A Proof of Twin Prime Conjecture by Using Twin Prime Model Table and Sieve Functions

Tae Beom Lee

join360@naver.com

Abstract

A twin prime is a pair of prime numbers of the form (p, p + 2). The assumption that twin primes exist infinitely is called Twin Prime Conjecture(TPC). But, up to date there is no valid proof of TPC. In this thesis we devised Twin Prime Model Table(TPMT) and Sieve Functions(SFs). We found that the sieve patterns of TPMT are equivalent to phased SFs. By using the periodicity of sinusoidal functions, we proved that TPC is true.

1. Introduction

TPC [1][2] states that there are infinitely many prime pairs with distance 2, like (3, 5), (11, 13). Whether there exist infinitely many twin primes has been one of the great questions in number theory for many years and still the question is not answered.

In 2013, Yitang Zhang [3][4] proved that there are infinitely many prime pairs that differ by $(p_{n+1} - p_n) < 7\times10^7$, where p_n is the n'th prime. After Zhang's discovery mathematicians like Terence Tao [5] and James Maynard [6][7] reduced the gap to 6 [2].

Instead narrowing the prime gaps, we devised Twin Prime Model Table(TPMT) and Sieve Functions(SFs), where we found enough sieve patterns to prove TPC. TPMT is a infinitely expandable 2-dimensional arrangement of all possible twin prime pairs, from which we could see how twin primes are found. SFs are functional representation of the sieve of Eratosthenes. By the use of TPMT and SFs, we could clearly see the underlying sieve patterns of TPC that became the basis of our proof.

2. Twin Prime Model Table(TPMT)

Lemma 2.1. Twin Prime Pairs(TPPs), except (3, 5), have the form $(6n - 1, 6n + 1)$, n = 1, 2, 3, ….

Proof. If odd multiples of 3, like 9, 15, 21, exist between two odd numbers, as in 5, 7, <u>9</u>, 11, 13, <u>15</u>, 17, 19, <u>21</u>, 23, 25, <u>27</u>, 29, …, the gap of two possible prime numbers will be greater than 2, failing to be a TPP. So, by removing multiples of 3 from odd number sequence, we get the form $(6n - 1, 6n + 1)$ like (5, 7), (11, 13), (17, 19), (23, 25). ■

TPMT is a 2-dimensional table that shows how twin primes are found. Our idea is that if we treat (*6n - 1, 6n + 1*) as a single vertical entity with a sequence number *n*, and arrange them in a 2-dimensional table, we may see some underlying patterns of TPC.

Definition 2.2. *Twin Prime Model Table(TPMT):* TPMT is an infinitely expandable 2-dimensional table, which arranges all possible twin prime pairs of the form (*6n - 1, 6n + 1*), *n* = 1, 2, 3, …, horizontally and vertically, and has the following properties. Table 1 shows the initial part of TPMT.

Table 1. Initial part of TPMT.

n	6n-1	6n+1	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	53	55	59	61	65	67	71	73	77	79	83	85	89	91	95	97	101	103	107	109
1	5	7	1d	1u																																		
2	11	13			1d	1u																																
3	17	19					1d	1u																														
4	23	25	5u						1d	1u																												
5	29	31									1d	1u																										
6	35	37	7d	5d									1d	1u																								
7	41	43													1d	1u																						
8	47	49		7u													1d	1u																				
9	53	55	11u		5u														1d	1u																		
10	59	61																			1d	1u																
11	65	67	13d			5d																	1d	1u														
12	71	73																							1d	1u												
13	77	79		11d	7d																						1d	1u										
14	83	85	17u				5u																						1d	1u								
15	89	91		13u		7u																									1d	1u						
16	95	97	19d					5d																									1d	1u				
17	101	103																																	1d	1u		
18	107	109																																			1d	1u

∗For larger TPMT, refer to http://re360.kr/$$$twin-6n.asp.

- **Horizontal arrangement.**
  - *n*: Sequence of possible twin prime pairs. TPPs are marked as red.
  - *6n – 1*: Smaller number of a possible TPP. Primes are marked as red.
  - *6n + 1*: Larger number of a possible TPP. Primes are marked as red.
  - (5, 7), (11, 13), (17, 19), … : Horizontal arrangement of all possible twin prime pairs.
- **Vertical arrangement.**
  - Sets {*n*, *6n - 1, 6n + 1*}, *n* = 1, 2, 3, … are vertically arranged. If (*6n - 1, 6n + 1*) is a TPP, {*n*, *6n - 1, 6n + 1*} is marked as red.

- **Inner cells.**
  - Inner cells are empty or marked as *md* or *mu,* where m = 1, 5, 7, 11, … and will be explained below.

Definition 2.3. *Seed value:* A number in the first row 5, 7, 11, 13, ... is called a seed value which is used in sieving out $6n – 1$ or $6n + 1$. We used s or s_i as a seed value variable.

