The Standard Model of Elementary Fermions and the Hilbert Repository

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Abstract

The word space is used in many ways and most of these applications give this word a different meaning. This makes the notion of space very obscure. Especially philosophers, mathematicians, and physicists have attributed a huge number of interpretations of the noun "space". This has led to a huge number of different forms of space. Humans live in an environment that is characterized by space and time. This paper focuses on the most elemental meanings that mathematicians and physicists attribute to the word "space". Next, the immediate extensions of this elementary space are investigated. Since physicists investigate our physical reality, the paper also investigates how physical reality treats the notion of space. This leads to a revolutionary new mathematical concept that is called the Hilbert repository. It exposes great similarity with part of the Standard Model of particle physics that concerns elementary fermions. This model exposes what experimenters have discovered about these elementary object types.

1 Mathematics versus reality

Mathematicians are humans and therefore they need names or symbols and extensive descriptions and recipes of the notions that they use. Without these linguistic extensions, for humans, mathematics would be unworkable. Physical reality does not require these additions. Reality does not use manuals or handbooks. Reality just applies the bare concepts. Still, it must obey the rules that are set by the structures and mechanisms. Physical reality does not intelligently obey rules. Probably, reality uses the trial-and-error approach. But that means that this approach must be efficient enough. The structures and mechanisms that reality applies must guide their usage automatically. Simple structures must automatically emerge into more complicated structures that offer restrictions that guide their usage. Mechanisms must limit the ways that they can be accessed.

The names and descriptions that humans use for mathematical and physical subjects are sometimes confusing. This is due to the history of these linguistic tools. It is impossible to cure these unhappy historical facts. It is important to be aware of the existence of these confusing habits.

In mathematics, spaces exist in many forms, and in combination with mechanisms they constitute dynamic systems. We will investigate these spaces and mechanisms to explain how these bare ingredients can successfully constitute dynamic systems.

The elemental spaces must emerge into more complicated spaces and the capabilities of these extensions must become automatically accessible.

The restrictions that go together with the extension of the model limit the structures and mechanisms that reality applies. This limits the part of mathematics that is suitable for comprehending the lower levels of the structure and behavior of physical reality. This does not imply that the current state of humanly developed mathematics covers all aspects of these lower levels. The lower levels of the structure and behavior of physical reality still contain incomprehensible mysteries. One of them is formed by the origin of the stochastic processes that control part of the dynamics of physical reality.

2 Demarcation

We will restrict our investigation to the simplest objects that can occupy space. These objects are point-like, or the objects are conglomerates of point-like objects. Space covered with a countable set of point-like objects behaves differently from space that is covered with an uncountable set of point-like objects. Uncountable sets form a sticky medium. It appears that these two different mediums can interact. The Hilbert repository forms a structure in which this interaction can be modeled. This document investigates what the Hilbert repository can explain.

3 Vector space

In human mathematics, space is not a well-defined concept. A vector space is considered as a quite elemental form of space. We start with a completely empty space. Completely empty space is a synonym for the ultimate nothingness. The next step involves the insertion of two point-like objects in the completely empty space. The first added point is the base point of a vector. The second point is the pointer of the vector. The vector has a length and a direction. The direction defines a direction line. The direction line contains at least two separate points. The integrity of the vector is conserved when it is shifted in parallel as one unit to a different location. This turns empty space into a vector space. Vectors can reach any available location in the vector space. If a vector is shifted in parallel until its beginning point coincides with the beginning point of another vector, then the difference in direction of the two vectors becomes apparent. We shall see that a sticky medium is synonymous with space that is completely covered with point-like objects. We apply the vector space to generate number systems. In the vector space, number systems define virtual locations. At the maiden state of the coordinate system, the coordinate markers turn these virtual locations into locations of point-like objects. In this way, the markers help to navigate in the set of point-like objects. The coordinate markers tell the life story of the point-like objects that start at the maiden state of the coordinate system.

4 The real number system

4.1 Counting and addition

We start with generating a simple number system. One possibility is that the vector is shifted along its direction line such that its base point takes the location of the pointer location of the original vector. This action creates a new vector that consists of the base point of the first vector and the pointer location of the second vector. The length of the third vector is twice the length of the first vector. All contributing points find a position at the same direction line. The contributing points act as counts and the shift installs the addition procedure.

