blanco 1975), as soon as I delve into developing $\mathbb{Q}$ like the result of recursive semiosis.

Moreover, recursive *Generalization* of sets (viz. abstracting the structure of a complex set like $\mathbb{Q}$ and acquiring it as an item, via the sign $\mathbb{Q}$) allows the *Symmetry principle* to suggested the idea of a set opposite to $\mathbb{Q}$ (e.g., $-\mathbb{Q}$), just like every item in $\mathbb{N}$ has its counterpart ($-\mathbb{N}$) in $\mathbb{Z}$: ***irrational numbers*** ($\mathbb{R}-\mathbb{Q}$) identify entities that cannot be expressed in terms of ratios. Hence irrational numbers fill the set of all the real numbers ($\mathbb{R}$, chap. 2.1.4), excluding the rational numbers ($-\mathbb{Q}$). Choike (1980) recorded in ancient Greece that idea had to be constructed via geometric representations of ratios between segments (viz. relating sides of triangles), in the absence of a formal symbolic language (long to be developed in the modernity). First of all, Choike proved how ancient philosophers believed that "all is number": a statement meaning a belief recurring throughout history in societies organized around a State, viz. on economics ideas of exchange and balance[56], that sprouting from the *Symmetry* innate to mathematics as a reflection of the unconscious, on which basis everybody deploys a set of (embodied) tools apt to grasp reality[57]. Then Choike proved how ancient scholars needed to deploy metaphors representing numbers (viz. perceptual cognitive stimuli), in order to manage mathematical concepts: that meaning mathematics is a semiosis because the management of Symbols is required in order to acquire mathematical Thoughts; and higher mathematics (from deployment of $\mathbb{Q}$ to quantum statistics) has been developed only on the basis of a structured specific language. Then Choike showed how ancient philosophers reiterated a series of fractions, operating what Lakoff/Nuñez (2000) called the *Basic Metaphor of Infinity*, that way seeding the soil for the "calculus of infinitesimals" that would sprout through the Age of Reason.

In the end, (the ancient reasoning of) reiterating infinite ratios (as an unconscious tendency to recursive semiosis) returned a contradiction, for certain numbers, appearing as to be commensurable or rational (${}^{s}/_{r}$), revealed themselves to be un-commensurable or irrational: e.g., the square root of 2 should be a rational number ($\sqrt{2}=\frac{s}{r}$) greater than 1 and smaller than 2 ($1 < \sqrt{2} < 2$) because $\{1^2 = 1\} \leftrightarrow \{1^2 \neq 2\}$; therefore $\{ 2=\frac{s^2}{r^2} \} \leftrightarrow \{2r^2 = s^2\}$, that implying both s^2 and r^2 are even numbers because $s^2 = 2r^2$ implies any $2n$ is even, therefore s is also even because any product *even*×*even* = *even*,

56 The Egyptian *Ahmes papyrus* stated that belief long before Greek philosophers.

57 On the other hand, societies "without a State" (Staid 2015) expressed balance and Symmetry deploying magical thinking.

thus (s being even) $s = 2n$ and $\{ 2=\frac{s^2}{r^2} \} \leftrightarrow \{ 2=\frac{(2n)^2}{r^2} \} \leftrightarrow \{2r^2 = 4n^2\} \leftrightarrow \{r^2 = 2n^2\}$, thus r^2 is even; hence $\sqrt{2}\neq\frac{s}{r}$ because the division of two even numbers must return another even number, which is not the case of $\sqrt{2}$, 2 being the smallest even natural number. That conclusion contradicts the initial assumption ($\sqrt{2}=\frac{s}{r}$), so that $\sqrt{2}$ cannot be expressed as a rational number: it is *irrational*, $\sqrt{2}\neq\frac{s}{r}$.

That *Asymmetric* conclusion is possible only delving into a rational management of semiosis, which is possible only sprouting a symbolic language from unconscious associations between real Referents and real Symbols. In the end, conscious mind claims for every entity to be *generalized* as a real phenomenon, via metaphors[58], even if an entity (like $-\mathbb{Q}$) has been conceived as a pure abstract *Condensation* of Thoughts.

2.1.4. Real numbers

Real numbers ($\mathbb{R}$) define *objective* numerical entities: numbers that can be found and grasped in real life, like digits (viz. fingers) and patterns inner to real phenomena, grasped through intellectual experiences developed around actual cognition. Also natural numbers ($\mathbb{N}$), integers ($\mathbb{Z}$) and rational numbers ($\mathbb{Q}$) result from cognitive and intellectual experiences, thus $\mathbb{R}$ includes $\mathbb{N}$ (fingers are real), $\mathbb{Z}$ (subtractions are real experiences) and $\mathbb{Q}$ (divisions and ratios are real processes). But elicitation of specific numbers from $\mathbb{R}$ requires complex intellectual activities: specific real numbers must be searched and acquired analyzing the structure of reality, where the senses collect data (Referents and Symbols) that the intellect interprets, assembling higher ideas and entities (Thoughts). Hence, real numbers represent inner core properties of nature, as they result disclosing inner patterns of the way human mind perceives and represents natural phenomena: real numbers provide conscious mind for reading the structure of unconscious or, at least, for grasping a bit of the unconscious, because $\mathbb{R}$ results from a volitional symbolic manipulation of other numbers sprouted from unconscious.

For instance, every time I take any line (e.g., a twine) and bend it in the shape of a circle (c), and then I measure its diameter and its radius ($d = 2r$), I always get the same *real* number (π) calculating the ratio of the circle and its diameter ($\pi = {}^{c}/_{d}$): knowing I cannot divide a real circle by a real radius (but I can divide the respective numerical lengths), I know that any geometrical length (a symbol or a number c) maintains a specific rapport (another symbol or number π) with another geometrical length (the number r): given any length (c), I can always compute a specific length ($r = {}^{c}/_{2\pi}$), so that π recurs *for real* in every measurement of circles and diameters and in every computation, transcending[59] rational calculus, for I cannot define π as a ratio between two specific numbers (while, e.g., $2 = {}^{8}/_{4}$), but I must define π as a ratio between two specific *entities* (c and r) translated in numbers. That way, π results as a property of the very inner structure of reality or as the way human mind sets up representations of reality: hence scholars, from Lambert (1768) on, defined π a ***transcendental number***, meaning real numbers represent properties of real processes operating (certain) rapports between mind and reality. Then *Symmetry principle*, *generalizing* $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ through the collective class of numbers, *condenses* them into $\mathbb{R}$: every number (viz. a Symbol) can represent something (viz. a Referent) revealing an idea (viz. a Thought), which is the foundation of numerology and Kabbalah[60].

58 That is the point in Lakoff/Nuñez (2000).

59 The verb *to transcend* blends Latin adverb *trans* ("beyond") and verb *scandēre* ("to ascend", "to climb"), hence it means "to go beyond" or "to exceed".

60 Hebrew noun *Qabālā* ("correspondence") means the ability and the habit of human mind in semiotics, linking Symbols (viz. numbers), Referents and Thoughts.

Moreover, certain real numbers (like α, π and e) convey evident symbolic properties, thus they fascinate scholars: specific real numbers develop infinite decimal strings (e.g., π = 3.14159265…), representing unconscious embodied and recursive abilities of processing infinite series (e.g., $e = \sum_{n=0}^{\infty} \frac{1}{n!}$); real numbers represent (or they visualize) an inner core property of the unconscious. That could be why Cantor wrote “I see it, but I don’t believe it” in his well known letter (1877 June 29) to Dedekind: he saw an argument proving a bijective relation between a finite set (viz. conscious) and an infinite set (viz. unconscious), but he needed some proof in order to believe it (Gouvêa 2011), i.e. in order to believe in the possibility to map infinite via finite tools, contradicting the then dominant framework provided by Kant (1781).

Yet conscious mind often returns results molded by the *Symmetric* structure of unconscious. E.g., Sommerfeld (1919) introduced a *fine-structure constant* (α), describing electrons (e) in the first orbit of hydrogen atoms: $\alpha = \frac{e^2}{4\pi\hbar}$, a ratio that scholars approximate such as $\alpha \approx \frac{1}{137}$; while calculus returns a very specific but uncountable value for the denominator, $4\pi\hbar$ = 137.035999084… ($\hbar = \frac{h}{2\pi}$ and π itself being infinite decimal strings). That approximation returns a number (α) with a very *Symmetric* structure, $\frac{1}{137}$ = 0.007299270072992700… where a double reflection elicits an infinite series: firstly the reflection of the zero (0.0), then the series of recursive reflections of a palindrome number, {07299270 | 07299270} ↔ {0729 | 9270}. Unconscious perceives the fine-structure constant α = 0.0072992700… as a *Symmetric* value: maybe scholars, like Barone/Dirac (2019: 80-84), should not be surprised in finding a *Symmetric* representation employed as a fundamental structure of the universe[61], because that is a conscious numerical translation of a property of unconscious structure of human representation of the universe, which is a *Symmetric* representation of that.

