A Proof of Twin Prime Conjecture by Using Twin Prime Model Table and Sieve Functions

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Abstract

Twin Prime Conjecture(TPC) states that there are infinitely many prime pairs (*p, p* + 2), where *p* is prime. But, up to date there is no valid proof of TPC. To prove TPC we devised Twin Prime Model Table(TPMT) and Sieve Functions(SFs). TPMT is a 2-dimensional table representation of all possible twin prime pairs(TPPs). SF is a sine function, $f_i(x) = sin\frac{\pi x}{p_i}$, where p_i is the i^{th} prime. SFs functionally represent the sieve of Eratosthenes because all zeros of $f_i(x)$ can't be prime exept the first zero. TPMT explicitly shows the mechanism of how TPPs are found from the possible twin prime pairs. To functionally represent this mechanism we introduced various sinusoidal functions. And by using properties of sinusoidal functions we proved TPC.

1. Introduction

TPC [1][2] states that there are infinitely many prime pairs with distance 2, like (3, 5), (11, 13). TPC remains unsolved since de Polignac, in 1849, made the more general conjecture that for every natural number *k*, there are infinitely many prime pairs such that (*p*, *p* + *2k*).

In 2013, Yitang Zhang [3][4] proved that there are infinitely many prime pairs that differ by $(p_{n+1} - p_n) < 7\times10^7$, where p_n is the n^{th} prime. After Zhang's discovery mathematicians like Terence Tao [5] and James Maynard [6][7] reduced the gap to 6 [2].

Our purpose is to prove TPC generally and easily. To do so, we devised Twin Prime Model Table(TPMT) and Sieve Functions(SFs). TPMT is a infinitely expandable 2-dimensional arrangement of all possible twin prime pairs, from which we can see the mechanism of how twin primes are found. This mechanism of sieving TPPs from TPMT is similar to sieving prime numbers from natural number sequence. SF is a sinusoidal function, $f_i(x) = sin\frac{\pi x}{p_i}$, where p_i is the i^{th} prime($p_1 = 2$). SFs functionally represent the sieve of Eratosthenes.

The details of our proof may seem to be somewhat lengthy and complicated, but the proof is general and not difficult.

2. Twin Prime Model Table(TPMT)

Lemma 2.1. Twin Prime Pairs(TPPs), except (3, 5), have the form (*6n - 1, 6n + 1*), *n* = 1, 2, 3, ….

Proof. If odd multiples of 3, like 9, 15, 21, exist between two odd numbers, as in 5, 7, <u>9</u>, 11, 13, <u>15</u>, 17, 19, <u>21</u>, 23, 25, <u>27</u>, 29, …, the gap of two possible prime numbers will be greater than 2, failing to be a TPP. So, by removing multiples of 3 from odd number sequence, we get the form ($6n$ - 1, $6n$ + 1) like (5, 7), (11, 13), (17, 19), (23, 25). ■

Definition 2.2. Twin Prime Model Table(TPMT): TPMT is an infinitely expandable 2-dimensional table, which arranges all possible twin prime pairs of the form ($6n$ - 1, $6n$ + 1). Table 1 shows the initial part of TPMT.

Table 1. Initial part of TPMT.

n	6n-1	6n+1	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53	55	59	61	65	67	71	73	77	79	83	85	89	91	95	97	101	103	107	109
1	5	7	1d	1u																																		
2	11	13			1d	1u																																
3	17	19					1d	1u																														
4	23	25	5u						1d	1u																												
5	29	31									1d	1u																										
6	35	37	7d	5d									1d	1u																								
7	41	43													1d	1u																						
8	47	49		7u													1d	1u																				
9	53	55	11u		5u														1d	1u																		
10	59	61																			1d	1u																
11	65	67	13d			5d																	1d	1u														
12	71	73																							1d	1u												
13	77	79		11d	7d																						1d	1u										
14	83	85	17u				5u																						1d	1u								
15	89	91		13u		7u																									1d	1u						
16	95	97	19d					5d																									1d	1u				
17	101	103																																	1d	1u		
18	107	109																																			1d	1u

The properties of TPMT are as follows.

- **Horizontal arrangement.**
  - n: Sequence of possible twin prime pairs.
  - $6n$ – 1: Smaller number of a possible TPP. Primes are marked as red.
  - $6n$ + 1: Larger number of a possible TPP. Primes are marked as red.
  - (5, 7), (11, 13), (17, 19), … : Horizontal arrangement of all possible twin prime pairs.
- **Vertical arrangement.**
  - (n, $6n$ - 1, $6n$ + 1) triples are vertically arranged. TPPs are marked as all red.

- **Inner cells.**
  - Inner cells are empty or marked as *md* or *mu,* where m = 1, 5, 7, 11, … and are explained below.