Repeating the shift and addition procedures generates the set of the ***natural numbers***. The procedure of addition can be reversed into subtraction until the base point of the first vector is passed. This is reason to identify this point as the condition in which space is back to being completely empty; For that reason, this point is called zero. If reverse addition is taken further, then this action introduces negative integer numbers. Together with zero and the natural numbers, this constitutes the set of ***integer numbers***.

4.2 Multiplication, division, and fractions

The following step is the introduction of multiplication by combining multiple additions of the same integer number. Multiplication with integer numbers does not introduce new numbers, but the reverse operation that we will call division can introduce new numbers that we call fractions or ratios. In this way, the number system is extended to the set of ***rational numbers***. All rational numbers except zero can be applied as a divisor. Scientists have shown that all rational numbers can be labeled with a natural number. This means that the set of rational numbers is still ***countable***. This also indicates that all rational numbers are still surrounded by empty space.

4.3 Superseding countability

Up to so far, all rational numbers take a location on the same direction line. The square of a rational number is a multiplication of that number with itself. The result is a rational number. The reverse operation is called square root and this operation does not always result in a ratio. However, a converging series of rational numbers can approach the result arbitrarily close. Many numbers exist that are not rational numbers and can be approached arbitrarily close by converging series of rational numbers. We call these numbers, ***irrational numbers***. The set of irrational numbers is not countable. If the set of the rational numbers is merged with the set of the irrational numbers, then the set of ***real numbers*** results. The set of all real numbers completely covers the same direction line. If the set covers all irrational numbers, then, on the direction line, around the real numbers no space is left. This fact drastically changes the behavior of the covering set of point-like objects.

5 Spatial dimensions

5.1 Different arithmetic

The direction line that is covered by all real numbers leaves no space to add extra numbers. If we want to add all square roots of negative real numbers, then we must use one or three new direction lines that are independent of the direction line that is occupied by the real numbers. The independent direction lines cross at point zero. The arithmetic on these new direction lines differs from the arithmetic of the real number direction line. We call the new direction lines spatial direction lines. The spatial arithmetic will automatically add a third independent direction line when a second spatial direction line is added. The real number direction line together with one spatial direction line forms the set of the complex numbers. The real number direction line together with three spatial direction lines form the set of the quaternions. Multiplying

spatial numbers with real numbers is straightforward. In handling the arithmetic of multidimensional number systems, it is wise to treat the combined number as a sum of a real number and a spatial number.

On spatial direction lines, the square of the spatial numbers results in a negative real number. Spatial numbers can be natural, rational, and irrational. Also, in spatial dimensions, the addition of all irrational numbers will supersede countability. The main difference between real numbers and spatial numbers lays in the value of the square of the numbers. In real numbers, the square is always a positive real number. In spatial numbers, the square is always a negative real number. The product of two arbitrary spatial numbers is the sum of a real scalar and a new spatial number that is perpendicular to both factors. The real scalar equals the inner product of the two spatial factors. The new spatial number equals the outer product of the two spatial factors.

5.2 Multidimensional arithmetic

In mathematics, a field is a set on which addition, subtraction, multiplication, and division are defined and behave mostly as the corresponding operations on rational and real numbers do. Division rings differ from mathematical fields in that their multiplication is not required to be commutative.

The real number arithmetic and the spatial number arithmetic can be mixed. Spatial numbers that reside on different spatial direction lines can be added and multiplied. This will make the spatial number space of the quaternions isotropic. The coordinate markers will capture the geometric symmetry and the location of the geometric center. Real numbers can be added and multiplied by spatial numbers.

The mix of real numbers and spatial numbers constitutes an associative division ring.

For multidimensional numbers, we will use boldface to indicate the spatial part and we will indicate the real part with suffix $_r$.