Indeed, *Symmetry principle*, *transferring* unconscious properties into the conscious mind, allows *real* arithmetic divisions by 0 and by ∞, even if radical theory refuses to compute divisions like ${}^{n}/_{0}$ and ${}^{n}/_{\infty}$ (chap. 1.5): that crucial remark shows how unconscious intuitions or suggestions strive to acquire an identification by the means of conscious mind. Scholars had to develop a specific tool, in order to admit both operations ${}^{n}/_{0} = \infty$ (returning a real value, rather than the trend given by $\lim_{r\to 0} {}^{n}/_{r} = \infty$ or the approximation given by ${}^{n}/_{0} \approx \infty$) and ${}^{n}/_{\infty} = 0$ (another real value, rather than the trend given by $\lim_{r\to\infty} {}^{n}/_{r} = 0$, thus ${}^{n}/_{\infty} \approx \infty$). The tool relies indeed on the idea of a *Symmetric* infinite continuum without boundaries, thus without + and – properties given in $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$ as the limits of ∞: **+** and **–** signs collapse through infinite limits ($-\infty$ and $+\infty$ collapse into ∞), since infinity has no boundaries, thus resulting tendencies ($\lim_{r\to 0} {}^{n}/_{r} = \infty$ and $\lim_{r\to\infty} {}^{n}/_{r} = 0$) are transferred to + and – properties of $\mathbb{R}$, so that + and – are acquired as tendencies, developing and pointing to the same only limit (∞, rather than $+\infty$ or $-\infty$). Hence, the linear representation given in fig. 7 can be reformulated as a ***projectively extended line*** (fig. 9), viz. a positional line (<——> from fig. 6 and 7) *projected*[62] into a circle, *condensing* the two infinite entities ($+\infty$ and $-\infty$) into one single entity (∞), collapsing (viz. *generalizing*) the distinction between $+\infty$ and $-\infty$. That way, ∞ being a point on a circle (rather than a tendency limit of an infinite line), ∞ can be managed as a real position opposed to 0 or as a *Symmetric* reflection of 0, just like –1 results as a reflection of 1 in $\mathbb{Z}$ (chap.

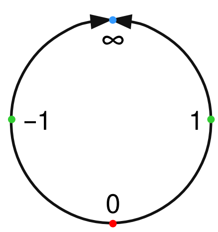

Figure 9: Projectively Extended Real Line

61 I delved into that topic in Rossi (2019-2020: 159-160).
62 Mathematicians employed indeed a Freudian expression here.

2.1.2)[63]. Those two examples (–1 and 1) in fig. 9 shows how an infinite set of rational numbers steps between every two integers in $\mathbb{R}$ (e.g., 0 and 1 contain an infinite set of rational numbers): the projectively extended real line (thus $\mathbb{R}$) identifies infinite infinite sets in one single infinite set (just like $\mathbb{Q}$ identifies different levels of infinity), like $0\rightarrow1$ (i.e. infinite positive real numbers smaller than 1), followed by $1\rightarrow2$ (viz. infinite positive real numbers greater than 1 and smaller than 2), then $2\rightarrow3$ and so on, until $r\rightarrow\infty$; and their opposite counterparts ($0\rightarrow-1$ and $-1\rightarrow\infty$). That way, $\mathbb{R}$ can be visualized in the accessible terms of $\mathbb{Z}$: recursive semiosis (*projecting* the visualization of $\mathbb{Z}$ into the structure of $\mathbb{R}$) employs conscious metaphors in order to acquire unconscious structure.

That *Projection* conveys a symbolic meaning, essential when operating with $\mathbb{R}$: every circularity *condenses* the idea of infinity and the idea of null void. On one hand, humanity always visualized recursive and never ending processes like cycles of mirroring items (e.g., the seasons of the year, with summer/winter and spring/autumn dyads), with every portion of a cycle mirroring its counterpart (e.g., day mirrors night, life mirrors death, etc.); on another hand, zero is a circle (prop. [2]), and every point (n) in an infinite ring structure has its *Symmetric* counterpart ($-n$) on the other side of the ring, the both counterparts annihilating one another ($n-n = 0$). That way, 0 and ∞ face one another as opposite natural poles: it is an unconscious structure of human cognition and epistemology[64], but that natural mindset rely on an inner complexity.

2.1.5. Complex numbers

Teachers use to teach that, developing quadratic equations ($ap^2+bp+c = 0$), scholars had to find a solution to problems like $p^2 = -n$, that has no real solution ($p \notin \mathbb{R}$) because the square root of a negative number ($p = \sqrt{-n}$) disregards elementary rules of arithmetic: the square of a number is always positive, $\{p^2 = n\} \leftrightarrow \{n = (-p)^2\}$ (chap. 1.4). Yet, decomposing the square root as $\{\sqrt{-n} = \sqrt{n}\times\sqrt{-1}\} \leftrightarrow \{\sqrt{-n} = \sqrt{n}\times i\}$, Cardano (1545) and Bombelli (1572) introduced an **imaginary unit** ($i = \sqrt{-1}$) implied in defining complex numbers ($\mathbb{C}$), making it possible to solve exponential equations.

Aside of that common teaching strategy, I think that semiotics catches the meaning of i directly through the meaning of exponentiation: quadratic exponentiation (p^2), for the sake of simplicity, even if the same argumentation validates higher exponentiation.

[23] $$\{p^2 = n\} \leftrightarrow \{1\times p\times p = n\}$$

The statement [23] means that $p^2 = p\times p$ is a transformation (chap. 1.4) turning 1 into n. E.g., the solution $p = (3, -3)$ to the problem $p^2 = 9$ represents a process (p) to be executed twice ($p^2 = 3\times3$) in order to transform 1 into 9.
Indeed: $\{1\times(3\times3) = 9\} \leftrightarrow \{9 = 1\times(-3)\times(-3)\}$.

The same idea applies to negative numbers resulting from exponentiation:

[24] $$\{p^2 = -n\} \leftrightarrow \{1\times p\times p = -n\}$$

The solution to the problem [24] (viz. finding some process $p\times p$ that transforms 1 into $-n$) exceeds the domain of $\mathbb{R}$, for every square, $\{p^2 = p\times p\}$ or $\{p^2 = (-p)\times(-p)\}$, returns a positive number (n): the solution ($p = i$) exceeds any

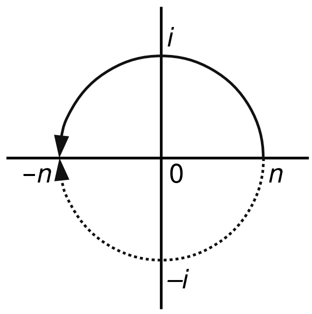

Figure 10: Imaginary Unit

63 That remark (given in fig. 9) explains complex numbers (chap. 2.1.5).
64 Lévi-Strauss evidenced that tendency in all his research.

real (viz. cognitive) phenomenon, therefor the solution must be based on some *imaginary* entity (i). Scholars visualized the solution as a spatial metaphor implying rotation (fig. 10): a 180° rotation *displaces* n to its *Symmetric* counterpart $-n$ (like –1 from 1 in fig. 7). Hence, the solution to the problem [24] is an **imaginary transformation** (i) consisting in a 90° rotation, so that $i^2 = i \times i$ operates a 180° rotation[65]. Namely:

[25] $$\{i^2 = \circlearrowleft_{180°}\} \leftrightarrow \{i = \circlearrowleft_{90°}\}$$

[26] $$\{i^2 = -1\} \leftrightarrow \{i = \sqrt{-1}\} \rightarrow i = \{\pm 1, \pm 1\}$$

The statement [26] implies $\{1 \times (-1) = -1\} \leftrightarrow \{-1 = (-1) \times 1\}$ (see chap. 2.3 about that *Symmetry*). That solution reveals the **symbolic meaning** of the proposition [6] about **×**, for i^2 multiplies $+1 \times (-1)$, applying the **+** of one ±1 to the **–** of the other ±1, thus symbolically interpolating a **+** toward a **–** ($^{+}\diagdown_{-}$) and vice versa ($_{-}\diagup^{+}$): the two slants combining one another into the multiplication sign (**** + **/** = **×**) via the *Symmetry principle* (chap. 1.4).