Definition 2.3. Seed value: A number from the first row like 5, 7, 101. It is used in sieving *6n – 1* or *6n + 1*. We use letter *s* as a seed value variable. s_i means i^{th} seed value.

Definition 2.4. Up-number and down-number: In (*6n - 1, 6n + 1*), *6n – 1* is down-number and denoted by *d*, *6n + 1* is up-number and denoted by *u*. So, (*6n - 1, 6n + 1*) can be denoted as (down-number, up-number) = (*d, u*).

If *6n – 1* or *6n + 1* is divisible by some seed value *s*, it is *sieved* and is marked as *md* or *mu,* where *m* = 1, 5, 7, 11, …. If down-number *6n – 1 = ms*, it is marked as *md*. If up-number *6n + 1 = ms*, it is marked as *mu*.

If *m* = 1, it is marked as (*1d, 1u*) pair, which means that (*6n - 1, 6n + 1*) pair is always divisible by itself. So, it does not mean sieved. If one or two numbers in (*6n – 1, 6n + 1*) pair is sieved, then (*6n – 1, 6n + 1*) can not be a TPP, because at least one of (*6n – 1, 6n + 1*) can not be prime.

Definition 2.5. Vertical pairs: Vertically arranged (*6n - 1, 6n + 1*) pairs.

Definition 2.6. Horizontal pairs: Horizontally arranged (*6n - 1, 6n + 1*) pairs.

From the above lemmas and definitions, we can see the mechanism of how TPPs are found in TPM, through sieve patterns between horizontal and vertical pairs.

- **Sieve patterns between horizontal and vertical pairs.**
  - **Diagonal (*1d, 1u*) pair pattern:** A horizontal (*6n - 1, 6n + 1*) pair always divides a vertical pair when they are same. So, (*1d, 1u*) pair always appears diagonally.
  - **Vertical pattern:** For a horizontal pair, the first downward non-empty pair is always (*1d, 1u*). For a seed value *s*, (*u, d*) or (*d, u*) pattern repeats vertically. This is because, for *m* = 1, 5, 7, 11*, …*, *ms* = *6n -1* or *ms* = *6n + 1* match occurs alternately.

From the above sieve patterns Lemma 2.7 follows.

Lemma 2.7. In TPMT, if all cells left side of a diagonal (*1d, 1u*) pair are empty, then the corresponding (*6n - 1, 6n + 1*) is a TPP.

Proof. If all cells left side of a diagonal (*1d, 1u*) pair are empty, the corresponding (*6n - 1, 6n + 1*) is divisible only by itself, so, it is a TPP. ∎

Definition 2.8. Phases: Phases are the values of n where the first downward *md* or *mu* appears. There are two kinds of phases. One is down-phase p_d, the value of n where the first *md* appears. The other is up-phase p_u, the value of n where the first *mu* appears.

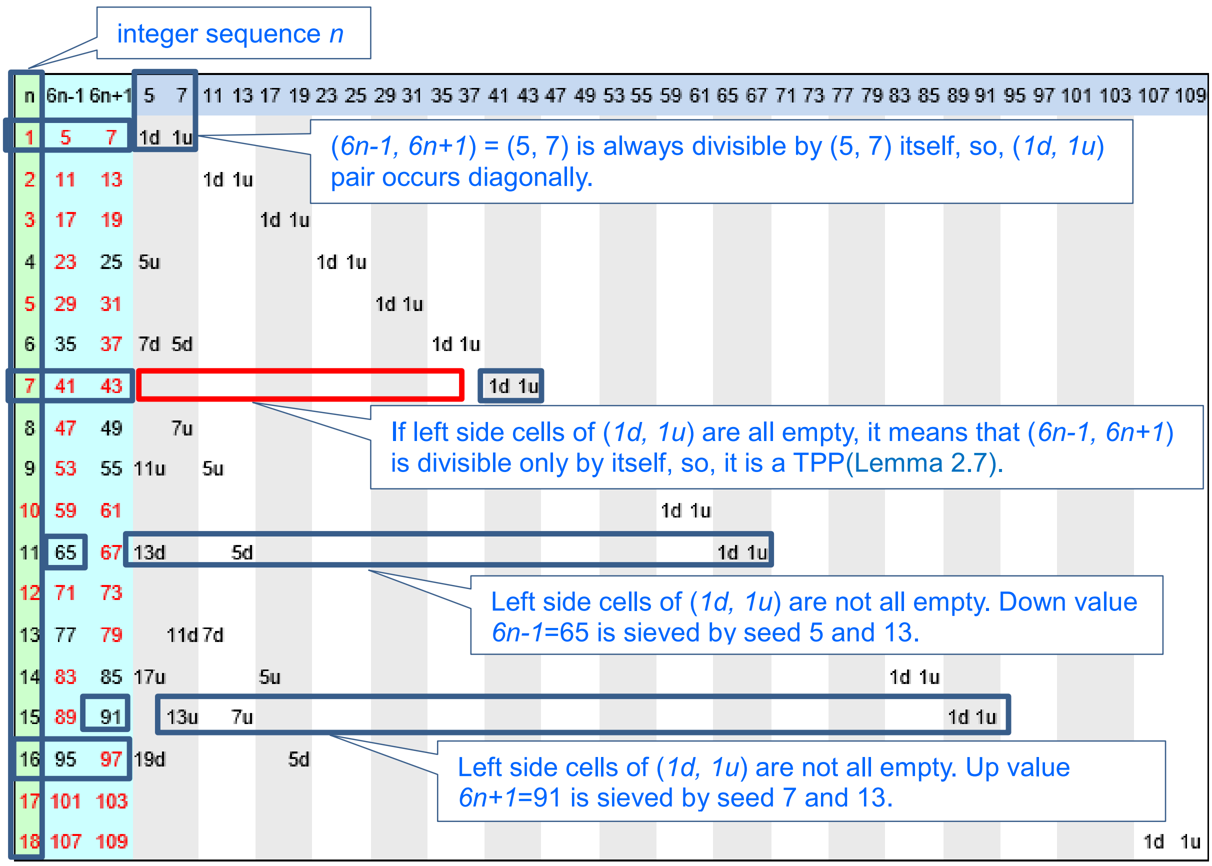
- **Phase examples:** The phase pair of seed 11 is $(p_d, p_u) = (2, 9)$. The phase pair of seed 13 is $(p_u, p_d) = (2, 11)$. Note that seed $s = p_d + p_u$. The phase pair will be used when defining phased Sieve Functions(pSFs). Table 2 shows some (p_d, p_u) or (p_u, p_d) pairs.

