Diophantine quadruples and ideal solutions of the Prouhet–Tarry–Escott problem of size four

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Abstract: A rational Diophantine m-tuple is a set of m rational numbers such that the product of any two is one less than a square. The Prouhet-Tarry-Escott problem seeks two different multisets of n integers such that the sums of like powers of each set are equal for all exponents up to some $k < n$. Here a new connection is established between rational Diophantine quadruples ($m = 4$) and ideal solutions of the Prouhet–Tarry–Escott problem of size 4 ($n = 4, k = 3$) Both problems are shown to be related to finding 3 by 3 singular matrices of integers whose 9 elements are all square.

Prouhet–Tarry–Escott

The Prouhet-Tarry-Escott (PTE) problem seeks two n-tuples of integers x_i and y_i such that the sums of like powers up to k are equal

$$\sum_{i=1}^{n} {x_i}^p = \sum_{i=1}^{n} {y_i}^p \,, 1 \le p \le k$$

k is called the degree, and n the size of the problem. If x_i and y_i are permutations of the same numbers then the solution is trivial. The system of equations is written with the notation $[x_i] =_k [y_i]$. For non-trivial solutions $k < n$. If $k = n - 1$, the solution is called ideal. Chains of n-tuples which are PTE solutions in pairs $[x_i] =_k [y_i] =_k [z_i] \dots$, are also of interest. The problem is equivalent to seeking pairs of polynomials which fully factorise over the integers and which differ by a polynomial of degree $n - k - 1$, a non-zero integer constant in the ideal case,

$$\prod_{i=1}^{n}(x - x_i) - \prod_{i=1}^{n}(x - y_i) = C$$

Versions of the PTE problem date back to Euler, but it was introduced in its current form by Prouhet in 1851 [1] and was studied in detail by Tarry who found solutions on size 6 and 8 [2] and Escott who found solution of size 7 [3] in the early twentieth century. Solutions of size 9 were found by Letac in 1942 [7]

Solutions are said to be equivalent if they differ only by permutations of the elements, addition of an integer constant, multiplication by a rational constant, or any combination of these operations.

In particular, negating each element on both sides leads to an equivalent solution. If these are the same up to permutations then the solution is called symmetric. The nature of symmetric solutions differs for odd vs even size n. If n is odd then the elements on the left side are the negatives of the elements on the right and the even power equations are automatically satisfied. If n is even, the elements on either side fall into pairs differing in sign, and the odd power equations are automatically satisfied.

In the current state of art of the PTE problem, ideal solutions are known for all sizes up to 10 [5] and also for size 12 [4,6].

Ideal PTE problems of low size

The ideal PTE problem of size two only requires that $x_1 + x_2 = y_1 + y_2$. Any solution is therefore equivalent to the symmetric form $[a, -a] =_1 [b, -b]$ which can be extended to an infinite chain of solutions. For the size two case all solutions are therefore equivalent to a symmetric form.

For size three, a linear and quadratic equation in three variables must be satisfied. A general solution is readily found and can be expressed in a symmetric form parameterised by six variables [13,14]

$$\begin{aligned} x_1 &= ap + bq + cr \\ x_2 &= aq + br + cp \\ x_3 &= ar + bp + cq \\ y_1 &= ap + br + cq \\ y_2 &= aq + bp + cr \\ y_3 &= ar + bq + cp \end{aligned}$$

This solution can be understood in terms of factorisations over the commutative ring generated by an element ω subject to $\omega^3 = 1$ so that a general element takes the form $= a + b\omega + c\omega^2$. The ring has a conjugation $\bar{u} = a + c\omega + b\omega^2$. With $v = p + q\omega + r\omega^2$, $x = x_1 + x_2\omega + x_3\omega^2$ and $y = y_1 + y_2\omega + y_3\omega^2$ the solution reduces $x = \bar{u}v$, $y = uv$.

If the additional relation $1 + \omega + \omega^2 = 0$ is imposed then the ring reduces to the Eisenstein integers and the general solution can be simplified to a four parameter form, but with less symmetry.

Solution chains of length 2^{l-1} can be generated using products of l Eisenstein integers with arbitrary selections of conjugates taken.

The general size 4 ideal case requires three equations, but the symmetric case reduces to one

$$[a, -a, b, -b] =_3 [c, -c, d, -d]$$

$$a^2 + b^2 = c^2 + d^2$$

This has the well-known general solution from products of Gaussian integers and their conjugates

$$a = pq - rs, b = pr + qs, c = pq + rs, d = pr - qs$$

Chains of symmetric solutions of size 4 can be formed using products of more Gaussian integers and their conjugates.

The general ideal case of size 4 was solved by "Crussol" in 1913 [15].