The unit i being an imaginary construct, a complex number can be thought of as a hybrid made out of real numbers ($\mathbb{R}$, represented on the horizontal axis in fig. 10 and 11) and of spatial *Transference* (the application of a geometric rotation to $\mathbb{R}$). Every complex number ($z \in \mathbb{C}$) identifies two parts: a *real part* ($a \in \mathbb{R}$) and an *imaginary part* (bi, with i being the *imaginary unit*, represented on the vertical axis in fig. 10 and 11, and $b \in \mathbb{R}$ stretching or shrinking i along the vertical axis). Argand (1806) visualized that *Condensation* of two different parameters into one single number as a vector resulting from the *Projection* of each parameter into a coordinate system (fig. 11):

[27] $$z = a+bi$$

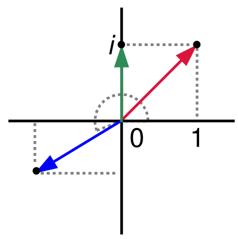

Figure 11: Complex Numbers

On that basis, scholars operate with complex numbers when they need to rotate (conceptual) objects: that way, $\{z = a+0i\} \leftrightarrow \{a+0i = \circlearrowleft_{0°}\}$ means no rotation at all (like the point 1 on real axis in fig. 11); while $\{z = 1+i\} \leftrightarrow \{1+i = \circlearrowleft_{45°}\}$ means a diagonal vector (like the red vector in fig. 11)[66]; then $\{z = 0+i\} \leftrightarrow \{0+i = \circlearrowleft_{90°}\}$ is an imaginary number located on the vertical axis (like the green vector pointing i in fig. 11); and $\{z = 0+i^2\} \leftrightarrow \{0+i^2 = \circlearrowleft_{180°}\} \leftrightarrow \{z = -1\}$ is just an example of general $-n$ point in fig. 10; and so on. While various combinations of $a+bi$ (e.g., the blue vector $z = -1-0{,}7i$ in fig. 11) locate peculiar degrees of the rotation process on a plane, indeed named after Argand (1806). All that meaning the sign **+** in complex numbers does not convey addition (as well as **–** does not convey subtraction), but it conveys a *Condensation* of two values (real and imaginary), even if syntax rules regarding **+** and **–** apply in $\mathbb{C}$ as well as in $\mathbb{R}$.

Incidentally, another *Generalization* process makes complex numbers collect real numbers (i.e. $\mathbb{R} \in \mathbb{C}$), for the imaginary part (bi) can be eliminated out of z (i.e. $a = a+0i$): e.g., $\{-4 = 4 \times (0+i^2)\} \leftrightarrow \{-4 = (0 \times 4)+(-1 \times 4)\} \leftrightarrow \{-4 = 0-4\}$.

On that basis, Lakoff/Nuñez (2000) discussed negative integers ($-\mathbb{N}$) and complex numbers ($\mathbb{C}$) as the results of a cognitive competence in rotating objects[67]. Nevertheless, the *Symmetry principle* operates as an unconscious **mirroring process**, while rotation operates as a conscious *Asymmetric* process. Both the rotation and the reflection

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65 See chap. 1.4 to verify why $180° = i^2$, rather than $180° = 2i$.

66 See chap. 1.4 about **×** meaning a 45° rotation of **+**.

67 They referred to Shepard/Metzler (1971) and Shepard/Cooper (1981).

rely on "embodied" experiences and abilities, and that could be a reason why of the confusion between the two, given the circular metaphor employed in visualizing $\mathbb{C}$.

In order to clarify the distinction between rotation and mirroring, I can validate the following procedure. If I think of *i* as of a two-dimensional symbolic metaphor applied to some asymmetric item (→), I see the following process at work: $0i$ = → (the item → goes under no process); i = →|← (the item → goes under reflection); i^2 = ← (the item → gets copied into its reflection ←, *likewise a 180° rotation* of the source item); i^3 = ←|→ (the reflected item ← goes under another reflection); i^4 = → (the reflected item ← gets copied into its reflection →, oriented just like the original item, *likewise a 360° rotation* of the source item). But, substituting the asymmetric item (→) with a slightly different asymmetric item (↗), I can evidence why and how reflection (fig. 13) differs from rotation (fig. 12).

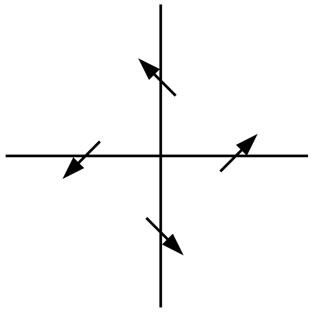

***Figure 12: Rotational* i**

Reflection operates a *Symmetrization* competence embodied in the unconscious: both the contralateral structure of nervous system[68] and the function of mirror-neuron system[69] operate on unconscious level (i.e. I perceive reflections as a result of embodied cognitive processes), while rotation operates on conscious level (i.e. I have to visualize a rotation in my mind[70]). Moreover, both the reflection and the rotation processes result from embodied experiences, especially from cognitive experiences involving my limbs and my hands, but rotation cannot help me "transforming" (as an internal representation) my right hand into my left hand, while I see and feel my left hand "reflecting" my right hand. The imaginary unit *i transfers* that embodied competences into an algebraic metaphor that visualizes a reflection as a result of a dual process, for $i = \{\pm 1, \pm 1\}$. I can deploy ↖ from ↗ (both on *real* axis) passing through two similar competences (like $-n$ results both via i or via $-i$ in fig. 10): on one hand, i represents the reflection property (on the upper side of fig. 13) and, on the other hand, $-i$ represents the rotation competence as a reflection of the reflection property (on the lower side of fig. 13), for $\{i^3 = i^2 \times i\} \leftrightarrow \{i^2 \times i = -i\}$.

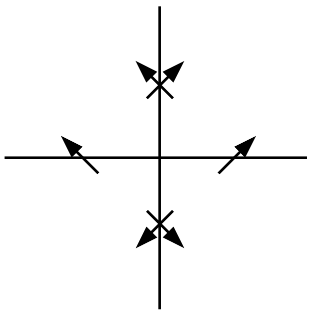

***Figure 13: Reflective* i**

Human visual system operates on the basis of the diagram depicted in fig. 13: apart from the symmetric function of the optic chiasm (which crosses optic nerves, sending left nerves to the right side of the visual cortex and vice versa), the retina receives an upside down reflected image of real objects[71], via the refraction properties of eye lenses; then the brain processes that information, assembling data and reconstructing (and recognizing) images as patterns of data stored and retrieved from memory. Eye lenses affect the path covered by rays of light, refracting (viz. flipping) and shrinking the visual images of real objects with respect to the focal properties of lenses: every reflection in eye lenses occurs through a focal point (*F* in fig. 14) that superposes input data to output results or, in other terms, there is a point in space where the input light data of an object, travel-

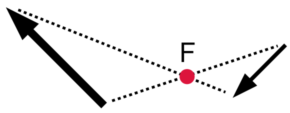

Figure 14: Symmetrization in Refraction

68 Loosemore (2011) suggested that contralaterality depends on the evolution of visual system.

69 Rizzolatti/Craighero (2004).

70 That is the point in Shepard/Metzler (1971) and Shepard/Cooper (1981).

71 A vertical reflection (fig. 14) works just like a horizontal reflection (fig. 13).

ing to the retina, get flipped around themselves[72]. The fig. 13 illustrates how *i* (on the vertical axis) operates as a superposition state or as a transition state, or a state happening in the middle of a *Transference* process between real items (on the horizontal axis): everybody experiences that *Transference* in the very act of seeing; *i* operates a reflective function, superposing inputs and outputs (±), just like the brain operates the same (embodied) function.

2.2. Letters

Mathematics being a system of signs (chap. 1.1), it expresses two essential functions of language: developing economics and playing games[73]. Brown (1959) explained that children acquire a spoken language as a tool for facilitating the satisfaction of physiological needs; a tool acquired playing with sounds and inter-acting with adults in a game apt to construct meanings, exploring connections between Symbols and Referents, developing Thoughts about the game itself[74], about existence[75], and about economics[76]. Then language reveals its aptitude for playing games in the frame of pure fun and pleasure[77], in the frame of economics[78], of social relationships[79], up to the frame of culture, which is also a game of combinations and permutations of signs, ideas and discourses.