Definition 2.4. *Up-number and down-number:* In ($6n - 1, 6n + 1$), $6n – 1$ is down-number and denoted by d, $6n + 1$ is up-number and denoted by u. So, ($6n - 1, 6n + 1$) can be denoted as (down-number, up-number) = (d, u).

If $6n – 1$ or $6n + 1$ is divisible by some seed value s, it is *sieved out* and is marked as *md* or *mu,* where m = 5, 7, 11, 13, …. If down-number $6n – 1 = ms$, it is marked as *md*. If up-number $6n + 1 = ms$, it is marked as *mu*.

If m = 1, it is marked as (*1d, 1u*) pair, which means that a ($6n - 1, 6n + 1$) pair is always divisible by ($6n - 1, 6n + 1$) pair itself. So, it does not mean sieved out. If one or two in ($6n – 1, 6n + 1$) pair is sieved out, then ($6n – 1, 6n + 1$) can not be a TPP, because at least one of ($6n – 1, 6n + 1$) can not be prime.

Definition 2.5. *Vertical pair:* A pair ($6n - 1, 6n + 1$) which is vertically arranged and mapped to an integer sequence n.

Definition 2.6. *Horizontal pair:* A pair ($6n - 1, 6n + 1$) in horizontal head row which are used as seed values.

From the above definitions, we can see the following sieve out patterns between horizontal and vertical pairs.

- **Sieve out patterns between horizontal and vertical pairs.**
  - **Diagonal (*1d, 1u*) pair pattern:** A horizontal ($6n - 1, 6n + 1$) pair always divides a vertical pair when they are same. So, (*1d, 1u*) pair appears diagonally.
  - **Vertical pattern:** For a horizontal pair, the first downward non-empty pair is always (*1d, 1u*). For a seed value s, (u, d) or (d, u) pattern repeats vertically. This is because, for m = 1, 5, 7, 11, …, $ms = 6n -1$ or $ms = 6n + 1$ match occurs alternately.

From the above sieve out patterns and divisibility between horizontal and vertical pairs, we can derive the following lemma.

Lemma 2.7. In TPMT, if all cells left side of a diagonal (*1d, 1u*) pair are empty, then the corresponding ($6n - 1, 6n + 1$) is a TPP.

Proof. If all cells left side of a diagonal (*1d, 1u*) pair are empty, the corresponding ($6n - 1, 6n + 1$) is divisible only by itself, so, it is a TPP. ■

Definition 2.8. Phases: Phases are the values of n where the first downward *md* or *mu* appears. There are two kinds of phases. One is down-phase p_d, the value of n where the first *md* appears. The other is up-phase p_u, the value of n where the first *mu* appears.

- **Phase examples:** The phase pair of seed 11 is $(p_d, p_u) = (2, 9)$. The phase pair of seed 13 is $(p_u, p_d) = (2, 11)$. Note that seed $s = p_d + p_u$. The phase pair will be used when defining phased Sieve Functions(pSFs). Table 2 shows some (p_d, p_u) or (p_u, p_d) pairs.

Table 2. Phase patterns.

Seed	5	7	11	13	17	19	23	25	29	31	35	37	41	43	47	49	...
Phases	1, 4	1, 6	2, 9	2, 11	3, 14	3, 16	4, 19	4, 21	5, 24	5, 26	6, 29	6, 31	7, 34	7, 36	8, 39	8, 41	...

The sieve patterns of TPMT is summarized in Figure 1.