Table 2. Phase patterns.

Seed	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	...
Phases	1, 4	1, 6	2, 9	2, 11	3, 14	3, 16	4, 19	4, 21	5, 24	5, 26	6, 29	6, 31	7, 34	7, 36	8, 39	8, 41	...

The sieve patterns of TPMT is summarized in Figure 1.

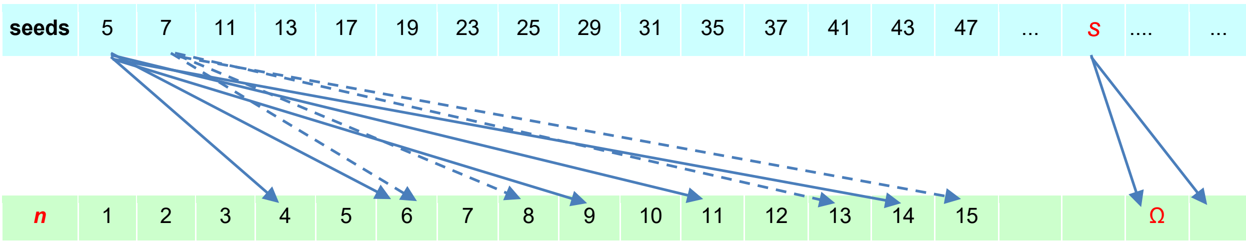
Figure 1. Sieve patterns of TPMT.



3. Sieve View of TPC

In Table 1, we can see that every vertical pair $(6n - 1, 6n + 1)$ is matched to a number sequence n = 1, 2, 3, …. So, we can transform sieving $(6n - 1, 6n + 1)$ vertical pairs to sieving number sequence n, …, as in Figure 2.

Figure 2. Patterns of sieving vertical pairs.



In Figure 2, arrows from each seed, sieve numbers from *n* = 1, 2, 3, … and the unsieved *n* represents a TPP. For example, for *n* = 1, (*6n – 1, 6n + 1*) = (5, 7) is a TPP. For *n* = 7, (*6n – 1, 6n + 1*) = (41, 43) is also a TPP. But for *n* = 6, (*6n – 1, 6n + 1*) = (35, 37) can't be a TPP, because 6 is sieved by seeds 5 and 7.

So, TPC can be stated as whether the seed values 5, 7, 11, 13, … can sieve all integers after some specific number *Ω,* which is marked as red in Figure 2. If such a specific number *Ω* exist, it means that TPPs are finite and so TPC is false.

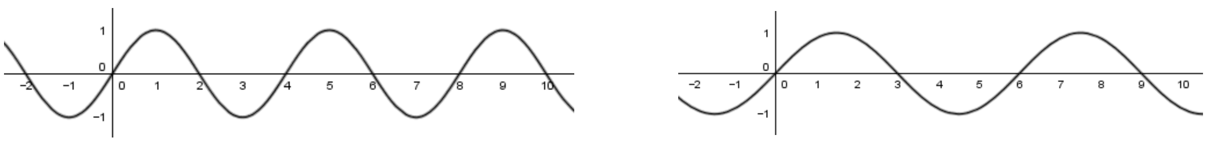
In this section, we transformed TPC view to a sieve view of an ordinary integer sequence *n* = 1, 2, 3, …, similar to the sieve of Eratosthenes.