Thus, the number a will be represented by the sum $a=a_r+\mathbf{a}$. This means that the product c=a b of two numbers a and b will split into several terms

$c = c_r+\mathbf{c} = a\,b = (a_r+\mathbf{a})\,(b_r+\mathbf{b}) = a_r\,b_r + a_r\,\mathbf{b} + \mathbf{a}\,b_r + \mathbf{a}\,\mathbf{b}$

The product d of two spatial numbers **a** and **b** results in a real scalar part d_r and a new spatial part **d**

$d = d_r+\mathbf{d} = \mathbf{a}\,\mathbf{b}$

$d_r = -\langle \mathbf{a},\mathbf{b} \rangle$ is the inner product of **a** and **b**

$\mathbf{d} = \mathbf{a} \times \mathbf{b}$ is the outer product of **a** and **b**

The spatial vector **d** is independent of **a** and independent of **b**. This means that $\langle \mathbf{a},\mathbf{d} \rangle=0$, and $\langle \mathbf{b},\mathbf{d} \rangle=0$

For the inner product and the norm, $\|\mathbf{a}\|$ holds $\langle \mathbf{a},\mathbf{a} \rangle = \|\mathbf{a}\|^2$

$\langle \mathbf{a},\mathbf{b} \rangle = \|\mathbf{a}\|\,\|\mathbf{b}\|\cos(\alpha)$

Mathematics often treats spatial numbers as vectors. Mathematics defines the inner product of vectors that represent spatial numbers as the above geometric scalar vector product. It is also called the dot product of two vectors. Hilbert spaces define a different kind of inner product. It is important to distinguish between the inner product in spatial number systems and the inner product in Hilbert spaces.

$\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\,\|\mathbf{b}\|\sin(\alpha)$

Only three mutually independent spatial numbers can be involved in the outer product.

These formulas still do not determine the sign of the outer product. Apart from that sign, the outer product is fixed.

The product of multidimensional numbers will split into five terms.

$$c = c_r + \mathbf{c} = a\, b \equiv (a_r + \mathbf{a})\,(b_r + \mathbf{b}) = a_r\, b_r - \langle \mathbf{a},\mathbf{b} \rangle + \mathbf{a}\, b_r + a_r\, \mathbf{b} \pm \mathbf{a}\times\mathbf{b}$$

Before these formulas are used, the sign of the outer product must be selected.

5.3 Symmetry

The number of mutually independent direction lines in a number system is called the ***dimension*** of the number system. The sequencing on a direction line can be done in one direction or the reverse direction. The direction of the first direction line is arbitrary. Also, the location of point zero is arbitrary. The coordinate system captures these choices at its maiden state.

Thus, the same number system exists in many versions that are distinguished by the selected coordinate system. The coordinate system reflects the geometric symmetry and the geometric center of the number system.

In the maiden state, the coordinate marker couples the identifier of the corresponding number to the point-like object that is identical to the pointer of the corresponding vector. The coordinate markers take care of the sequencing along direction lines. Sequencing in the real number dimensions is independent of sequencing in spatial number dimensions.

The real numbers, the complex numbers, and the quaternions appear to be the only three division rings that offer an associative multiplication. Hilbert spaces can only cope with associative division rings. Our purpose is to apply Hilbert spaces. So, we do not look for other number systems. Hilbert systems apply a private version of a chosen number system. The private coordinate system selects which version is tolerated. The selected version of the number system is maintained by a dedicated operator that we will call the reference operator. In its eigenspace, this operator provides a private parameter space, which settles the private geometric symmetry and the geometric center of the Hilbert space. The private parameter space turns the Hilbert space into a corresponding function space. The eigenvectors of the reference operator form an orthogonal base for the Hilbert space. This allows a special trick that abstracts a complex-number-based Hilbert space from a quaternionic Hilbert space.

A complex-number-based Hilbert space can be abstracted from a quaternionic Hilbert space by taking all eigenvectors of its reference operator that belong to the same spatial direction together with the real number eigenvectors and use these vectors as an orthogonal base of the new complex-number-based Hilbert space. This shows that complex-number-based Hilbert spaces can be considered subspaces of quaternionic Hilbert spaces.

The reverse trick is only possible if in the quaternionic Hilbert space, locally, the conditions are sufficiently isotropic. The stickiness of the space coverage can disturb the local isotropy. This only occurs in non-separable Hilbert spaces. The exact match is established during the maiden state of the coordinate system. Deformations are captured by natural operators. The natural operators represent fields. For moderate deviations, the coordinate markers can help to find a suitable map of the complex-number-based coordinate markers to the quaternionic coordinate markers.