A peculiar trait of mathematical language, that specifically sprouted from the soil of culture, is that mathematics exists only in the domain of writing, whereas every other language sprouted in the domain of sound: people speaking and talking, artists playing music and plays, animals calling love and crying help, soldiers deploying orders in battlefileds, etc. Written language assembles all sorts of auditory and visual data in the mold of culture, enhancing social memory and preserving knowledge; whereas mathematics *relies* on **written information**, for it cannot be discussed outside of a system of graphical signs because mathematical ideas sprout from visual management of signs[80]. Thus, mathematics developed a full language, assembled from atomic symbols: numbers (chap. 2.1) and operative signs (chap. 1) that algebra *projects* out of the domain of arithmetic and into the domain of spoken languages, borrowing letters from alphabets and deploying them as Symbols that *condense* multiple and *generalized* Referents (e.g., $\boldsymbol{x}$ usually denotes variable numbers, Greek $\boldsymbol{\delta}$ denotes a difference, Hebrew $\aleph$ denotes

72 The mathematical concept of *image* evokes that fact: the image of a function is given by all the outputs produced by the calculations, just like every lens (viz. a function) returns an output image of an object (viz. an input).

73 Game theory *condenses* that two functions under mathematical models, acknowledging the relevance of information and communication or, generally, of language.

74 Children learn that random combinations of sounds and imitation processes pay social rewards.

75 Wittgenstein (1921) suggested why language affects the perception of reality.

76 Berne (1953) explained why every transaction is mediated firstly by transactions of words.

77 Brown (1959) suggested why people gain satisfaction of their unconscious needs via gaming, implying the very multiple meaning of the verb *to play* (a record or a musical instrument, a sport or a match, a role or an act, etc.).

78 Von Neuman/Morgenstern (1944) overlooked how their theory of games conflicts with the psychology of organized fun and intrinsic pleasure of playing: Caillois (1967) explained how mathematics applied to games destroys the principle of pleasure innate to games, as long as it destroys the uncertainty underlying the very urgency of players.

79 Luhmann (1982) explained why love is a medium or a specific symbolic code apt to satisfy individuals while granting perpetuation and stability of social systems: love is a game of communication, developing a semantic and a syntax, that (in Freudian topics) grants pleasure principle with respect to reality principle.

80 Lakoff/Nuñez (2000) discussed conceptual metaphors as essential tools for acquiring mathematics because spatial metaphors operate as visual clues needed to access numbers and calculations.

the size of infinite sets, etc.); and, conversely, letters lose their natural function (viz. combining sounds into words) in order to gain a new symbolic function, for they no longer are phonetic components of objects, as they are mathematical objects themselves[81].

That way, algebra operates another recursive semiosis, *displacing* Referents of given Symbols and condensing Thoughts. That is a process necessary to manage unknown values and variables, represented via letters: numbers are given in algebra as known values, being Thoughts embodied in cognitive experience of Referents, while unknown values or generic values are undefined Thoughts (viz. meta-Thoughts or Thoughts of Thoughts) of undefined Referents (viz. meta-Referents or Thoughts of Referents) to be represented via Symbols given outside of the domain of arithmetic (viz. meta-Symbols or Symbols of meta-Thoughts and meta-Referents). The letters *condensate* every possible number into one symbol, *generalizing* the idea of a numerical entity: apart from what they represent (e.g., $\boldsymbol{a}$ for a datum, $\boldsymbol{x}$ for unknown values, $\boldsymbol{\Sigma}$ for a summation), **the letters represent qualities** or symbolic properties common to every mathematical item, *displacing* arithmetic out of the domain of actual experience and into the domain of abstract thinking, entering algebra.

Algebraic letters reveal an undefined domain of thinking. Managing letters in algebraic expression, I consciously visualize and manipulate an undefined or fuzzy level of thinking, where semiosis codifies and allows to process *Asymmetric* relations on a *Symmetric* continuum[82]: equations visualize in conscious (viz. intellectual) terms the blurry (fuzzy, hazy, etc.) property of unconscious, where everything stands (or could stand) for everything else, where every Symbol stands for every Referent (viz. a letter stands for every quality and for every number), *condensing* every Thought, where every item is *generalized* into a common class, and *Transference* and *Displacement* collapse[83], for there is no difference passing from one unconscious item to another, every idea being *projected* into one another.

2.3. Equations

Arithmetic manages numbers (chap. 2.1) on the basis of elementary operations (chap. 1), while algebra operates meta-processes on meta-entities (chap. 2.2): algebra operates in the domain of pure semiosis, for it *generalizes* cognitive Symbols out of the domain of Referents and into the domain of Thoughts (fig. 2), because algebra *displaces* quantities into the domain qualities; and it *displaces* calculations out of the domain of counting and ordering, into the domain of symbolic associations via the equality.

Algebra is the manipulation of equations: expressions employing = in order to describe some arithmetic relation (+, –, ×, ∕) between elements of a proposition (letters and numbers located on a specific position with respect to the sign =) and other elements (letters and numbers located on the other side of =). An equality is a representation of **unconscious *Symmetry principle* via conscious *Asymmetry***: an equality *generalizes* (elements of) different classes, *projecting* them into one common set and

81 *Projection* is a dual process: it always transforms at least two different items into a third item, *transferring* some characteristic of one item into the other.

82 E.g., $\{x = ay+b\} \leftrightarrow \{{}^{x-b}/_{a} = y\}$ employ different processes expressing an equality (chap. 1.6; 2.3).

83 Unconscious *Transference* and *Displacement* operate only with regard to conscious experience because inside the domain of unconscious any point or any item is the same: that is the idea of "infinite sets" behind Matte Blanco (1975), for a single dimensionless point is an infinity because every atomic part of infinity is that same point.

abolishing (incidental) differences between classes. An equation operates what Brown (1959/1985) defined as the historical conflict of human unconscious: it shows differences, negating unity via differentiation of elements, just in order to evidence their identity, via the *Absence of negation*. That meaning, the equivalence function is a semiosis operated by conscious mind via the *Generalization* of different Thoughts into single items or Symbols[84].

Matte Blanco (1975) explained the unconscious *Symmetry principle* consists in acquiring one information, then mirroring it as a representation (viz. a conceptual symbol) opposite to the information itself, assuming a dual structure for the organization of information: the *Symmetry principle* operates as an unconscious representation of information, as it follows the elementary limbic processes[85] based on aggression/getaway behaviors or fight/flight instincts or appreciation/refusal opinions and so on[86]. For instance: perceiving the symbolic information ←, my unconscious elicits its meaning (viz. *left-oriented*) comparing that information with its opposite or mirror element (→), so that the *Asymmetric* dyad ←|→ (necessary for acquiring meanings) collapses into the *Symmetric* continuum ⟵⟶ condensed into the unconscious[87] (chap. 2.1.2). The equality sign forces a representation of some identity between different elements, disposed on both sides of the sign (e.g., $x = y$): in the terms of Matte Blanco (1975), equality sign elicits (abstract) *Symmetry* where I perceive (factual) *Asymmetry*.

[28] $$\{x = y\} \leftrightarrow \{\boldsymbol{x} \neq \boldsymbol{y}\}$$

The proposition [28] represents how mathematics correlates conscious and unconscious mind via semiosis (fig. 1 and 2): two different Symbols ($\boldsymbol{x} \neq \boldsymbol{y}$) signify a substantial identity of Thoughts ($x = y$), that meaning the superficial difference between $\boldsymbol{x}$ and $\boldsymbol{y}$ signifies an undifferentiated continuum (=), implying the collapse of Referents. In other terms: different Symbols (***x** and **y***) express a single Thought (*x or y*) pertaining multiple Referents (*x and y*). Hence, via mathematics, conscious mind traces cognitive differences back to a continuous unity underlying the unconscious.

Simple algebraic expressions, like $x = y$, elicit a fundamental paradox, apt to represent (viz. to visualize or to objectify) the *Symmetry principle* that drives the unconscious: equations (=) imply differences (≠) between items that unconscious treats like identical and different *in the same time*, along with the principle of non-existence of time through unconscious mind, stated by Freud (1920). Moreover, in order to grasp unconscious processes, the conscious mind has to treat identical elements in equations like *Asymmetric* elements or different items, just because algebraic elements (must) occupy different positions in equations: the statement [28] clarifies the implicit difference in the equation "$x = y$", but it also implies differences in identities like "2 = 2", conveying **a positional difference**, because $\mathbf{2}_{\mathrm{Left}} \neq \mathbf{2}_{\mathrm{Right}}$. In more general terms:

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84 E.g. the word *frame* conveys different meanings (nouns and verbs), as well as the memory of a specific melody conveys different emotions and different images in my mind, as well as my memory of that specific melody simplifies the melody itself, omitting notes, harmony and timbres.