4. Sieve Functions

To functionally treat the sieve of Eratosthenes, we introduce SFs and Composite Sieve Functions(CSFs).

Definition 4.1. Sieve Function(SF): A sine function $f_i(x) = sin\frac{\pi x}{p_i}$, where p_i is the i^{th} prime number, p_1 = 2. Examples of SFs are shown in Figure 3.

Figure 3. Example SFs.



(a) SF for p_1 = 2, $f_1(x) = sin\frac{\pi x}{2}$. **(b)** SF for p_2 = 3, $f_2(x) = sin\frac{\pi x}{3}$.

In Figure 3, the multiples of p_i are the zeros of $f_i(x)$. So, we can say that numbers $n = tp_i$, *t* = 2, 3, 4, … are sieved by $f_i(x)$. When, *t* = 1, $n = p_i$ is considered unsieved.

Let's see how SFs give functional view of the sieve of Eratosthenes.

Lemma 4.2. In sieving numbers less than or equal to *N*, we need to sieve numbers which are the multiples of prime numbers only up to $\sqrt{N}$.

Proof. Suppose $xy = N = \sqrt{N}\sqrt{N}$. If $x \geq \sqrt{N}$, then $y \leq \sqrt{N}$ and vice versa. Thus, if $xy = N$, then one of *x* or *y* must be less than or equal to $\sqrt{N}$. This means that if *N* can be factored, one of the factors which will sieve *N*, must be less than or equal to $\sqrt{N}$. ■

For example, to find out prime numbers between 1 to 51, we sieve multiples of prime numbers between 2 and $\sqrt{50} > 7.07$, i.e., 2, 3, 5, 7. In functional view, four SFs, $f_1(x) = sin\left(\frac{\pi x}{2}\right), f_2(x) = sin(\frac{\pi x}{3}), f_3(x) = sin(\frac{\pi x}{5}), f_4(x) = sin(\frac{\pi x}{7})$ are required to sieve all composite numbers between 1 and 51.

Definition 4.3. Composite Sieve Function(CSF): A product of SFs, $F_k(x) = \prod_{i=1}^{k} f_i(x)$. An example CSF is depicted in Figure 4.

Figure 4. Example CSF, *k* = 2, dotted or dashed graphs are SFs.

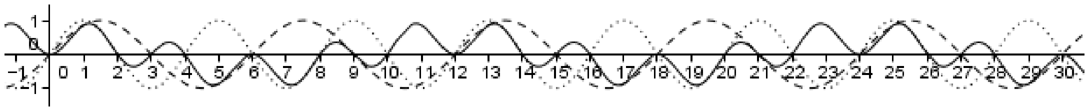


Figure 4 depicts an example CSF, $F_2(x) = \prod_{i=1}^{2} f_i(x) = sin(\frac{\pi x}{2})sin(\frac{\pi x}{3})$. The dotted and dashed graphs are $f_1(x) = sin(\frac{\pi x}{2})$ and $f_2(x) = sin(\frac{\pi x}{3})$, respectively. Numbers $N = 2t$ and $N = 3t, t = 2, 3, 4, \ldots$ are sieved by $f_1(x)$ and $f_2(x)$. The unsieved numbers between 1 and 25 are all prime numbers, because 3 is the largest prime number less than $5 = \sqrt{25}$.

Definition 4.4. Sieved *n*: Integers that are sieved by some sieve function.

Definition 4.5. Unsieved *n*: Integers that are not sieved by some sieve function.

Lemma 4.6. A CSF $F_k(x) = \prod_{i=1}^{k} f_i(x)$ is a periodic function with period $\prod_{i=1}^{k} p_i$.

Proof. A CSF $F_k(x)$ is the product of *k* periodic sine functions with period p_i, so, a CSF is also a periodic function with period $\prod_{i=1}^{k} p_i$ [8][9]. ■

Lemma 4.7. Any CSF $F_k(x) = \prod_{i=1}^{k} f_i(x)$ can not sieve all integers after some specific number *Ω*.

Proof 1. If some CSF $F_k(x)$ can sieve all integers after a specific number *Ω*, it means that the prime numbers are finite, which contradicts. ■

Proof 2. $F_k(x)$ has a finite-period with unsieved *n* within its period. So, it will repeats infinitely many times its period, which means that it is impossible to sieve all integers after some specific number Ω. ■

Lemma 4.7 states that any CSF can't sieve all integers after some specific number *Ω*, leaving infinite prime numbers. It means that any multiplication of sine functions with period greater than 1, can't have all integers after some specific number as zeros. Only $sin(\pi x)$ can sieve all integers after a specific number, which means that to sieve all numbers after a specific number, all integers should be sieved. This also implies that $sin(\pi x)$ can't be expressed by multiplying or adding any other sine functions with period greater than 1.