5.4 Stickiness

If space is covered with point-like objects that act as markers of a coordinate system, then the behavior of the combination is determined by the cardinal number of the set of point-like objects. If the set is countable, then the set of point-like objects acts as an ensemble of discrete objects. Every member of the set seems to be surrounded by empty space. However, if the set is no longer countable, then the behavior of the combination of space and point-like objects changes from an ensemble of discrete objects to a coherent sticky medium. It looks as if the combination occupies all available space. The combination becomes deformable and mathematically the medium acts as a differentiable continuum. This switch in behavior happens if number systems containing all integer numbers and all rational numbers are suddenly extended by adding all irrational numbers. It means that the coordinates besides concerning integer markers and rational markers also concern irrational markers. The coordinate system puts the numbers in the correct sequence. It means that some coordinate markers merge into the same point. All converging series of markers end in a limit that is also a coordinate marker. A single deformation does not change the sequencing that the coordinate markers indicate. Each dynamic deformation takes response time for the reaction of the sticky medium.

6 Sticky coordinates

The set of the complex numbers covers two dimensions. For complex numbers, the outer product does not exist. Two extra independent lines can offer a location to other roots of negative numbers. Together the four direction lines constitute the number system of the quaternions. Both the complex numbers and the quaternions contain a one-dimensional subspace that obeys the arithmetic of the real numbers. In the real number system, all squares of numbers deliver a positive scalar. In the spatial dimensions of the number system, all squares of numbers deliver a negative real number. If the real numbers are interpreted as timestamps, then stickiness can be interpreted as a dynamic behavior that covers all spatial dimensions. The stickiness of the medium leads to a particular dynamic behavior of the medium. Any sudden local deformation is quickly spread in all directions over the full medium until the disturbance vanishes at infinity. The spread occurs with a fixed finite speed. Finally, each sudden local deformation expands the medium. The deformations do not touch the number systems. Instead, in the maiden state, the coordinate system reflects the geometric symmetry and the geometric center of the number system. The coordinate markers will be used to follow the deformations and the vibrations of the medium. In the maiden state, the coordinate markers locate at the same locations as the corresponding numbers. The relation between the number system and the coordinate system corresponds to the relation between a parameter space and the function that applies the parameter space.

At the scale of elementary particles, the deformations caused by these particles are recurrently regenerated. This is implemented by the ongoing hopping path of the particle. The hopping path recurrently regenerates a coherent hop landing location swarm that can be described by a stable location density distribution. If the hop landings cause a reaction of the sticky medium, then that reaction blurs the hop landing location distribution. The blur smooths the effect of the hop landing location swarm. Consequently, the deformation can be described by a smooth function, which is a blurred version of the location density distribution that describes the hop landing location swarm.

Humans often have problems comprehending what an infinite set is and are not familiar with uncountable sets. That is why the switch in behavior works counterintuitively.

Functions can describe the deformations and vibrations of the sticky medium. Differential calculus describes the corresponding change of the coordinate markers in fine detail. Mathematicians can interpret the solutions of quaternionic differential equations. Second-order partial differential equations treat the interaction between sticky mediums and point-like actuators.

7 Embedding in underlying vector space

7.1 Map of vector space

If two vector spaces have the same number of mutually independent vectors, then they have the same dimension. This enables constructing a map of the first vector space onto the second vector space. This map introduces relations between the original vectors and their maps. It is possible to map a vector space onto itself. In that case, one of the relations is called the inner product, and the vector space is called inner product space. This naming is confusing because this inner product differs considerably from the inner product that exists between spatial parts in number systems.

The resulting inner product space features the astonishing capability that its maps can archive the numbers that are delivered by the inner product of vectors that map onto the original vector direction. For that reason, the maps are also called ***operators***. The archived numbers are called ***eigenvalues*** and the involved vectors are called ***eigenvectors***. The operators manage the archived numbers in their ***eigenspaces***. The inner product space is a direct extension of the underlying elemental vector space.

This investigation disregards the interesting question of why vector spaces exist that can map onto other vector spaces or themselves and what activates these spaces to construct that map. This paper leaves that question open. The value of the inner product of vector pairs appears to be restricted to members of an ***associative division ring***. A restriction to a real number would be easily comprehensible because that can be interpreted as scaling or inversion of the original vector. But apart from the real numbers also the complex numbers and the quaternions constitute associative division rings. These numbers are not so easily interpreted as scaling factors.

