A logarithmic integral and related integrals

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abstract

In this note we give some formulas related with the integral:

$$\int_0^\infty \frac{-\ln x}{\sqrt{x}\,(1+x^2)(1+x^3)}\,dx = \frac{2\pi^2}{3\sqrt{3}}$$

keywords: logarithmic integral , number Pi , series

I. Introduction

Theorem.

$$\int_0^\infty \frac{-\ln x}{\sqrt{x}\,(1+x^2)(1+x^3)}\,dx = \frac{2\pi^2}{3\sqrt{3}} \tag{1}$$

Proof. In Gradshteyn - Ryzhik , p.377 , 3.524.12 , appears

$$\int_0^\infty x\,\frac{\sinh a x}{\cosh b x}\,dx = \frac{\pi^2}{4b^2}\sin\left(\frac{a\pi}{2b}\right)\sec^2\left(\frac{a\pi}{2b}\right)\,,\quad b>|a| \tag{2}$$

for $a=2$, $b=3$ we have

$$\begin{aligned}\frac{2\pi^2}{3\sqrt{3}} &= 4\int_0^\infty \frac{x\sinh 2x}{\cosh 3x}\,dx = 4\int_0^\infty \frac{x\sinh 2x\cosh 2x}{\cosh 2x\cosh 3x}\,dx = 2\int_0^\infty \frac{x\sinh 4x}{\cosh 2x\cosh 3x}\,dx = \\ &\int_{-\infty}^\infty \frac{x\sinh 4x}{\cosh 2x\cosh 3x}\,dx = \int_{-\infty}^\infty \frac{x\,e^{4x}}{\cosh 2x\cosh 3x}\,dx = 4\int_0^\infty \frac{x^3\ln x}{(x^2+x^{-2})(x^3+x^{-3})}\,dx = 4\int_0^\infty \frac{x^8\ln x}{(1+x^4)(1+x^6)}\,dx = \\ &2\int_0^\infty \frac{x^4\ln\sqrt{x}}{(1+x^2)(1+x^3)\sqrt{x}}\,dx = \int_0^\infty \frac{x^4\ln x}{\sqrt{x}\,(1+x^2)(1+x^3)}\,dx = -\int_0^\infty \frac{\ln x}{\sqrt{x}\,(1+x^2)(1+x^3)}\,dx\end{aligned} \tag{3}$$

remark: $\pi = 4\sum_{n=0}^\infty (-1)^n (2n+1)^{-1}$.

II. Related integrals

$$\int_0^\infty \frac{-\ln x}{(1+x^4)(1+x^6)}\,dx = \frac{\pi^2}{6\sqrt{3}} \tag{4}$$

$$\int_0^\infty \frac{x^8 \ln x}{\left(1+x^4\right)\left(1+x^6\right)}\, dx = \frac{\pi^2}{6\sqrt{3}} \tag{5}$$

$$\int_0^\infty \frac{\sqrt{x}\ x^3 \ln x}{\left(1+x^2\right)\left(1+x^3\right)}\, dx = \frac{2\pi^2}{3\sqrt{3}} \tag{6}$$

$$-\int_0^\infty \frac{\left(1-x^4\right)\ln x}{1+x^6}\, dx = \frac{\pi^2}{3\sqrt{3}} \tag{7}$$

$$-\int_0^\infty \frac{\left(1-x^2\right)\ln x}{1-x^2+x^4}\, dx = \frac{\pi^2}{3\sqrt{3}} \tag{8}$$

$$-\int_0^1 \frac{\left(1-x^2\right)\ln x}{\sqrt{x}\ \left(1+x^3\right)}\, dx = \frac{2\pi^2}{3\sqrt{3}} \tag{9}$$

$$-\int_0^1 \frac{(1-x)\ln x}{\sqrt{x}\ \left(1-x+x^2\right)}\, dx = \frac{2\pi^2}{3\sqrt{3}} \tag{10}$$

$$\int_1^\infty \frac{(x-1)\ln x}{\sqrt{x}\ \left(1-x+x^2\right)}\, dx = \frac{2\pi^2}{3\sqrt{3}} \tag{11}$$

$$\int_0^\infty \frac{x\ln(1+x)}{\sqrt{1+x}\ \left(1+x+x^2\right)}\, dx = \frac{2\pi^2}{3\sqrt{3}} \tag{12}$$

$$-\int_0^\infty \frac{1}{\left(1+x+x^2\right)\sqrt{x(1+x)}} \ln\left(\frac{x}{1+x}\right) dx = \frac{2\pi^2}{3\sqrt{3}} \tag{13}$$

$$-\int_0^1 \frac{x^3}{\left(1-2x+2x^2\right)\left(1-3x+3x^2\right)} \sqrt{\frac{x}{1-x}}\ \ln\left(\frac{1-x}{x}\right) dx = \frac{2\pi^2}{3\sqrt{3}} \tag{14}$$

$$-\int_0^1 \frac{(1-x)^3}{\left(1-2x+2x^2\right)\left(1-3x+3x^2\right)} \sqrt{\frac{1-x}{x}}\ \ln\left(\frac{x}{1-x}\right) dx = \frac{2\pi^2}{3\sqrt{3}} \tag{15}$$

$$\int_0^1 \frac{x}{1+14x^2+x^4} \ln\left(\frac{1+x}{1-x}\right) dx = \frac{\pi^2}{48\sqrt{3}} \tag{16}$$

$$\int_{-\infty}^\infty \frac{x\, e^{4x}}{\cosh(2x)\cosh(3x)}\, dx = \frac{2\pi^2}{3\sqrt{3}} \tag{17}$$