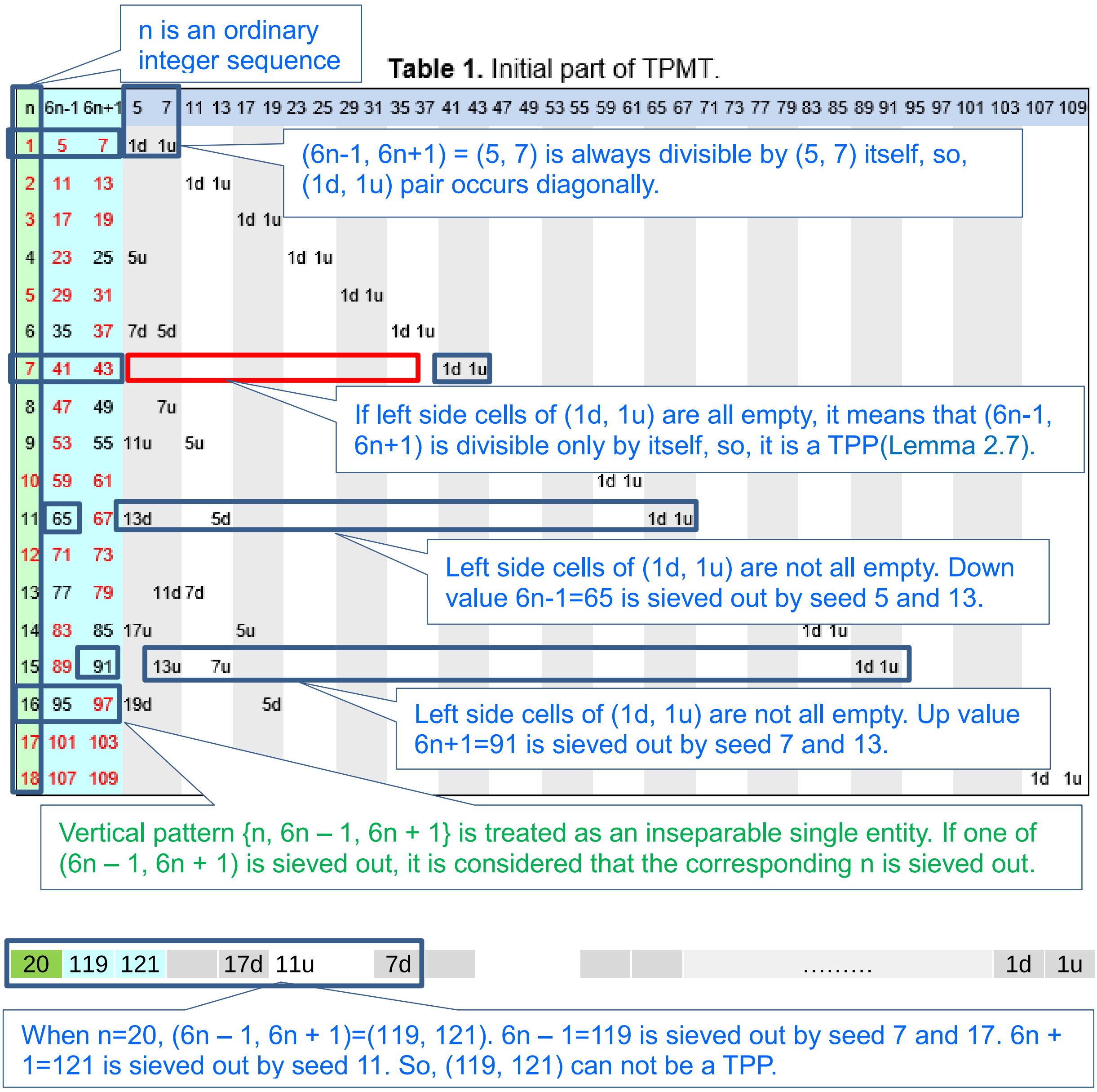


Figure 1. Sieve patterns of TPMT.

3. Sieve View of TPC

In Table 1, we can see that every vertical pair (*6n - 1, 6n* + *1*) is matched to a number sequence *n* = 1, 2, 3, …. So, we can transform sieving out vertical pairs (*6n - 1, 6n* + *1*) to sieving out numbers from *n* = 1, 2, 3, …, as in Figure 2.

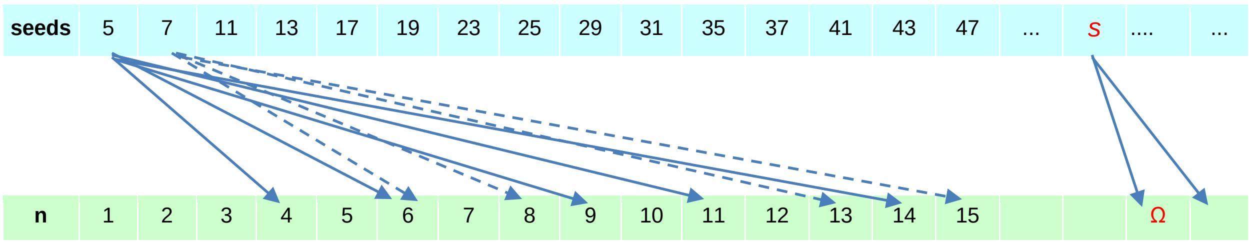


Figure 2. Sieving out numbers from a natural number sequence *n*.

In Figure 2, arrows from each seed, sieve out numbers from a natural number sequence *n* = 1, 2, 3, … and the unsieved *n* represents a TPP. For example, for *n* = 1, (*6n – 1, 6n* + *1*) = (5, 7) is a TPP. For *n* = 7, (*6n – 1, 6n* + *1*) = (41, 43) is also a TPP. But for *n* = 6, (*6n – 1, 6n* + *1*) = (35, 37) can't be a TPP, because 6 is sieved out by seeds 5 and 7.