Defining the inner product space differently than via a map solves the dilemma. We start from an associative division ring and attach an independent vector that is taken from a vector space to each of the members of the selected version of the chosen number system. We prepared this by attaching a vector that points to a coordinate marker to each of the members of the chosen associative division ring. The applied vector space will become ***the underlying vector space*** of the inner product space. Next, the attached vectors are considered as an orthogonal base of the infinite-dimensional vector space that underlies the inner product space. This base is constituted by the eigenvectors of a special normal operator, whose eigenspace represents the natural parameter space of the inner product space. In other words, the final inner product space is generated from a selected set of coordinate systems without the explicit need to consider a map of the vector space onto itself. The inner product space is constituted by superpositions of the eigenvectors of the referenced special operator. The superposition coefficients are taken from the selected version of the chosen number system. The operator is considered as the owner of the eigenspace and defines the set of eigenvectors that belong to the eigenvalues. The superposition coefficients take over part of the role of the inner products of vector pairs. The value of the inner product of a vector and a base vector, which is an eigenvector of the special operator that owns the natural parameter space as its eigenspace is a superposition coefficient and becomes a coordinate marker value at the maiden state of the coordinate system.

A century ago, a group of mathematicians discovered the existence of such special vector spaces.

7.2 Hilbert space

At the beginning of the last century David Hilbert and others discovered the special behavior of inner product spaces. John von Neumann, the assistant of David Hilbert introduced the name Hilbert space for inner product spaces that are complete. The most important aspect of Hilbert spaces is their capability to archive sets of numbers inside the eigenspaces of operators. The eigenvalues of all operators of a Hilbert space must be a member of a selected version of an associative division ring [2]. Coordinate systems determine the selected version. This selected version supplies the Hilbert space with a private parameter space that determines the geometric symmetry and the geometric center of the Hilbert space. This private parameter space is the ***natural*** parameter space of the Hilbert space. It is the parameter space of functions for which the target values populate the eigenspaces of a class of ***natural operators***. Other operators can exist in a Hilbert space that manages a different parameter space of a corresponding function in their eigenspace. These are not natural operators. The private parameter space is the eigenspace of a special operator that in this document will be called the ***reference operator***. The eigenvectors of the reference operator archive the elements of the selected version of the chosen number system and embed them in this way into the underlying vector space.

7.3 Symmetry and geometric center

The definition of a Hilbert space hardly ever mentions that the Hilbert space selects a single version and not all available versions of the chosen number system. Other treatments of Hilbert spaces usually only mention that the Hilbert space selects between the real number system, the complex number system, and the quaternionic number system. With the version, the Hilbert system also selects the Cartesian and polar coordinate system, and via that choice, the Hilbert space selects its inherent ***geometric symmetry*** and its ***geometric center***. The selected version defines what natural operators are and which parameter space is the natural parameter space. This particular asset acts as the ***root geometry*** of the Hilbert space.

7.4 Bra's and ket's

Paul Dirac introduced a handy notation for the relationship that exists between an original vector and its map. This relation applies to a bra and a ket [1]. This section treats the case that the inner product space applies quaternions to specify the values of its inner products.

The bra $\left\langle \vec{f} \right|$ is a covariant vector, and the ket $\left| \vec{g} \right\rangle$ is a contravariant vector. The inner product $\left\langle \vec{f} \mid \vec{g} \right\rangle$ acts as a metric. It has a quaternionic value. Since the product of quaternions is not commutative, care must be taken with the format of the formulas.

7.4.1 Ket vectors

The addition of ket vectors is commutative and associative.

$$\left| \vec{f} \right\rangle + \left| \vec{g} \right\rangle = \left| \vec{g} \right\rangle + \left| \vec{f} \right\rangle = \left| \vec{f} + \vec{g} \right\rangle \tag{7.4.1}$$

$$\left(\left| \vec{f} + \vec{g} \right\rangle \right) + \left| \vec{h} \right\rangle = \left| \vec{f} \right\rangle + \left(\left| \vec{g} + \vec{h} \right\rangle \right) = \left| \vec{f} + \vec{g} + \vec{h} \right\rangle \tag{7.4.2}$$

Together with quaternions, a set of ket vectors forms a ket vector space. Ket vectors are covariant vectors.