85 LeDoux (2000); Morgane/Galler/Mokler (2005).

86 Hall/Bodenhamer (1997) recorded a wide series of oppositions, distinctive of cognition and data processing, like up/down, left/right, towards/away (the three dyads correlate directly to space perception as symbolized in Cartesian coordinate system), in/out, good/bad, and so on. All of the dyadic categories can be identified in the structure of emotions discussed by Ekman/Friesen (1975), like the dyads happiness/sadness, anger/fear and surprise/disgust.

87 The same principle works with every dual experience, like ↑ and ↓ resolving into ↕, or + and – resolving into ±, or ○ and ● resolving into ●. Lakoff/Nuñez (2000) explained how *in* and *out* states resolve into a conceptual intersection (*in* ∩ *out*). Rossi (2019-2020) discussed how Taoist *tàijítú* Symbol (☯) recapitulates the unitary and the symmetric Thoughts relative to all opposing Referents.

[29] $\{\boldsymbol{x}\} \leftrightarrow \{x = x\} \leftrightarrow \{\boldsymbol{x}_{\mathrm{Left}} \neq \boldsymbol{x}_{\mathrm{Right}}\}$

Every letter ($\boldsymbol{x}$) conveying any meaning (x) can express a tautology ($x = x$), but the expression of that tautology employs more instances of that letter ($\boldsymbol{x}_{\mathrm{Left}} \neq \boldsymbol{x}_{\mathrm{Right}}$)[88]. The statement [29] means that unconscious *Symmetry* ($x = x$) has to be visualized via conscious *Asymmetry* ($\boldsymbol{x}_{\mathrm{Left}} \neq \boldsymbol{x}_{\mathrm{Right}}$) whenever the intellect tries to grasp it. *Displacement* is an inevitable conscious process, when the intellect delves into unconscious, because Symbols must be manipulated and referred one another in order to represent unconscious processes; but *Displacement* is also an inevitable unconscious process affecting cognition, because unconscious treats many different (perceived) items like if they were identical items or one continuum; and referring conscious mind to unconscious is inevitable when I operate mathematics, because the acquisition of numbers and of operations relies on unconscious processes (chap. 1 and 2.1). For instance, a tautology like "4 = 4" (exemplifying prop. [29]) can be integrated with arithmetical operations, like "4–3 = 4–3": that way, both "4 = x" and "4–3 = x" are legit statements, with respect to the general case $\{x = x\}$, because that means that I can *generalize* an arithmetic entity (4 or 4–3) via any Symbol ($\boldsymbol{x}$).

That *Generalization* of numbers (e.g., 4) into entities (x) via Symbols ($\boldsymbol{x}$) makes it possible to operate with Symbols as well as with numbers: a statement "4 = 4" implies an equation "4–4 = 0" as well as $\{x = x\} \leftrightarrow \{x–x = 0\}$ because of the *Symmetrization* and of the *Displacement* operated by unconscious on entities. That semiotic process is evidenced by **Dedekind-Peano axioms**: a set of definitions of $\mathbb{N}$ expressed in formal or symbolic logic, rather than in actual experience of items. Dedekind (1888) and Peano (1889) introduced the evidence of *Symmetry principle* being a framework for human ability to think of numbers and to process numbers. Their axiom 2, $\{x = x\}$, states that equality is a *reflexive* property of arithmetic: the key trait of *Symmetry principle* requires *Displacement* (substituting one item with another item), just like the embodied experiences acquaint ourselves with the evidence of right hand reflecting left hand (Matte Blanco 1975), as much as every other mirroring experience does. Axiom 2 states ubiquity of identity, *displacing* x through the two sides of the equation, like stated in the proposition [29]: the reader identifies and "accepts" $\boldsymbol{x}$ to be put in two different places at the same time, on the basis of our innate competence in thinking of *Symmetric* states and in looking for similarities in different things; "different things" meaning "objects occupying different coordinates in spacetime", which is a common cognitive experience; i.e. the unconscious continuity makes it possible to think $x = x$ even if $\boldsymbol{x}_{\mathrm{Left}} \neq \boldsymbol{x}_{\mathrm{Right}}$. Reading a statement "$x = x$", the reader sees the Symbol $\boldsymbol{x}$ in two different positions, recognizing multiple instances of $\boldsymbol{x}$, actually different one another in spacetime[89], while semiosis (fig. 1) elicits one same Thought for x.

Dedekind-Peano axiom 3, $\{x = y\} \leftrightarrow \{y = x\}$, states that equality is a *Symmetric* property of algebra: the *Displacement* mirrors identities, implying *Symmetry* of perceptions[90]; different places (the right sides of equations with respect to the left sides) identify same items, like different Symbols identify same Thoughts (as discussed about axiom 2, above, and as in considering, e.g., **0,999** = 1). The unconscious *transfers* cognitive experiences (recognizable in the body and in other mirroring experiences) from

88 That idea leads to the special case of single Symbols *condensing* different Thoughts ($x \neq x$).

89 See the *many* concept discussed in chap. 2.1.1.

90 I believe that mirror-neuron system (Rizzolatti/Craighero 2004) provided a partial evidence of how the *Symmetry principle* operates, as discussed by Matte Blanco (1975), with respect to the unconscious competence in mirroring both actual and ideal items: it bases our innate conscious competence in mentally visualizing rotations of objects (Shepard/Metzler 1971; Shepard/Cooper 1981), thus in representing negative numbers (Lakoff/Nuñez 2000).

actual events to internal mind representations, depicted and organized as a continuum mixing up polarities: left and right, as much as up and down, good and evil, etc. disappear in the unconscious continuity; while conscious experience differentiates unconscious continuum creating opposite polarities, in order to manage and to interact with reality (i.e. the Freudian *reality principle*).

Dedekind-Peano axiom 4, $\{\{x = y\}, \{y = z\}\} \leftrightarrow \{x = z\}$, states that equality is *transitive* or, in other terms, it applies to every Symbol involved in a proposition: semiosis (chap. 1.1) is the unconscious process that *transfers* Symbols, like $\boldsymbol{x}$, from an entity, like $\{x = y\}$, to another entity, like $\{y = z\}$, returning a new entity $\{x = z\}$; just like the unconscious *transfers* Symbols from one side to the other side of equations, like $\{x = x\} \leftrightarrow \{x–x = 0\}$. That axiom evidences two remarks: firstly, *displacing* an item leaves a void location (0, revealing the meaning of **0**); secondly, the *Displacement* affects items (numbers, letters) *and* processes (operation signs), but *displaced* processes get turned in their opposite polarity, i.e. $\{+x = +x\} \leftrightarrow \{+x–x = 0\}$, for addition and subtraction, as well as $\{x = ax\} \leftrightarrow \{{}^{x}/_{x} = a\}$, for multiplication and division. That is because of the *Symmetrization* of processes (chap. 2.1.2): an item (x) is visualized as a point in a continuum (where every item stands for every other item), while a process (+, –, ×, ∕) gives a meaning to an item, establishing a relation between that point and another point.

Matte Blanco (1975) stated that sequences are impossible when unconscious *Symmetry* is applied, that is because a sequence is a juxtaposition of items, meaning an order only outside of an undifferentiated continuum (——); whereas conscious processes alter the continuum, altering its intellectual representation, thus imposing an order or a meaning (<——>). When the conscious mind *displaces* Symbols along equations (e.g., $\{x = x\} \leftrightarrow \{x–x = 0\}$) the arithmetic operations must be mirrored (viz. *Asymmetry* must be consciously imposed), in order to preserve the meaning of a proposition, because the reflection preserves the orientation of conscious mind along the “spatial metaphors” that the unconscious employs to represent mathematical concepts (Lakoff/ Nuñez 1999): paradoxically, **conscious *Symmetry* renders *Asymmetry***.