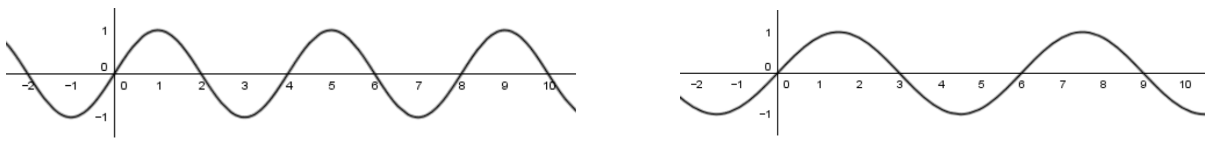
So, TPC is same as whether the seed values 5, 7, 11, 13, … can sieve out all integers after some specific number *Ω* which is marked as red in Figure 2. If such a specific number *Ω* exist, it means that TPPs are finite, and TPC is false.

In this section, we transformed TPC view to a sieve view of an ordinary integer sequence *n* = 1, 2, 3, …, similar to the sieve of Eratosthenes.

4. Sieve Functions

To prove the infinitude of TPPs by using the periodicity of sinusoidal functions, we introduce SFs and Composite Sieve Function(CSF).

Definition 4.1. *Sieve Function(SF):* A sine function, $f_i(x) = sin(\pi x/p_i)$ where p_i is the *i*'th prime number, $p_1 = 2$. Examples of SF are shown in Figure 3.



(a) SF for $p_1 = 2$, $f_1(x) = sin(\pi x/2)$. **(b)** SF for $p_2 = 3$, $f_2(x) = sin(\pi x/3)$.

Figure 3. Example SFs.

In Figure 3, the multiples of p_i are the zeros of $f_i(x)$. So, we can say that numbers $N = tp_i$, t = 2, 3, 4, … are sieved out by $f_i(x) = sin(\pi x/p_i)$. When, t = 1, $N = p_i$ remains unsieved.

To show how SFs give functional view of the sieve of Eratosthenes, let's invoke a well known lemma.

Lemma 4.2. In sieve of Erastothenes of *N* number, we need to sieve out numbers which are the multiples of prime numbers only up to $\sqrt{N}$.

Proof. Suppose $xy = N = \sqrt{N}\sqrt{N}$. If $x \geq \sqrt{N}$, then $y \leq \sqrt{N}$ and vice versa. Thus, if $xy = N$, then one of *x* or *y* must be less than or equal to $\sqrt{N}$. This means that if *N* can be factored, one of the factors which will sieve out *N*, must be less than or equal to $\sqrt{N}$. ■

For example, to find out prime numbers between 1 to 50, we sieve out multiples of prime numbers between 2 and $\sqrt{50} > 7.07$, i.e., 2, 3, 5, 7. In functional view, four SFs, $f_1(x) = \sin(\pi x/2)$, $f_2(x) = \sin(\pi x/3)$, $f_3(x) = \sin(\pi x/5)$, $f_4(x) = \sin(\pi x/7)$ are required to sieve out all composite numbers between 1 and 51.

Definition 4.3. *Composite Sieve Function(CSF):* A product of SFs, $F_k(x) = \prod_{i=1}^{k} f_i(x)$. An example CSF is depicted in Figure 4.

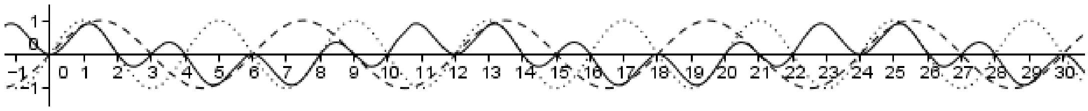


Figure 4. Example CSF, *k* = 2, dotted or dashed graphs are SFs.

Figure 4 depicts an example CSF, $F_2(x) = \prod_{i=1}^{2} f_i(x) = sin(\frac{\pi x}{2})sin(\frac{\pi x}{3})$. The dotted or dashed graphs are $f_1(x) = sin(\frac{\pi x}{2})$ and $f_2(x) = sin(\frac{\pi x}{3})$. Numbers $N = 2t$ and $N = 3t, t = 2, 3, 4, \ldots$ are sieved out by $f_1(x)$ and $f_2(x)$. The unsieved numbers between 1 and 25 are all prime numbers, because 3 is the largest prime number less than $5 = \sqrt{25}$.

Definition 4.4. *Sieved n:* Integers that are sieved out by some sieve function.

Definition 4.5. *Unsieved n:* Integers that are not sieved out by some sieve function.

Lemma 4.6. A CSF $F_k(x) = \prod_{i=1}^{k} f_i(x)$ is a periodic function with period $\prod_{i=1}^{k} p_i$.

Proof. A CSF $F_k(x)$ is the product of *k* periodic sine functions with period p_i, so, a CSF is also a periodic function with period $\prod_{i=1}^{k} p_i$ [8][9]. ■

Lemma 4.7. Any CSF $F_k(x) = \prod_{i=1}^{k} f_i(x)$ can not sieve out all integers after some specific number Ω.