The linear equation is resolved by making use of an additive constant to set both sides to zero

$$x_1 + x_2 + x_3 + x_4 = y_1 + y_2 + y_3 + y_4 = 0$$

The remaining quadratic and cubic power equations are transformed by a linear substitution

$$\begin{gathered}
x_1 = X_1 + X_2 + X_3 \\
x_2 = X_1 - X_2 - X_3 \\
x_3 = -X_1 - X_2 + X_3 \\
x_4 = -X_1 + X_2 - X_3 \\
y_1 = Y_1 + Y_2 + Y_3 \\
y_2 = Y_1 - Y_2 - Y_3 \\
y_3 = -Y_1 - Y_2 + Y_3 \\
y_4 = -Y_1 + Y_2 - Y_3
\end{gathered}$$

This reduces the system of equations to

$$X_1{}^2 + X_2{}^2 + X_3{}^2 = Y_1{}^2 + Y_2{}^2 + Y_3{}^2$$

$$X_1 X_2 X_3 = Y_1 Y_2 Y_3$$

Taking account of the common factors in the second equation, a general solution parameterised by 9 integers $t_{ij}, i = 1,2,3, j = 1,2,3$ is given by

$$X_i = t_{i1} t_{i2} t_{i3} \quad i = 1,2,3$$

$$Y_j = t_{1j} t_{2j} t_{3j} \quad j = 1,2,3$$

Leaving just one equation to be resolved

$$t_{11}{}^2 t_{12}{}^2 t_{13}{}^2 + t_{21}{}^2 t_{22}{}^2 t_{23}{}^2 + t_{31}{}^2 t_{32}{}^2 t_{33}{}^2 = t_{11}{}^2 t_{21}{}^2 t_{31}{}^2 + t_{12}{}^2 t_{22}{}^2 t_{32}{}^2 + t_{13}{}^2 t_{23}{}^2 t_{33}{}^2$$

Crussol and other authors since have completed the solution by treating it as a quadratic in the three variables t_{ii} with the remaining variables taken as givens. Standard methods can be used to parameterise all solutions over the rational numbers, which can then be transformed to integers by multiplying through by all denominators. This provides a solution which is complete, but opaque and unsymmetrical. An alternative approach is to recognise the equation as a determinant expression

$$\begin{vmatrix} {t_{11}}^2 & {t_{33}}^2 & {t_{22}}^2 \\ {t_{23}}^2 & {t_{12}}^2 & {t_{31}}^2 \\ {t_{32}}^2 & {t_{21}}^2 & {t_{13}}^2 \end{vmatrix} = 0$$

The ideal PTE problem of size 4 is therefore equivalent to seeking 3 by 3 singular matrices with all square elements. Note that a solution being symmetric is equivalent to one of the elements being zero.

The singularity of a matrix is equivalent to there being a linear relationship between the rows or columns. I.e there exist integers a, b, e such that

$$\begin{gathered} a{t_{11}}^2 + b{t_{33}}^2 + e{t_{22}}^2 = 0 \\ a{t_{23}}^2 + b{t_{12}}^2 + e{t_{31}}^2 = 0 \\ a{t_{32}}^2 + b{t_{21}}^2 + e{t_{13}}^2 = 0 \end{gathered}$$

For example, in the specific case of $a = b = 1, e = -1$ the problem requires three Pythagorean triples.

Multiplication or division of the elements in any row or column to an integer only affects the overall solution by a constant multiplier. If we are interested in constructing solutions up to equivalence then we can freely apply such factors. It can be arranged that no element in the bottom row is zero. By this means it is possible to reduce the last row of the matrix to all unit elements while keeping the other two rows in integer form.

$$\begin{vmatrix} {t_{11}}^2 & {t_{33}}^2 & {t_{22}}^2 \\ {t_{23}}^2 & {t_{12}}^2 & {t_{31}}^2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

The linear relationship is then subject to the condition $a + b + e = 0$ and it can be assumed that the three coefficients are relatively prime in pairs. In this case the general solution in integers to $ax^2 + by^2 + ez^2 = 0$ can be parameterised up to a common factor by

$$x = bu^2 + ev^2, \qquad y = ew^2 + au^2, \qquad z = av^2 + bw^2$$

$$u + v + w = 0$$

A general solution up to equivalence is therefore given by

$$X_1 = (bu^2 + ev^2)(er^2 + ap^2)$$

$$X_2 = (av^2 + bw^2)(bp^2 + eq^2)$$

$$X_3 = (ew^2 + au^2)(ar^2 + bp^2)$$

$$Y_1 = (bu^2 + ev^2)(ar^2 + bp^2)$$

$$Y_2 = (av^2 + bw^2)(er^2 + ap^2)$$

$$Y_3 = (ew^2 + au^2)(bp^2 + eq^2)$$

$$a + b + e = u + v + w = p + q + r = 0$$

A further reduction of the determinant equation can be made if the elements are allowed to be rational. Dividing the first and second rows by squares ${t_{22}}^2$ and ${t_{31}}^2$ which can be assumed to be non-zero gives

$$\begin{vmatrix} x^2 & z^2 & 1 \\ t^2 & y^2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$(x^2 - 1)(y^2 - 1) = (z^2 - 1)(t^2 - 1)$$

This brings us to the problem of Diophantus.