On the other hand, unconscious *Symmetryzation* and *Displacement* operate allowing multiple arrangements of items throughout equalities:

[30] $$\{2 = 1+1\} \leftrightarrow \{–1–1 = –2\}$$

[31] $$\{2 = 1+1\} \leftrightarrow \{1+1 = 2\}$$

[32] $$\{2 = 1+1\} \leftrightarrow \{\mathbf{2+2 = 1}\}$$

[33] $$\{2 = 1+1\} \leftrightarrow \{\mathbf{–2–2 = –1}\}$$

The equivalences above here mirror one another via *Symmetric* statements: [30] reflects the positioning of every item and it inverts the meaning of the operation signs, thus it is legit in algebra because it preserves meanings via mirror images in $\mathbb{Z}$ (chap. 2.1.2); [31] reflects the positioning of items only, preserving signs, thus it is legit in algebra because it preserves meanings in $\mathbb{N}$ (chap. 2.1.1), evidencing how equality abolishes the criteria of order (viz. left-to-right or right-to-left order of reading collapse through a *Symmetric* continuum); [32] reflects positioning of items and switches Symbols of current numbers taken from clusters (on the left side one cluster being “2” and the other cluster being “1+1”), not validating in algebra (because 2+2 = 4), but symbolically legit (because of recursive semiosis structure); then [33] inverts operation signs on [32], applying algebraic rules to symbolic continuity.

Even though statements [32] and [33] are inconsistent with arithmetic and logic, unconscious assumes them as legitimate or "symbolically possible": I need to consciously focus on the meaning of the statement [32], in order to grasp its fallacy, while I can accept its *Symmetric Gestalt*, given as an instance of a generic $\{\boldsymbol{x} = \boldsymbol{y}\} \leftrightarrow \{\boldsymbol{y} = \boldsymbol{x}\}$, given $\{\boldsymbol{x} = 1\textit{item}\}$ (**1** or **2**) and $\{\boldsymbol{y} = 2\textit{items}\}$ (**1±1** or **2±2**), that evidencing how Symbols can represent clusters and, conversely, how a cluster can be thought of as a Symbol via recursive semiosis. Moreover, should I translate the structure of the statements above here in a *generalized* formulation like $\{x = y+y\} \leftrightarrow \{x+x = y\}$, I return a legitimate statement in algebra because the variables (x and y) express two different equivalences ("$x = y+y$" and "$y = x+x$"), given x and y represent unknown numbers: $x = y+y$ could express $2 = 1+1$, which can be expressed too as $x+x = y$, because I can assume $x_{\text{Left}} = y_{\text{Right}}$ (being $\boldsymbol{x}_{\text{Left}} \neq \boldsymbol{x}_{\text{Right}}$) or, in other terms, I can impose symbolic *Asymmetry* reassuming $\{x = y+y\} \leftrightarrow \{z+z = y\}$ via "self-semiosis", i.e. via the same $\boldsymbol{x}$ and $\boldsymbol{y}$.

Korzybski (1933/1994: 577-579) showed how differential calculus relies on that self-semiotic process, implying $x = x_0$ and $x = x_1$ (or x_{Left} and x_{Right} or, more generally, x_a and x_b) at the same time: any single variable (x) represents a whole continuum, i.e. unconscious *condenses* a whole continuum into a Symbol. Equations represent general *functions* or processes applied to items: expressing "$y = f(x)$" means that I can compute any output value (y taken from a continuum) applying a specific function (f being an operation or a set of operations) to a specific input value (x taken form a continuum). Thus, **differential calculus** describes the inner structure of any given function (f), analyzing how its actual values (x and y) change (a "change" being the difference y_1-y_0) or, in other terms, analyzing how the function operates upon variables: recursive semiosis returns $\{\Delta x = x_1-x_0\} \leftrightarrow \{\Delta y = y_1-y_0\}$, Greek letter Δ meaning a *difference* in the continuity of a variable (x or y); and symbolic logic, operating on $y = f(x)$, gives $\{\Delta y = f(\Delta x)\} \leftrightarrow \{{}^{\Delta y}/_{\Delta x} = f\}$. Hence, differential calculus returns the rate (${}^{\Delta y}/_{\Delta x}$) describing the operative structure of a function (f): and a rate is an *Asymmetric* process (chap. 1.5) needed to describe a *Symmetric* structure.

Moreover, given minimal differences in continuity (represented by lowercase δ, instead of capital Δ), infinitesimal calculus measures very tiny rates of change in values ($\delta y = f(\delta x)$, given $\delta x \approx 0$), that meaning the possibility to approach the extreme limit given by ${}^{\delta y}/_{0} = \infty$: every number divided by zero returns the same result (∞), as discussed in chap. 2.1.4. Korzybsky (1933/1994: 582) suggested that "the whole psychologics of this process is intimately connected with the activities of the *nervous structure* and also with the structure of science": suggesting that (higher) mathematics represents or reflects the inner language of human mind; and suggesting that developing mathematics is a natural process given by unconscious; a process that required ages of symbolic manipulation throughout history, in order to deploy (and to keep deploying) a complete formal language that humanity acknowledges as a tool apt to acquire or "to possess" the world, as Brown (1959/1985) stated. Sciences acquired mathematics as *the* means apt to control nature (viz. "to sublimate" and "to alienate" it, in Freudian terms): hence, discussing the way natural sciences (like physics) and social sciences (like economics) employ mathematics could highlight peculiar traits of the unconscious structure of mainstream modern culture.

3. Inferences

Mathematics is **a language structured on duality**, expressing the dual structure of human mind (every dual polarization being structured on reflections or on mirror patterns, viz. on the *Symmetrization* of *Asymmetry*): positive and negative numbers (thus rapports between greater and smaller values[1]), odd and even numbers (a *quality* of *quantities* based on the possibility of dividing numbers by 2), equality and inequality, left and right members of expressions, etc., they all reflect (viz. they visualize and metaphorize) embodied spatial categories of perceptions, like left and right, up and down, towards and away, in and out, etc.; and that correlations are reflections pertaining both the conceptual nature of mathematics and its material body (viz. its aesthetic), while natural (spoken) languages are polysemous and they are developed on a very complex structure (e.g., paraverbal inflections affect meaning; single words convey multiple meanings; different syntactic structures convey identical meanings; etc.).

Taken the assumption of dual perceptions of reality[2], Durand (1963) reduced the categories of human imagery to ascending and descending polarities, respectively recalling Freudian *Sublimation* and *Libidinous* processes: mathematics, with its dual structure, via *Displacement* and *Transference*, evidences the same operation, conceptualizing cognitive data (viz. *sublimating* actual experiences, "moving up" matter, detaching spirit from flesh) and, on the other hand, measuring and recording abstract conceptions (viz. materializing and possessing ideal structures, "moving down" thoughts, visualizing them as real objects in order to manipulate them).

Visualizing equations, the intellect becomes aware of multiple Thoughts *condensed* into single Symbols; and transforming or manipulating equations, visualizing *Transference* and *Displacement*, the intellect becomes aware of the continuity of Referents (underlying perceptions) being differentiated by semiosis. In the same time, language (as a symbolic medium) *differentiates* and *generalizes* reality: Possati (2020) explained how language gives rise to subjectivity, any instrument or technology coming from an unconscious dynamic; a "resistance" to raw cognitive data (a resistance given by embodied mirroring abilities and by sensory-neural codification) results in unconscious *Symmetryzation* of *Asymmetric* cognition. Mathematics (being a language) is a tool risen from unconscious dynamics, yet showing clearly and, moreover, highlighting the evidences of the original "resistance": *Condensation*, *Displacement*, *Generalization*, *Projection*, *Transference*, *Absence of negation* and *Symmetryzation* are visible in mathematical signs and syntax (chap. 1 and 2); **mathematics evidences** that the "unconscious is at the same time what guides the game of combination and recombination of signifiers […] and what is repressed and censored by signifiers" (Possati 2020: 8). That is why mathematics is a language expressing the structure of unconscious processes operated on raw perceptions.

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1 Plato (IV sec. BC: 24) introduced the mathematical problem of intellectual relations between differences in qualities, implying differences in quantities: "if they did not abolish quantity, but allowed it and measure to make their appearance in the abode of the more and less, the emphatically and gently, those latter would be banished from their own proper place. When once they had accepted definite quantity, they would no longer be hotter or colder; for hotter and colder are always progressing and never stationary; but quantity is at rest and does not progress. By this reasoning hotter and its opposite are shown to be infinite".

2 Recently, Hameroff/Penrose (2014) developed their *Orchestrated Objective Reduction* model of consciousness essentially on the basis of dual states of tubulines (i.e. compounds structuring the cytoskeleton of neurons), caused by the electric polarization (+ or –) of the monomer of each tubuline.