$$\int_{-\infty}^\infty \frac{x\sinh(4x)}{\cosh(2x)\cosh(3x)}\, dx = \frac{2\pi^2}{3\sqrt{3}} \tag{18}$$

$$\int_0^\infty \frac{x\sinh(4x)}{\cosh(2x)\cosh(3x)}\, dx = \frac{\pi^2}{3\sqrt{3}} \tag{19}$$

$$\int_0^\infty \frac{x\sinh(2x)}{\cosh(3x)}\, dx = \frac{\pi^2}{6\sqrt{3}} \tag{20}$$

$$\int_0^\infty \frac{x\left(1-(\tanh x)^2\right)\tanh x}{1+14(\tanh x)^2+(\tanh x)^4}\, dx = \frac{\pi^2}{96\sqrt{3}} \tag{21}$$

$$\int_0^\infty \frac{(\sinh x)^3 \ln \cosh x}{7+\cosh(4x)}\, dx = \frac{\pi^2}{48\sqrt{3}} \tag{22}$$

$$\int_0^\infty \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}\,\left(1+4x^2\right)}\, dx = \frac{\pi^2}{12\sqrt{3}} \tag{23}$$

$$\int_0^\infty \frac{x \ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}\,\left(\left(\sqrt{1+x^2}+x\right)^{3/2}+\left(\sqrt{1+x^2}-x\right)^{3/2}\right)}\, dx = \frac{\pi^2}{3\sqrt{3}} \tag{24}$$

$$\int_1^\infty \frac{\left(\left(x+\sqrt{x^2-1}\right)^{2/3}-\left(x-\sqrt{x^2-1}\right)^{2/3}\right)\ln\left(x+\sqrt{x^2-1}\right)}{x\sqrt{x^2-1}}\, dx = \pi^2\sqrt{3} \tag{25}$$

III. Related Series

$$\frac{2\pi^2}{3\sqrt{3}} = \frac{16}{9}\cdot\sum_{n=0}^{\infty} 3^{-n} \sum_{k=0}^{n}\sum_{m=0}^{n-k}\binom{n-k}{m}(-2)^m \sum_{r=0}^{k}\binom{k}{r}(-2)^r\left(\frac{1}{(4m+6r+1)^2}-\frac{1}{(4m+6r+9)^2}\right) \tag{26}$$

$$\frac{\pi^2}{6\sqrt{3}} = \sum_{n=0}^{\infty}(-1)^n \sum_{k=0}^{n}\left((4n+2k+1)^{-2}-(4n+2k+9)^{-2}\right) \tag{27}$$

$$\frac{2\pi^2}{3\sqrt{3}} = \sqrt{2}\,\ln 2 \sum_{n=0}^{\infty}(-1)^n 2^{-3n}\left(\frac{1}{6n+1}-\frac{1/4}{6n+5}\right) + 2\sqrt{2}\sum_{n=0}^{\infty}(-1)^n 2^{-3n}\left(\frac{1}{(6n+1)^2}-\frac{1/4}{(6n+5)^2}\right) + \sum_{n=0}^{\infty}(-1)^n 2^{-3n-4}\sum_{k=0}^{n}\frac{(-1)^k 2^{2k}}{k+1}\sum_{m=0}^{n-k}\binom{2m}{m}(-1)^m\left(\frac{2}{3n-2k-2m+3}+\frac{1}{3n-2k-2m+4}\right) \tag{28}$$

$$\frac{\pi^2}{3\sqrt{3}} = \frac{1}{\sqrt{e}}\sum_{n=0}^{\infty}(-1)^n e^{-2n}\sum_{k=0}^{n} e^{-k}\left(\frac{4n+2k+3}{(4n+2k+1)^2}-\frac{e^{-4}(4n+2k+11)}{(4n+2k+9)^2}\right) + 2\int_0^{1/2}\frac{x\sinh(2x)}{\cosh(3x)}\, dx \tag{29}$$

For $0 < z < 1$, we have

$$\frac{2\pi^2}{3\sqrt{3}} = 4\sqrt{z}\sum_{n=0}^{\infty}\left(-z^2\right)^n\sum_{k=0}^{n}\frac{z^k}{(4n+2k+1)^2} - 2\sqrt{z}\,\ln z\sum_{n=0}^{\infty}\left(-z^2\right)^n\sum_{k=0}^{n}\frac{z^k}{4n+2k+1} + \int_z^\infty \frac{-\ln x}{\sqrt{x}\,\left(1+x^2\right)\left(1+x^3\right)}\, dx \tag{30}$$

$$\frac{\pi^2}{6\sqrt{3}} = 24\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{\left(36n^2+36n+5\right)^2} \tag{31}$$

$$\frac{\pi^2}{6\sqrt{3}} = \sum_{n=0}^{\infty}\left(\frac{1}{(12n+1)^2}-\frac{1}{(12n+5)^2}-\frac{1}{(12n+7)^2}+\frac{1}{(12n+11)^2}\right) \tag{32}$$

References

A. Gradshteyn, I.S., and Ryzhik, I.M. : Table of Integrals, Series, and Products. Seventh edition, edited by Alan Jeffrey and Daniel Zwillinger, Academic Press , 2007 .

B. Reynolds, R. and Stauffer, A. : Note on an integral by Anatolii Prudnikov, AIMS Mathematics, 6(3): 2680-2689. DOI: 10.3934/math-.2021162 , 2020 , http://www.aimspress.com/journal/Math .