A quaternion α can be used to construct a covariant linear combination with the ket vector $\left| \vec{f} \right\rangle$

$$\left| \alpha \vec{f} \right\rangle = \left| \vec{f} \right\rangle \alpha \tag{7.4.3}$$

7.4.2 Bra vectors

For bra vectors hold

$$\left\langle \vec{f} \right| + \left\langle \vec{g} \right| = \left\langle \vec{g} \right| + \left\langle \vec{f} \right| = \left\langle \vec{f} + \vec{g} \right| \tag{7.4.4}$$

$$\left(\left\langle \vec{f} + \vec{g} \right| \right) + \left\langle \vec{h} \right| = \left\langle \vec{f} \right| + \left(\left\langle \vec{g} + \vec{h} \right| \right) = \left\langle \vec{f} + \vec{g} + \vec{h} \right| \tag{7.4.5}$$

Bra vectors are contravariant vectors.

$$\left\langle \alpha \vec{f} \right| = \alpha^* \left\langle \vec{f} \right| \tag{7.4.6}$$

Quaternions can constitute linear combinations with bra vectors.

A set of bra vectors form the vector space that is adjunct to the vector space of ket vectors that are the origins of these maps. If the map images the adjunct space onto the original vector space, then the bra vectors may be mapped onto the same ket vector.

7.4.3 Inner products

For the inner product holds

$$\left\langle \vec{f} \mid \vec{g} \right\rangle = \left\langle \vec{g} \mid \vec{f} \right\rangle^* \tag{7.4.7}$$

For quaternionic numbers α and β hold

$$\left\langle \alpha \vec{f} \mid \vec{g} \right\rangle = \left\langle \vec{g} \mid \alpha \vec{f} \right\rangle^* = \left(\left\langle \vec{g} \mid \vec{f} \right\rangle \alpha \right)^* = \alpha^* \left\langle \vec{f} \mid \vec{g} \right\rangle \tag{7.4.8}$$

$$\left\langle \vec{f} \mid \beta \vec{g} \right\rangle = \left\langle \vec{f} \mid \vec{g} \right\rangle \beta \tag{7.4.9}$$

$$\begin{aligned} \left\langle (\alpha + \beta)\, \vec{f} \mid \vec{g} \right\rangle &= \alpha^* \left\langle \vec{f} \mid \vec{g} \right\rangle + \beta^* \left\langle \vec{f} \mid \vec{g} \right\rangle \\ &= (\alpha + \beta)^* \left\langle \vec{f} \mid \vec{g} \right\rangle \end{aligned} \tag{7.4.10}$$

This corresponds with (7.4.3) and (7.4.6)

$$\left\langle \alpha \vec{f} \right| = \alpha^* \left\langle \vec{f} \right| \tag{7.4.11}$$

$$\left| \alpha \vec{g} \right\rangle = \left| \vec{g} \right\rangle \alpha \tag{7.4.12}$$

We made a choice. Another possibility would be $\left\langle \alpha \vec{f} \right| = \alpha \left\langle \vec{f} \right|$ and $\left| \alpha \vec{g} \right\rangle = \alpha^* \left| \vec{g} \right\rangle$

7.4.4 Operator construction

$\left| \vec{f} \right\rangle \langle \vec{g} |$ is a constructed operator.

$$|\vec{g}\rangle \left\langle \vec{f} \right| = \left(\left| \vec{f} \right\rangle \langle \vec{g} | \right)^\dagger \tag{7.4.13}$$

The superfix † indicates the adjoint version of the operator.

For the orthonormal base $\{|\vec{q}_i\rangle\}$ consisting of eigenvectors of the reference operator, holds

$$\langle \vec{q}_n | \vec{q}_m \rangle = \delta_{nm} \tag{7.4.14}$$

The ***reverse bra-ket method*** enables the definition of new operators that are defined by quaternionic functions.

$$\langle \vec{g} | F | \vec{h} \rangle = \sum_{i=1}^{N} \{ \langle \vec{g} | \vec{q}_i \rangle F(q_i) \langle \vec{q}_i | \vec{h} \rangle \} \tag{7.4.15}$$