That is to say how perceptions identify $\mathbb{N}$ as a cognitive *Asymmetric* set, under the metaphor —> (chap. 2.1.1), that allows to develop the *Symmetric* set $\mathbb{Z}$, under the metaphor <—>, resulting as a reflection or a mirror image of the basic property of $\mathbb{N}$ (chap. 2.1.2), i.e. reflecting additions via i^2 (chap. 2.1.5). That basic evolution in conceiving numbers evidences how perceiving real *Asymmetries* results in an innate tendency to *generalize* phenomena in the terms of unconscious *Symmetry*. Another evidence of that innate tendency is the *Symmetry* implied in the quotients resulting from basic fair divisions (chap. 1.4): given $\{\mathbb{Z} = {}_{\leftarrow}\mathbb{N}_{\rightarrow}\}$, $\mathbb{Z}$ results as a continuum split in two opposite or mirror continua ($-\mathbb{N}$ and $+\mathbb{N}$), where negative numbers rise in $\mathbb{Z}$ just like the adverb “no” rises in spoken language, both being a peculiarity of human beings. That is to say human psychology identifies a peculiar trait through the exclusive adoption of specific (linguistic) tools like, for instance, negative numbers and ratios (while other animals acquire only natural numbers[3]) or negative statements (while other animals cannot express negative statements[4]).

Brown (1959/1985) pointed out how the sciences based on mathematics (viz. the only dissertations acknowledged as truly scientific dissertations) describe a specific rapport between human consciousness and perception of reality: when algebra elicits a new formulation of a problem, it describes the structure of a new rapport between human mind and reality, rather than describing the structure of reality itself; reality being still an ambiguous (foggy, chaotic, blurry, etc.) structure of data, even after millennia of evolution in critical thinking. Indeed, mathematics evidences the evolution of a universal language, thus a path in the evolution of the human being or, at least, of human cultures: mathematical language has been affected by native languages of scholars all around the world and through the ages[5], as Cajori (1928) pointed out. Thus, the evolution of science (viz. the development of scientific knowledge) results as **a linguistic game**: playing with mathematical expressions and Symbols, human being developed complex Thoughts throughout history, getting through seminal milestones as the Pythagorean theorem or the law of universal gravitation or quantum statistics; all that milestones, speculating on Referents, created a world of new Referents (fig. 2), discussed and analyzed in their turn as real objects of reality itself.

3.1. Physics

Speculations in sciences represent a clear example of the recursive semiosis discussed in the latter paragraph. For instance, in describing reality, classical physics faces simultaneous processes, occurring independent one another, thus $pq = qp$ both if I measure the position (q) of a cannonball and then I measure its momentum (p) and if I reverse the order of the operations (viz. measuring the momentum first, then the position, pq); but quantum physics also faces sequential processes, occurring in a specific order, thus $pq \neq qp$, for I get some results measuring firstly the position and then the momentum (qp) of *wavicles*[6], as I get different results if I reverse the order of measurements (pq) because one measurement affects the other[7], on the basis of observations conducted on *double slit* experiments for light and electrons[8]. Thus $pq–qp \neq 0$: there is a difference

3 Capelewicz (2021).
4 Ben-Yami (2017).
5 That is why chapters 1 and 2 of this paper insisted on etymology.
6 Eddington (1928: 201) blended the nouns *wave* and *particle* in order to express the *duality* of matter on quantum scales.
7 Heisenberg (1927) speculated about that *uncertainty principle*.
8 Taylor (1909), Davisson/Germer (1927), Jönsson (1961).

(between the two processes) that can be measured[9] in order to reveal the level of uncertainty intrinsic to a wavicle system.

Sequential experiences in quantum mechanics moved Dirac (1930: 24) on noting that the "commutative axiom of multiplication does not hold for linear operators": linear operators **do not commute** ($pq \neq qp$). That is to say that quantum physics manages the multiplication of certain operators just like a division, that is the only non-commutative elementary operation (chap. 1.5): fundamentally, quantum physics, under specific circumstances, *transfers* on the multiplication a property of the division.

That anticommutative approach to multiplication came from Lie (1874)[10] defining a tool apt to transform equations of spheres into equations of lines, in order to solve differential equations via linear operators[11], convenient in computing results: he had to find some linear parameter that preserved the *Symmetry* of quadratic equations[12], and he found it in the *Poisson bracket*[13], stating $pq = -qp$ as an operation on matrices: fundamentally, rows and columns of a symmetric square matrix (representing coordinates in a vector space) can be transposed (inverting positive and negative signs of items), preserving the *Symmetry* of the square matrix, if the *determinant* of the matrix[14] remains the same in both the representations of the matrix[15]. Mathematicians have always been guided by *Symmetry*: here the equality of $pq = -qp$ is arithmetically possible only for $i(pq) = i(qp)$. That is to say that, **given the reflection properties** of i (with $i = \{\pm 1, \pm 1\}$ from prop. [26] in chap. 2.1.5), the *Asymmetry* evident in $\{\boldsymbol{pq = -qp}\}$ is possible only under the *Symmetric* property of $i(pq) = i(qp)$.

That way, quantum mechanics introduced the *Asymmetry* of actual experience in mathematics, generating special items like *bra* vectors and *ket* vectors (Dirac 1930) that evidence an interest of scholars in underlying meanings of algebraic Symbols: anticommutativity of Poisson "*bra*(c)*ket*" survives explicitly in Dirac algebra, where $|A\rangle$ is a column vector (named *ket*) and $\langle B|$ is a row vector (named *bra*), both listing data representing some particular state of wavicles. Their products clearly do not commute both on graphical and conceptual levels: $\langle B|A\rangle = c$ is the *inner product* resulting in a real number, while $|A\rangle\langle B| = \hat{C}$ is the *outer product* resulting in a matrix: Moreover, bras and kets are subject to linear operators (taken from Lie algebra): "the product $\alpha\beta$ is defined as the linear Operator which, operating on any ket $|A\rangle$, changes it into that ket which one would get by operating first on $|A\rangle$ with β, and then on the result of the first Operation with α" (Dirac 1930: 23); that is the reason why "the ket vector must always be put on the right of the linear operator" (ibid.). Quantum mechanics algebra relies on the order of operations, preserving the inner *Symmetry* of Thoughts via the superficial *Asymmetry* of Symbols. That meaning that quantum algebra *transfers* the non-

9 Kennard (1927) and Weyl (1928) calculated $\delta q \cdot \delta p \geq {}^{\hbar}/_{2}$: half the reduced Planck constant (${}^{\hbar}/_{2}$) is the minimum ($\geq$) level of uncertainty about the relation ($\cdot$) between the uncertainty (δ) of the position (q) and the uncertainty of the momentum (δp). That is because $\hbar = {}^{h}/_{2\pi}$ is the reduction of Planck constant (h) to cyclic events ($2\pi = 360°$), like waves: quantum cycles ($\hbar$) are measured in *angles* (e.g., α), ranging from $-\alpha$ to $+\alpha$, thus the range of uncertainty about δq and δp must be $2(\delta q \cdot \delta p)$ because each ranges $-\alpha$ and $+\alpha$. Korzybski (1933/1994: 714-715) explained that when the Planck constant (h) "is made to approach zero, pq approaches qp, and so we pass to the classical mechanics", hence quantum mechanics *generalize* classical mechanics.

10 Thus anticommutative algebra is called *Lie algebra*.

11 Helgason (1994) resumed that approach.

12 E.g., a circle centered at the origin of axes (like that in fig. 10) has equation $x^2+y^2 = r^2$.

13 Poisson (1809).

14 The determinant is the difference between the products of the elements in each diagonal.

15 *Hermitian matrices* (or self-adjoint matrices) map rotations of coordinate systems as transpose conjugate matrices (implying complex numbers), following the work of Hermite (1855).

commutation property of the division (chap. 1.5) to the multiplication for the sake of *Symmetry* underlying *Asymmetry*.

3.2. Economics

Brown (1959/1985) showed that economics are intrinsic to Freudian human psychology: *libido* is a process of accumulation and management of accumulated resources, taken both as actual and symbolic items. The chapter 1 of this paper discussed how economy is strictly bound to arithmetic, for gathering and sharing resources deal with elementary operations. That remark points out the unconscious bound linking mathematics to psychology. Moreover, economy being a tool apt to manage resources, it is a framework of power, which is an actualization of the *libido*: dividing preys, sharing food, composing ratios, etc. are different ways of "administrating[16]" the power through a community (allocating goods, collecting taxes, marrying people, etc.) via transactions, balancing or unbalancing reciprocity (the power being the ability of unbalancing reciprocity with the approval of allies, via that social contract that is the Freudian *totemic brotherhood*[17]).