We also can prove the infinitude of prime numbers by induction.

Lemma 4.8. There are infinitely many prime numbers.
Proof. Let's prove by induction.
Step 1: At k = 1, there are infinitely many unsieved n of the sieve function $F_1(x)$.
Step 2: Suppose, at k, there are infinitely many unsieved n of the sieve function $F_k(x)$.
Step 3: Then, at $k + 1$, adding a new seed p_{k+1}, can not change that $F_{k+1}(x)$ is a finite-period function, with unsieved n within its period. So, by Lemma 4.7, there are infinitely many unsieved n of the sieve function $F_{k+1}(x)$, leaving infinitely many prime numbers. ■

In this section, we introduced SFs and CSFs and showed that the sieve of Eratosthenes can be functionally treated by the sinusoidal sieve functions. The periodicity of sinusoidal sieve functions can give visible understanding on how prime numbers are found and why the last prime number can not exist.

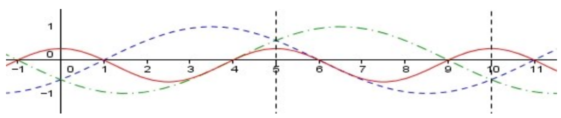
5. phased SFs

The sieve patterns of TPMT have non-zero phases as in Definition 2.8. So, we need to define phased SFs.

Definition 5.1. phased Sieve Function(pSF): A product of two sine functions, $h_i(x) = d_i(x)u_i(x)$, where $d_i(x) = sin\frac{\pi(x - p_d)}{s_i}$ is down-sieve-function and $u_i(x) = sin\frac{\pi(x - p_u)}{s_i}$ is up-sieve-function, and s_i is the i^{th} seed of sequence s = 5, 7, 11, …, and (p_d, p_u) is a phase pair of i^{th} seed. The example graphs of $h_i(x)$ are depicted in Figure 5 (a) and (b).

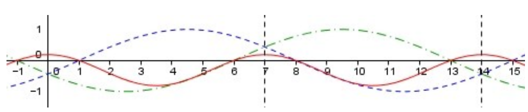
Definition 5.2. Composite phased Sieve Function(CpSF): The product of pSFs, $H_k(x) = \prod_{i=1}^{k} h_i(x), k = 1, 2, 3, \ldots$, as in Figure 5 (c).

Figure 5. pSF and CpSF examples.



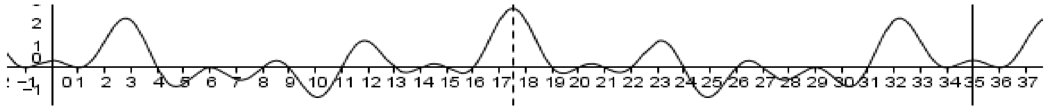
- red: $h_1(x) = sin\frac{\pi(x-1)}{5}sin\frac{\pi(x-4)}{5}$, period=5.
- dotted: $d_1(x) = sin\frac{\pi(x-1)}{5}$, p_d = 1.
- dashed: $u_1(x) = sin\frac{\pi(x-4)}{5}$, p_u = 4.

(a) $h_1(x)$, s_1 = 5.



- red: $h_2(x) = sin\frac{\pi(x-1)}{7}sin\frac{\pi(x-6)}{7}$, period=7.
- dotted: $u_2(x) = sin\frac{\pi(x-1)}{7}$, p_u = 1.
- dashed: $d_2(x) = sin\frac{\pi(x-6)}{7}$, p_d = 6.

(b) $h_2(x)$, s_2 = 7.



(c) $H_2(x) = 5h_1(x)h_2(x)$ (5 is multiplied to enlarge the graph), period *Q* = 5*7 = 35.

Figure 6 shows how the sieve patterns of TPMT can be functionally represented for s_1 = 5 of Figure 5 (a).

Figure 6. Functional representation of the sieve patterns of TPMT, s_1 = 5.