Diophantine Quadruples

In the third century AD, Diophantus of Alexandria wrote a series of books on mathematical problems dealing with finding solutions to indeterminate problems in rational numbers. Most of his problems in the surviving books are motivated by geometric constructions or real-life situations, but one problem is more esoteric. Diophantus sought sets of rational numbers such that the product of any two is one less than a rational square. He found examples of sets of four such numbers. It is not known what inspired this problem, but it was well chosen and has been the source of much interesting research.

A Diophantine m-tuple is defined as a set of m positive integers a_i such that the product of any two is one less than a square $a_i a_j + 1 = {x_i}^2, i \neq j$. Solutions to the rational case (negative or positive but not zero) as originally studied by Diophantus are rational Diophantine m-tuples.

Fermat found the set quadruple 1, 3, 8, 120 and thereby set the long-standing problem of deciding the existence of Diophantine quintuples. Euler found a fifth rational number that could be added to form a rational Diophantine quintuple. In 1999 I found the first examples of rational Diophantine sextuples [8].

The integer case has been extensively studied using methods of Diophantine approximation. Dujella set an upper bound on the size of elements in a Diophantine quintuple [9] before their existence was finally shown to be impossible [10]. A stronger conjecture that all Diophantine quadruples are solutions of a regularity equation remains unresolved [11]. Continued interest in the problem is spurred by its relationship to elliptic curves with torsion and high rank [12].

The case of Diophantine quadruples a, b, c, d requires

$$\begin{aligned} ab+1 &= x^2, \qquad & cd+1 &= y^2 \\ ac+1 &= z^2, \qquad & bd+1 &= t^2 \\ ad+1 &= u^2, \qquad & bc+1 &= v^2 \end{aligned}$$

The first two pairs of equations imply

$$(x^2-1)(y^2-1) = (z^2-1)(t^2-1) = abcd$$

From the analysis of the ideal PTE problem of size four, it can now be seen that a rational Diophantine quadruple provides a solution.

$$(X_1, X_2, X_3, Y_1, Y_2, Y_3) = (x^2y^2, z^2, t^2, z^2t^2, x^2, y^2)$$

Indeed it provides three separate solutions from different choices of the squares. Consider for example Fermat's quadruple

$$(a,b,c,d) = (1,3,8,120)$$

$$(x,y,z,t,u,v) = (2,31,3,19,11,5)$$

There are three singular matrices

$$\begin{vmatrix} 2^2 & 3^2 & 1 \\ 19^2 & 31^2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2^2 & 5^2 & 1 \\ 11^2 & 31^2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 3^2 & 5^2 & 1 \\ 11^2 & 19^2 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

From this we derive three solutions to the problem

$$[88,22,-26,-84] =_3 [78,46,-56,-68]$$

$$[73,41,-63,-51] =_3 [77,33,-71,-39]$$

$$[84,40,-46,-78] =_3 [90,24,-86,-28]$$

As a bonus, if the three pairs are amalgamated into a single pair then the fourth, fifth and sixth powers also give equal sums providing a solution for $n = 12, k = 6$

$$[88,22,-26,-84,73,41,-63,-51,84,40,-46,-78] =_6$$
$$[78,46,-56,-68,77,33,-71,-39,90,24,-86,-28]$$

It can be checked algebraically that this works for any Diophantine quadruple

The inverse derivation can also be followed. Given an ideal PTE solution of size 4 a 3x3 singular matrix can be formed with elements that are all square. If the solution is not symmetric then none of the elements are zero. For any choice of row and column (nine possibilities) the matrix can be normalised by multiplying rows and columns until the elements in the row and column are all equal to one and the remaining square of four elements are rational squares. This provides nine rational solutions to the equation

$$(x^2-1)(y^2-1) = (z^2-1)(t^2-1)$$

For each of these it is possible to find a quadruple of rationals (a, b, c, d) such that

$$ab + 1 = x^2, \qquad cd + 1 = y^2$$
$$ac + 1 = z^2, \qquad bd + 1 = t^2$$

The solution is undetermined up to a rational r so that the general solution is given in terms of one starting solution as $(a_0 r, \frac{b_0}{r}, \frac{c_0}{r}, d_0 r)$

To complete a Diophantine quadruple it is then necessary to solve

$$a_0 d_0 r^2 + 1 = u^2, \qquad b_0 c_0 / r^2 + 1 = v^2$$

For the general solution this can be reduced to an elliptic curve. However, two special cases can be found using the concept of regularity for Diophantine quadruples. A regular rational Diophantine quadruple is one which satisfies the equation

$$(a + b - c - d)^2 = 4(ab + 1)(cd + 1)$$

It is well-known that if (a, b, c) is a Diophantine triple, then this equation can be solved for d to give a Diophantine quadruple. There are in general two rational solutions one of which may be zero. It is perhaps less well-known that this equation can also be solved to complete the Diophantine quadruple given (x, y, z, t) as above. Again there are in general two rational solutions.

In conclusion, one ideal non-symmetric solution to the PTE problem of size four provides nine partial rational Diophantine quadruples and therefore up to 18 regular Diophantine quadruples. Each of these in turn would provide three solutions to the PTE problem, one of which would be the original and the two others would be new.

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