Proof 1. If some CSF $F_k(x)$ can sieve out all integers after a specific number Ω, it means that the prime numbers are finite, which contradicts. ■

Proof 2. $F_k(x)$ is a finite-period function with unsieved *n* within its period. So, it will repeats infinitely many times its period, which means that it is impossible to make all integers after some specific number Ω as sieved *n*. ■

Lemma 4.7 states that any CSF $F_k(x)$ can not sieve out all integers after some specific number Ω, leaving infinite prime numbers, even though $k \to \infty$. Only $sin(\pi x)$ can sieve out all integers after a specific number Ω = 2.

We also can prove the infinitude of prime numbers by induction.

Lemma 4.8. There are infinitely many prime numbers.

Proof. Let's prove by induction.

Step 1: At *k* = 1, there are infinitely many unsieved *n* of the sieve function $F_1(x)$.

Step 2: Suppose, at *k*, there are infinitely many unsieved *n* of the sieve function $F_k(x)$.

Step 3: Then, at *k + 1*, adding a new seed p_{k+1}, can not change that $F_{k+1}(x)$ is a finite-period function, with unsieved *n* within its period. So, by Lemma 4.7, there are infinitely many unsieved *n* of the sieve function $F_{k+1}(x)$, leaving infinitely many prime numbers. ■

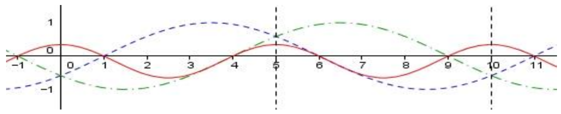
In this section, we introduced SFs and CSFs and showed that the sieve of Eratosthenes can be functionally treated by the sinusoidal sieve functions. The periodic property of sinusoidal sieve functions can give visible understanding on how prime numbers are found and why the last prime number can not exist.

5. phased SFs

In previous section we introduced SFs and CSFs with zero phase. That is to say, the phase of $f_i(x) = sin(\pi x/p_i)$ is zero. But, the sieve patterns of TPMT have non-zero phases as in Definition 2.8. So, we need to define phased SFs.

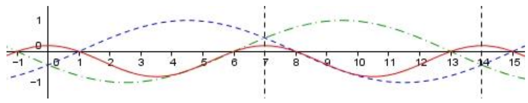
Definition 5.1. *phased Sieve Function(pSF):* A product of two sine functions, $h_i(x) = d_i(x)u_i(x)$, down-sieve-function $d_i(x) = sin\{\pi(x - p_d)/s_i)\}$ and up-sieve-function $u_i(x) = sin\{(\pi(x - p_u)/s_i\}$, where s_i is the *i*'th seed of sequence *s* = 5, 7, 11, …, and (p_d, p_u) is a phase pair of *i*'th seed. The example graphs of $h_i(x)$ are depicted in Figure 5 (a) and (b).

Definition 5.2. *Composite phased Sieve Function(CpSF):* The product of pSFs, $H_k(x) = \prod_{i=1}^{k} h_i(x)$, *k* = 1, 2, 3, …, as in Figure 5 (c).



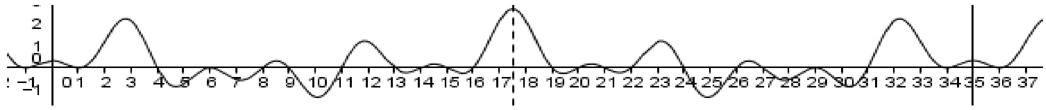
(a) $h_1(x)$, $s_1 = 5$.

- red: $h_1(x) = \sin\{\pi(x - 1)/5\}\sin\{\pi(x - 4)/5\}$, period=5.
- dotted: $d_1(x) = \sin\{\pi(x - 1)/5\}$, $p_d = 1$.
- dashed: $u_1(x) = \sin\{\pi(x - 4)/5\}$, $p_u = 4$.



(b) $h_2(x)$, $s_2 = 7$.

- red: $h_2(x) = \sin\{\pi(x - 1)/7)\sin(\pi(x - 6)/7)\}$, period=7.
- dotted: $u_2(x) = \sin\{\pi(x - 1)/7\}$, $p_u = 1$.
- dashed: $d_2(x) = \sin\{\pi(x - 6)/7\}$, $p_d = 6$.



(c) $H_2(x) = 5h_1(x)h_2(x)$ (5 is multiplied for visual effect), period $Q = 5*7 = 35$.

Figure 5. pSF and CpSF examples.

Following Figure 6 shows how the sieve patterns of TPMT can be functionally implemented for $s_1 = 5$, as in Figure 5 (a).