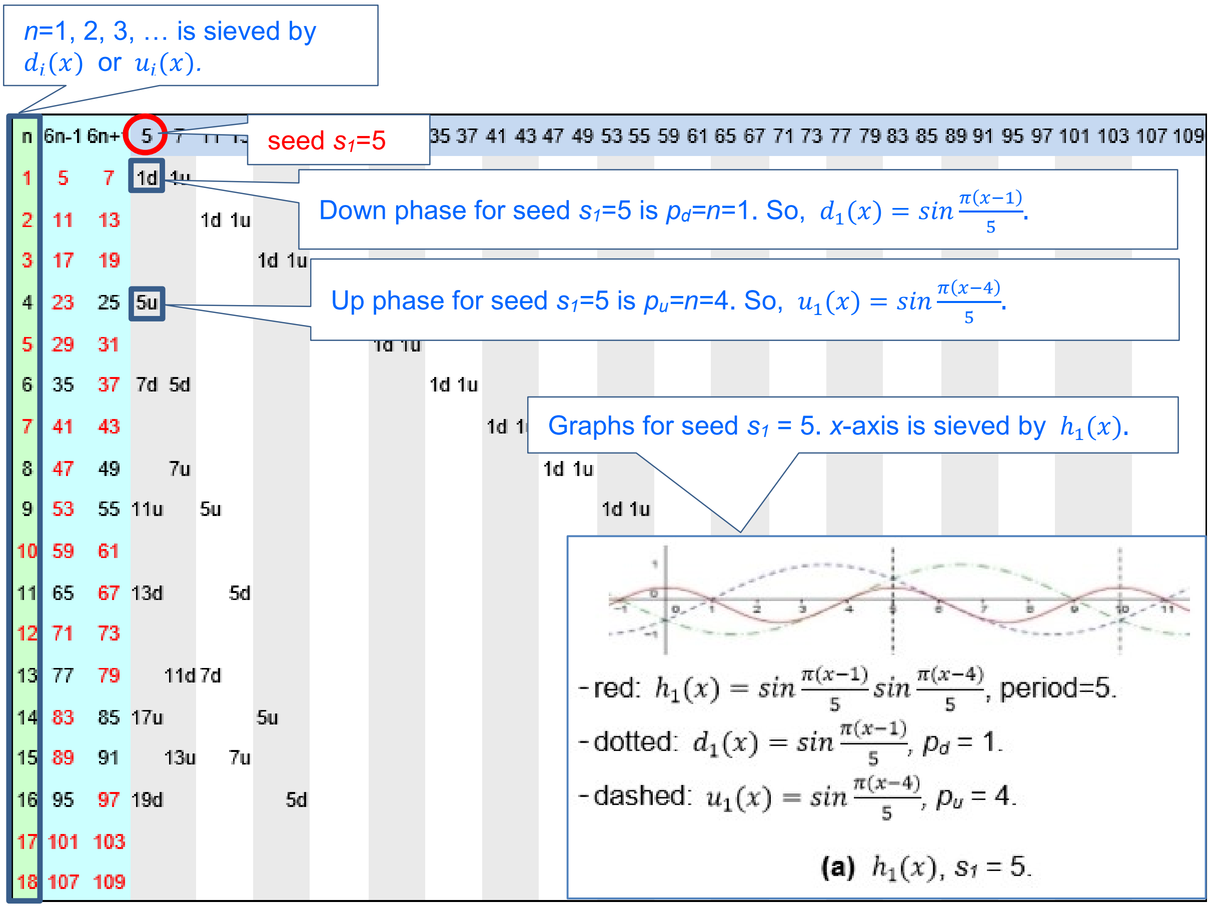
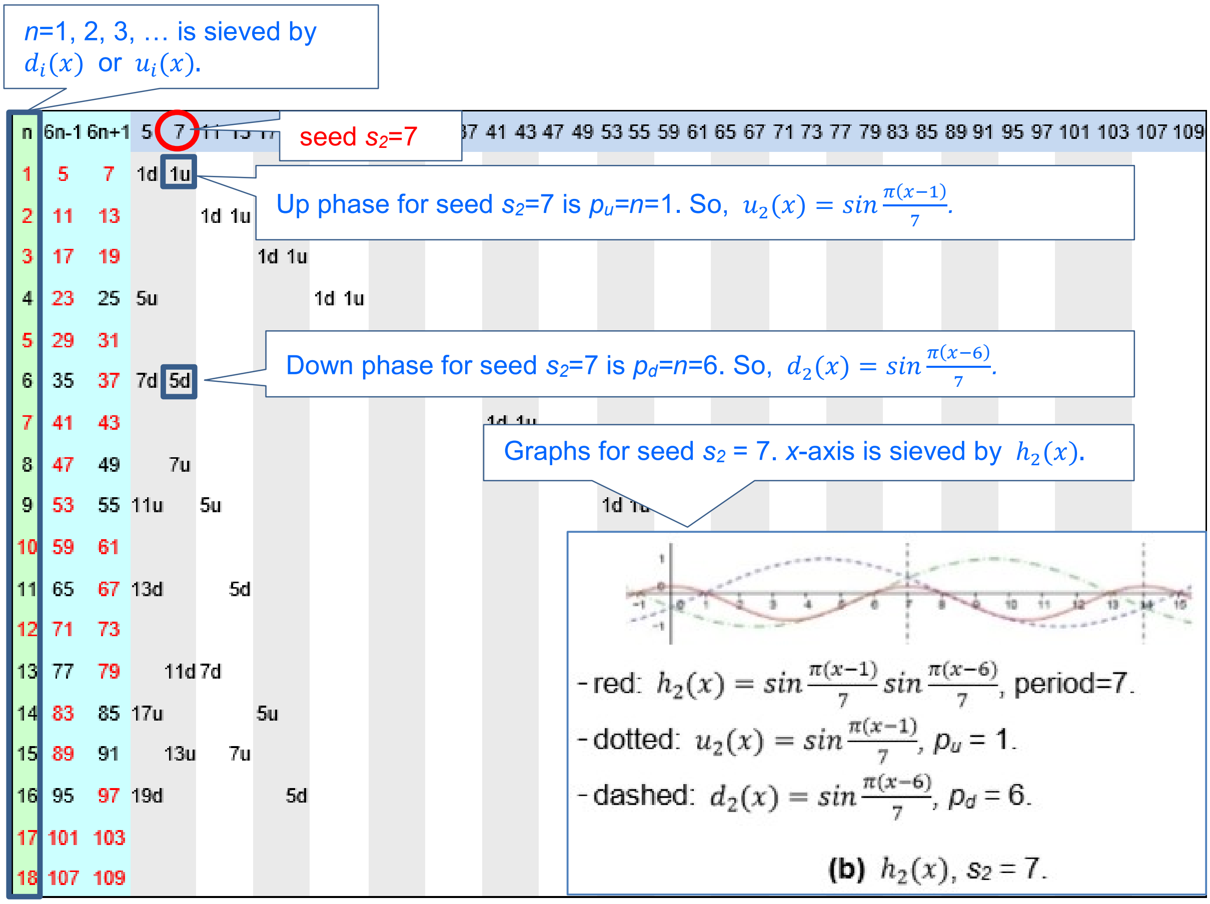