In that frame, Brown (1959/1985) pointed out a seminal remark about the nature of economics: monetary economies (viz. societies mediated by money[18]) developed the Sacred and the Symbolic along scientific patterns of culture, while gift economies (viz. societies "without a State"[19]) developed the Sacred and the Symbolic through magical thinking. Both social models rely on unconscious patterns of Freudian *sublimation*: a strategy apt to detach or to dissociate the unity of unconscious continuum, *projecting*, *transferring* and *displacing* sexual impulses into intellectual speculations of conscious mind. Remarking that, assuming economics and sciences as expressions of a rational framework, thus expressions of mathematical thinking, dominant advanced cultures strove (and still strive) to possess (viz. to acquire and manage) nature and reality, developing a symbolic map that alienated (and still alienates) individuals: Brown pointed out that peculiarity of civilization as a most relevant paradox revealed by psychoanalysis applied to economics and, generally, to sociopolitical matters. But that tendency towards alienation depends on mathematics being an expression of unconscious conflicts.

What is missing in that polarization (viz. progressed cultures opposed to primitive cultures[20]) is that progressed societies, alienated in the compulsory repetition of the impossible task of *extinguishing* the primordial guilt through capitalist economy of exchange, differ from primitive societies "without a State", alienated in the compulsory repetition of the continual task of *sharing* the primordial guilt through gift economies. Capitalist societies are **"mathematized" societies**: money, markets and accumulation of surplus sprouted through space and time in societies that developed mathematics, whereas gift economies, communions and *potlach* habits sprouted in societies disregarding mathematics. I find monetary economics and interest rates everywhere and whenever I find higher mathematics. Moreover, societies and cultures based on writing developed mathematics, whereas societies based on oral tradition did not, because simple arithmetic is an embodied experience (chap. 1), whereas algebra and higher mathematics require symbolic manipulations, allowed by written language only.

16 Brown (1959/1985) evidenced that economy sprouts from religion, thus *administrating* goes along with *managing*.
17 Freud (1913).
18 Graeber (2011) recorded at least 5,000 years of monetary economy.
19 Staid (2015) discussed gift economies throughout an examination of anthropology.
20 The opposition validates from the point of view of progressed cultures only.

Economics went through an evolution based on discursive dissertations, approaching algebraic formulations only around XIX century[21]: nevertheless, arithmetic reasoning is evident in "basic" economics, for accumulation and distribution of resources, and computation of surplus and taxation rely on elementary operations (chap. 1 and 2). But higher economics sprouted from the implementation of algebraic reasoning: manipulating equations (viz. *displacing* items through expressions, *projecting* qualities of entities onto other entities, *transferring* properties of objects to other objects, etc.) scholars saw Thoughts hidden in Symbols[22]. And that process reveals also hidden psychological meanings often overlooked and neglected by economists.

For instance (but economics brim of controversial theories like the following), the contribution in understanding economic growth[23] given by Solow (1956) revealed political implications widely ignored by mainstream economics, nevertheless mathematics implied in that model (viz. its Symbols) convey critical remarks (viz. Thoughts) widely visible in everyday life (viz. Referents) and, curiously, coherent with the critics moved by Marx (1867) to capitalism itself. In brief terms, that neoclassical model of growth acquires two assumptions. First: $Y = (1–s)Y+I$ means that a community demands for goods and services (a real Y on the left side of the equation) coherent with (=) the investments (I, necessary for producing Y) and with the portion $(1–s)$[24] of incomes (a nominal Y on the right side of the equation) exceeding savings; and transforming the equation it results in $I = sY$[25], meaning that savings (sY) sustain investments (I)[26]. Second assumption: investments (I taken from sY) increase the capital stock (K), so that $K = sY–dK–nK$ means that capital deteriorates itself via some percentage share ($–d$), as a consequence of the work (but sY reintegrates it), and the capital depreciates via some percentage share ($–n$) that increases as the population is increasing (because every new individual worker needs new capital in order to perform new work); and transforming that second equation it results in $K(1+d+n) = sY$[27], meaning that savings will never pay debts (viz. investments) back, because of the increase in population (n), and that labour power will always keep on working (d) because of the necessity to save money needed for reintegrating capital. That way, economics proves the scientific reality of the biblical *work sentence*: "By the sweat of your face you shall eat bread until you return to the ground" (Genesis 3:19). And it seems that Brown (1959/1985) was right in stating that humanity developed the division of labour, the accumulation of surplus, the money and economics in general as a sublimation of a death instinct, given as a tool apt to expunge guilt: economics is a religion, worshipped in atavistic temples (ancient priests and ministers managed offers and money exchanged in markets[28]), like ancient sects and cults worshipped mathematics[29].

21 Cournot (1838), Jevons (1871), Walras (1881) and Fisher (1892) "mathematized" economics.

22 Note 5 in chap. 2 exemplifies that procedure.

23 Economic growth has been considered an index of social wealth until bioeconomy revealed the environmental impact of global negative externalities: comparing per-capita incomes of different nations (via ratios) has been a parameter widely accepted in assessing stability, reliability and political power of nations, neglecting how that sublimated comparison reflected a brutal comparison of strengths.

24 The variable s identifies a share percentage ($0 < s < 1$) of Y allocated to savings: thus "$1–s$" identifies the percentage share of whole Y (1) left over to be allocated to consumption.

25 Namely: $\{Y = (1–s)Y+I\} \leftrightarrow \{Y = Y–sY+I\} \leftrightarrow \{Y–Y+sY = I\} \leftrightarrow \{sY = I\}$.

26 In per-capita terms, it means that the savings of each worker sustain the capital assigned to that same worker in order to perform the work.

27 Namely: $\{K = sY–dK–nK\} \leftrightarrow \{K+dK+nK = sY\} \leftrightarrow \{K(1+d+n) = sY\}$.

28 Wall Street, the "temple" of modern financial markets, has been built in the very shape of a classical temple.

29 For instance, Kahn (2001) described Pythagoreans as a sect following specific rules, performing rituals and observing religious practices.

The refusal of the evident (social) conflict intrinsic to that idea of "social wealth" is a collective act of repression (viz. *Displacement*): that is why Brown (1959/1985) and Girard (1972) discuss economics as a process developed in order to expiate the collective guilt, sublimating collective violence (via money and exchange), but that way reaffirming violence as *the* tool opted by modern societies to erase the guilt via modern totems and taboos[30].

Western culture waited until XX century in order to recognize how much the international economic linkages defuse the risk of warfare[31]; but still it has to acquaint itself with its need for conflicts, intrinsic to the atavistic social contract (while the modern declination of that contract is just a sublimation apt to repress that unconscious remark). And economics cannot solve that paradox, until it won't recognize mathematics as a means to reveal inner Thoughts (characterized by *Symmetric* continuity) hidden underneath algebraic Symbols (visualizing *Asymmetric* discontinuity) and beyond mere arithmetic Referents (relying on sensory-neural cognition).

Conclusions

Mathematics is an asymmetric representation of symmetry, visualizing how conscious mind deals with unconscious: algebraic expressions imply balance and associations between different entities, meaning a symmetric structure underlying the complexity of natural phenomena perceived as asymmetric differences. That way Mathematics, evidencing and symbolizing categories, allows the conscious mind to explore the unconscious, transferring and projecting complex thoughts into specific and essential signs. That semiosis allows the intellect to speculate about infinite continuity, as well as the transference allows the intellect to substitute the unconscious with conscious mind via psychotherapy, projecting unconscious onto the external world. That is why mathematics operates exactly on the same basis of the therapeutic process discussed by Freud (1923), when he stated that unconscious becomes conscious via verbal representations (viz. words), and by Brown (1959/1985: 148), when he suggested that unconscious becomes conscious via perceptions: chap. 1.1 (fig. 1) shows why Freud focused on symbols and Brown focused on referents, and why scholars recently evidenced how those ideas can be joint together in the category of cognitive symbols (fig. 2).

Having applied mathematics fundamentally to every possible branch of human knowledge, culture is following that purpose of "[preserving] in its symbol systems a map of the lost reality, guiding the search to recover it", as discussed by Brown (1959/1985: 167) with regard to psychoanalysis. Hence, the acquisition of mathematics means the acquisition of a tool that allows conscious mind to synthesize the asymmetry operated on symmetric unconscious: moreover, mathematics represents how conscious mind acquires unconscious via semiosis.

30 Rossi (2019).
31 Rummel (1979).

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