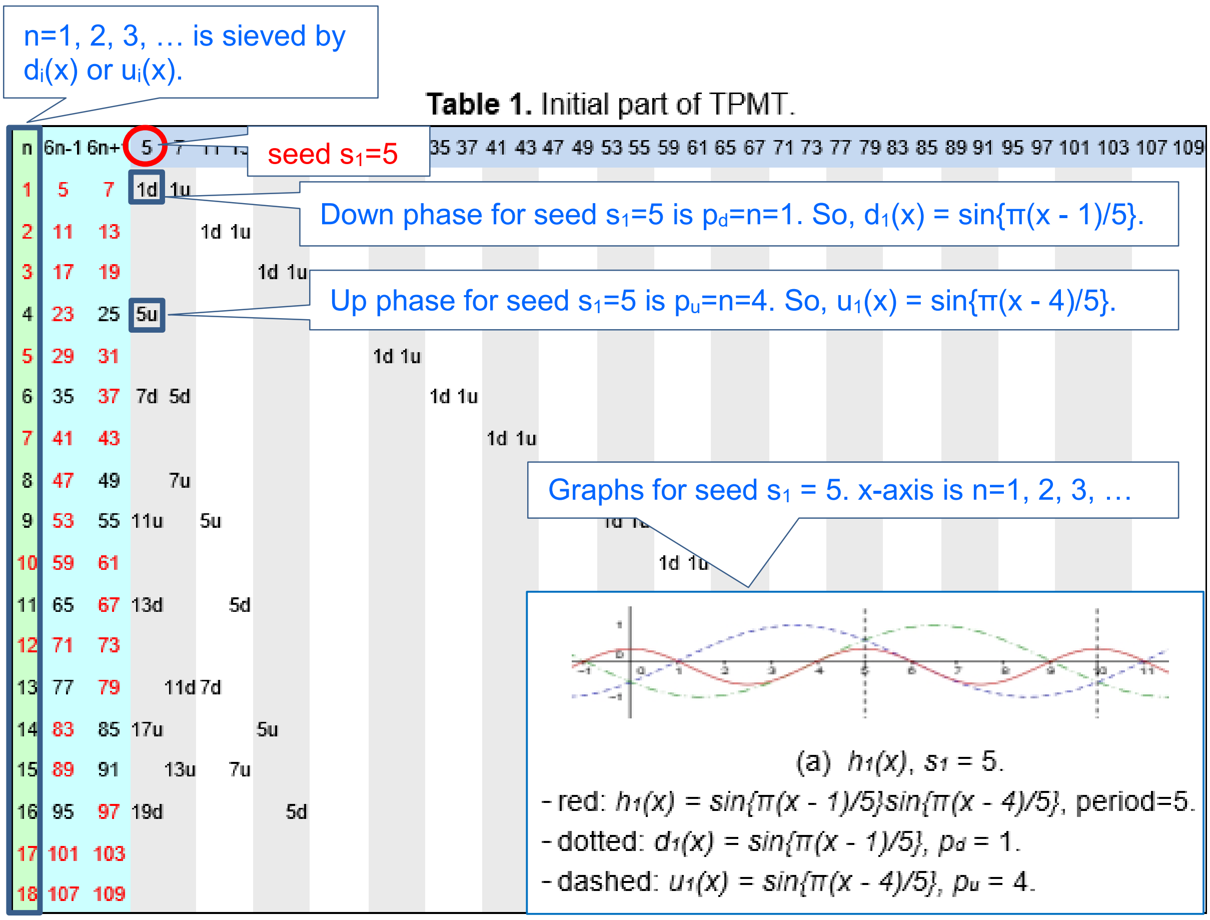


Figure 6. Functional implementation of the sieve patterns of TPMT, $s_1 = 5$.

Following Figure 7 shows how the sieve patterns of TPMT can be functionally implemented for s_2 = 7, as in Figure 5 (b).