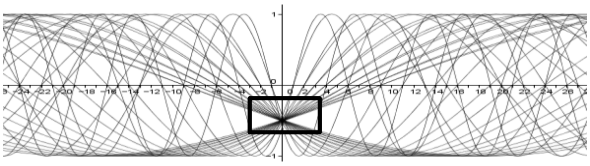
Figure 7 shows how the sieve patterns of TPMT can be functionally represented for s_2 = 7 of Figure 5 (b).

Figure 7. Functional representation of the sieve patterns of TPMT, s_2 = 7.

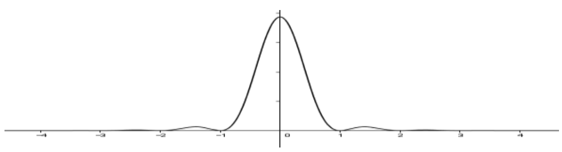


Like SFs and CSFs, pSFs and CpSFs are also periodic functions. Figure 8 depicts pSFs and CpSF with adjusted amplitude for visual effect. (a) depicts the graphs of $d_i(x)$ and $u_i(x)$ for 18 seeds s_i = 5, 7, 11, …, 55, 1 ≤ i ≤ 18. (b) depicts $H_{18}(x) = h_1(x) \dots h_{18}(x)$. (c) depicts enlarged view of some zeros of (a). (d) depicts enlarged view of two concentration points inside the rectangle of (a).

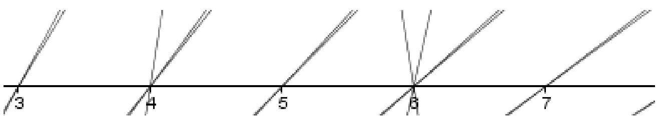
Figure 8. Graphs of pSFs and CpSF for the first 18 seeds.



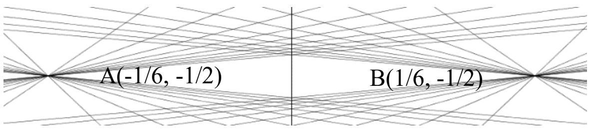
(a) $d_i(x)$ and $u_i(x)$, $1 \le i \le 18$.



(b) CpSF $H_{18}(x)$.



(c) Enlarged view of some zeros of (a).



(d) Enlarged box in (a).

In Figure 8 (c), at least two graphs will cross n = 1, 2, 3, …, which stands for diagonal (*1d, 1u*) pair in Table 1. Extra crosses stand for *md* or *mu* in Table 1, $m \neq 1$. For example, for n =

6, all the left side cells of (*1d, 1u*) are not empty. There are two non-empty cells with *7d* and *5d*, So, 4 graphs cross 6, and (*6n – 1, 6n + 1*) = (35, 37) can't be a TPP.

- **Meaning of zeros:** Figure 8 (c) is enlarged view of some zeros of (a). Two graphs cross 3, 5 and 7, meaning that all cells left side of (*1d, 1u*) in Table 1 are empty for *n* = 3, 5, 7, resulting TPPs. For *n* = 4, 6, more than two graphs cross them, meaning that in Table 1, left side cells of (*1d, 1u*) are not all empty for *n* = 4, 6, failing to be TPPs.

So, in view of phased SFs, <u>*sieved n*</u> are integers that are crossed more than twice by graphs of $d_i(x)$ and $u_i(x)$, and let's call it *sieved n of the sieve function* $H_k(x)$. Likewise, <u>*unsieved n*</u> are integers that are crossed twice by the graphs of $d_i(x)$ and $u_i(x)$, and let's call it *unsieved n of the sieve function* $H_k(x)$.

Each graphs of (a) crosses one of the two points, we call them concentration points.

- **Concentration points:** Figure 8 (d) shows two concentration points where all graphs of (a) should pass. Which point to pass is determined by the following rules.
  - **Rule 1:** For $u_i(x) = sin\frac{\pi(x-p_u)}{s_i}$, where p_u is up-phase of seed s_i and $s_i = 6i + 1$, $u_i\left(-\frac{1}{6}\right) = sin(-\frac{\pi}{6}\frac{6p_u+1}{6i+1})$. So, when $i = p_u$ it will be $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$. This means that graphs with up-phase will pass $A(-\frac{1}{6}, -\frac{1}{2})$. For example, for seed value 5, (p_d, p_u) = (1, 4), $u_i\left(-\frac{1}{6}\right) = sin\frac{\pi\left(-\frac{1}{6}-\ 4\right)}{5} = sin\left(-\frac{25\pi}{30}\right) = sin\left(-\frac{5\pi}{6}\right) = sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$.
  - **Rule 2:** For $d_i(x) = sin\frac{\pi(x-p_d)}{s_i}$, where p_d is down-phase of seed s_i and $s_i = 6i - 1$, $d_i\left(\frac{1}{6}\right) = sin(-\frac{\pi}{6}\frac{6p_d-1}{6i-1})$. So, when $i = p_d$ it will be $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$. This means that graphs with down-phase will pass $B(\frac{1}{6}, -\frac{1}{2})$. For example, for seed value 5, (p_d, p_u) = (1, 4), $d_i\left(\frac{1}{6}\right) = sin\frac{\pi\left(\frac{1}{6}-\ 1\right)}{5} = sin\left(-\frac{5\pi}{30}\right) = sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$.

In this section, we introduced pSFs and related functions with non-zero phases. The periodic properties of pSFs and CpSFs are similar to those of SFs and CSFs, because they all inherit the properties of sinusoidal functions. By doing so, the sieve patterns of TPMT can be represented by pSFs and CpSFs, and we are ready to prove TPC.

6. Proof of TPC

Lemma 6.1. A CpSF $H_k(x)$ is periodic with period $Q = \prod_{i=1}^{k} s_i$.

Proof. $H_k(x)$ is the product of *k* periodic pSFs $h_i(x)$ with period s_i, so, it is also periodic with period $Q = s_1 s_2 \dots s_k = \prod_{i=1}^{k} s_i$. ■

Lemma 6.2. A CpSF $H_k(x)$ can't make all integers after some specific number *Ω* as sieved *n*, i.e., as its zeros.

Proof. $H_k(x)$ is a finite-period function with non-sieved *n* within its period. So, it will repeats infinitely many times its period, which means that it is impossible to make all integers after a specific number *Ω* as sieved *n*, i.e., as its zeros. ■

Lemma 6.3. There are infinitely many TPPs.

Proof. Let's prove by induction.

Step 1: At *k* = 1, there are infinitely many unsieved *n* of the sieve function $H_1(x)$.

Step 2: Suppose, at *k*, there are infinitely many unsieved *n* of the sieve function $H_k(x)$.

Step 3: Then, at *k + 1*, adding a new seed s_{k+1}, can not change that $H_{k+1}(x)$ is a finite-period function, with unsieved *n* within its period. So, by Lemma 6.2, there are infinitely many unsieved *n* of the sieve function $H_{k+1}(x)$, leaving infinitely many TPPs. ■

As in Lemma 4.7, Lemma 6.2 and 6.3 state that any CpSF can not sieve all integers after some specific number *Ω*, leaving infinite TPPs. Only $sin(\pi x)$ can sieve all integers after a specific number.

7. Conclusion

In this thesis, we devised TPMT and SFs to prove TPC. In TPMT a TPP is found where all left cells of (*1d, 1u*) are empty. SFs and CSFs functionally represent the sieve of Eratosthenes and any CSFs can't sieve all integers after some specific number *Ω*. The functional representations of sieve patterns of TPMT are pSFs and CpSFs. The properties of pSFs and CpSFs are similar to those of SFs and CSFs and any CpSF $H_k(x)$ can not sieve all integers after some specific number *Ω*, leaving infinite TPPs. Only $sin(\pi x)$ can sieve all integers after a specific number. A periodic function repeats values within a period infinitely many times. If values within a period are TPPs, they will also repeat infinitely many times, proving that there can't be last TPP.

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