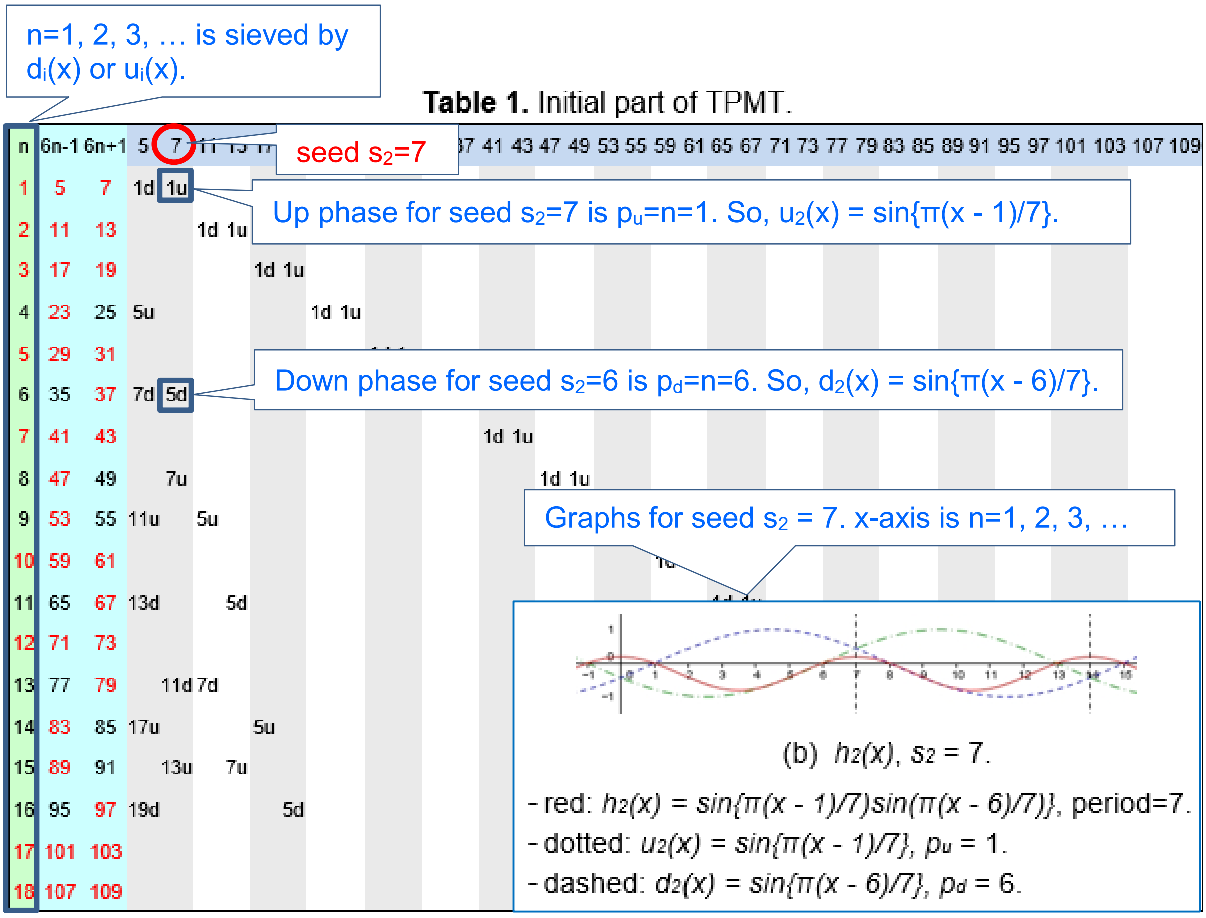
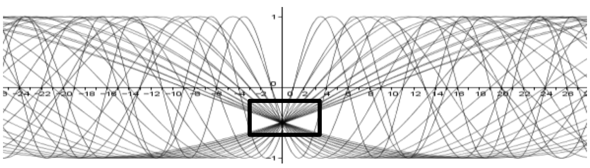
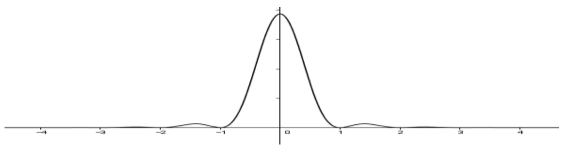


Figure 7. Functional implementation of the sieve patterns of TPMT, s_2 = 7.

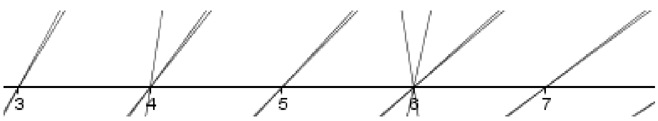
Like SFs and CSFs, pSFs and CpSFs are also periodic functions. Figure 8 depicts pSFs and CpSF with adjusted amplitude for visual effect. (a) depicts the graphs of $d_i(x)$ and $u_i(x)$ for 18 seeds s_i = 5, 7, 11, …, 55, 1 ≤ i ≤ 18. (b) depicts $H_{18}(x) = h_1(x)\ldots h_{18}(x)$. (c) depicts enlarged view of some zeros of (a). (d) depicts enlarged view of two concentration points inside the rectangle of (a).



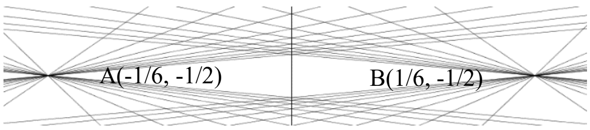
(a) $d_i(x)$ and $u_i(x)$, $1 \leq i \leq 18$.



(b) CpSF $H_{18}(x)$.



(c) Enlarged view of some zeros of (a).



(d) Enlarged box in (a).

Figure 8. Graphs of pSFs and CpSF for the first 18 seeds.

In Figure 8 (c), at least two graphs will cross n = 1, 2, 3, …, which stands for diagonal (*1d, 1u*) pair in Table 1. Extra crosses stand for *md* or *mu* in Table 1, $m \neq 1$. For example, for n = 6, all the left side cells of (*1d, 1u*) are not empty. There are two non-empty cells with *7d* and *5d*, So, 4 graphs cross 6, and ($6n - 1$, $6n + 1$) = (35, 37) can't be a TPP.

- **Meaning of zeros:** Figure 8 (c) is enlarged view of some zeros of (a). Two graphs cross 3, 5 and 7, meaning that all cells left side of (*1d, 1u*) in Table 1 are empty for n = 3, 5, 7, resulting TPPs. For n = 4, 6, more than two graphs cross them, meaning that in Table 1, left side cells of (*1d, 1u*) are not all empty for n = 4, 6, failing to be TPPs.

So, in view of phased SFs, *sieved n* are integers that are crossed more than twice by graphs of $d_i(x)$ and $u_i(x)$, and let's call it *sieved n of the sieve function* $H_k(x)$. Likewise, *unsieved n* are integers that are crossed twice by the graphs of $d_i(x)$ and $u_i(x)$, and let's call it *unsieved n of the sieve function* $H_k(x)$.

Each graphs of (a) crosses one of the two points, we call them concentration points.

- **Concentration points:** Figure 8 (d) shows two concentration points where all graphs of (a) should pass. Which point to pass is determined by the following rules.
  - **Rule 1:** For $u_i(x) = \sin\{\pi(x - p_u)/s_i)\}$, where p_u is up-phase of seed s_i and $s_i = 6i + 1$, $u_i(-1/6) = \sin\{-\pi/6(6\cdot p_u + 1)/(6i + 1)\}$. So, when $i = p_u$ it will be $\sin(-\pi/6)$ = -1/2. This means that graphs with up-phase will pass A(-1/6, -1/2). For example, for seed value 5, (p_d, p_u) = (1, 4), $\sin\{\pi(x - 4)/5\} = \sin\{\pi(-1/6 - 4)/5\} = \sin(-\pi 5/6) = \sin(-\pi/6)$.
  - **Rule 2:** For $d_i(x) = \sin\{\pi(x - p_d)/s_i)\}$, where p_d is down-phase of seed s_i and $s_i = 6i - 1$, $d_i(1/6) = \sin\{-\pi/6(6\cdot p_d - 1)/(6i - 1)\}$. So, when $i = p_d$ it will be $\sin(-\pi/6)$ = -1/2. This means that graphs with down-phase will pass B(1/6, -1/2). For example, for seed value 5, (p_d, p_u) = (1, 4), $\sin\{\pi(x - 1)/5\} = \sin\{\pi(1/6 - 1)/5\} = \sin(-\pi/6)$.

In this section, we introduced pSFs and related functions with non-zero phases. The periodic properties of pSFs and CpSFs are similar to those of SFs and CSFs, because they all inherit the properties of sinusoidal functions. By doing so, the sieve patterns of TPMT can be implemented as the pSFs and CpSFs, and we are ready to prove TPC.

6. Proof of TPC

Lemma 6.1. A CpSF $H_k(x)$ is periodic with period $Q = \prod_{i=1}^{k} s_i$.

Proof. $H_k(x)$ is the product of k periodic pSFs $h_i(x)$ with period s_i, so, it is also periodic with period $Q = s_1 s_2 \dots s_k = \prod_{i=1}^{k} s_i$. ■

Lemma 6.2. A CpSF $H_k(x)$ can't make all integers after some specific number Ω as sieved n.

Proof. $H_k(x)$ is a finite-period function with non-sieved n within its period. So, it will repeats infinitely many times its period, which means that it is impossible to make all integers after a specific number Ω as sieved n. ■

Lemma 6.3. There are infinitely many TPPs.

Proof. Let's prove by induction.

Step 1: At k = 1, there are infinitely many unsieved n of the sieve function $H_1(x)$.

Step 2: Suppose, at k, there are infinitely many unsieved n of the sieve function $H_k(x)$.

Step 3: Then, at $k + 1$, adding a new seed s_{k+1}, can not change that $H_{k+1}(x)$ is a finite-period function, with unsieved n within its period. So, by Lemma 6.2, there are infinitely many unsieved n of the sieve function $H_{k+1}(x)$, leaving infinitely many corresponding TPPs. ■

7. Conclusion

In this thesis, we devised TPMT and SFs to prove TPC. In TPMT a TPP is found where all left cells of (*1d, 1u*) are empty. We also transformed TPC view to the sieving out of a natural number sequence n = 1, 2, 3, …. By introducing SFs and CSFs, we can have functionally equivalent representations of the sieve of Eratosthenes. We showed that the proof of Euclid for the infinitude of prime numbers, is equivalent to the fact that the CSFs can not sieve out all integers after some specific number *Ω*. To functionally implement the sieve patterns of TPMT, we introduced pSFs and CpSFs, where the phase values are not all zeros. The periodic properties of pSFs and CpSFs are similar to those of SFs and CSFs, even though phase values are not all zeros. A periodic function repeats values within a period infinitely many times. If the values within a period are TPPs, they will also repeat infinitely many times, proving that TPPs can not be finite.

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