The symbol F is used both for the operator F and the quaternionic function $F(q)$. This enables the shorthand

$$F \equiv |\vec{q}_i\rangle F(q_i) \langle \vec{q}_i| \tag{7.4.16}$$

for operator F. It is evident that for the adjoint operator

$$F^\dagger \equiv |\vec{q}_i\rangle F^*(q_i) \langle \vec{q}_i| \tag{7.4.17}$$

For ***reference operator*** $\mathfrak{R}$ holds

$$\mathfrak{R} = |\vec{q}_i\rangle q_i \langle \vec{q}_i| \tag{7.4.18}$$

If $\{q_i\}$ consists of all rational values of the version of the quaternionic number system that Hilbert space $\mathfrak{H}$ applies then the eigenspace of $\mathfrak{R}$ represents the natural parameter space of the separable Hilbert space $\mathfrak{H}$. It is also the parameter space of the function $F(q)$ that defines the natural operator F in the formula (7.4.16).

7.4.5 Operator types

I is used to indicate the identity operator.

For normal operator N holds $NN^\dagger = NN^\dagger$.

The normed eigenvectors of a normal operator form an orthonormal base of the Hilbert space.

For unitary operator U holds $UU^\dagger = U^\dagger U = I$

For Hermitian operator H holds $H = H^\dagger$

A normal operator N has a Hermitian part $\dfrac{N + N^\dagger}{2}$ and an anti-Hermitian part $\dfrac{N - N^\dagger}{2}$

For anti-Hermitian operator A holds $A = -A^\dagger$

7.5 Separable space

In mathematics a topological space is called separable if it contains a countable dense subset; that is, there exists a sequence $\left\{\left| \vec{f}_i \right\rangle\right\}_{i=\infty}^{i=0}$ of elements of the space such that every nonempty open subset of the space contains at least one element of the sequence.

Its values on this countable dense subset determine every continuous function on the separable inner product space.

The Hilbert space $\mathfrak{H}$ is separable. That means that a countable row of elements $\left\{\left| \vec{f}_n \right\rangle\right\}$ exists that spans the whole space. In the quaternionic Hilbert space, the quaternions are chosen as the private associative division ring.

If $\left\langle \vec{f}_m \mid \vec{f}_n \right\rangle = \delta(m,n)$ [1 if n=m; otherwise 0], then $\left\{\left| \vec{f}_n \right\rangle\right\}$ is an orthonormal base of Hilbert space $\mathfrak{H}$.

A ket base $\left\{\left|\vec{k}\right\rangle\right\}$ of $\mathfrak{H}$ is a minimal set of ket vectors $\left|\vec{k}\right\rangle$ that span the full Hilbert space $\mathfrak{H}$.

Any ket vector $\left|\vec{f}\right\rangle$ in $\mathfrak{H}$ can be written as a linear combination of elements of $\left\{\left|\vec{k}\right\rangle\right\}$.

$$\left|\vec{f}\right\rangle = \sum_k \left|\vec{k}\right\rangle\left\langle\vec{k} \mid \vec{f}\right\rangle \tag{7.5.1}$$

A bra base $\left\{\left\langle\vec{b}\right|\right\}$ of $\mathfrak{H}^\dagger$ is a minimal set of bra vectors $\left\langle\vec{b}\right|$ that span the full Hilbert space $\mathfrak{H}^\dagger$.

Any bra vector $\left\langle\vec{f}\right|$ in $\mathfrak{H}^\dagger$ can be written as a linear combination of elements of $\left\{\left\langle\vec{b}\right|\right\}$.

$$\left\langle\vec{f}\right| = \sum_b \left\langle\vec{f} \mid \vec{b}\right\rangle\left\langle\vec{b}\right| \tag{7.5.2}$$

Often, a base selects vectors such that their norm equals 1. Such a base is called an orthonormal base. The normed eigenvectors of a normal operator form an orthonormal base.

Separable Hilbert spaces do not support closed sets of irrational numbers as eigenvalues of an operator. The eigenspaces of their operators are countable.

For any subspace S let $S^\perp$ be the orthogonal complement of S. Call the subspace "closed" if $S^{\perp\perp} = S$

Call this whole vector space, and the Hermitian form (7.4.7), "orthomodular" if for every closed subspace S we have that $S + S^\perp$ is the entire space. (The term "orthomodular" derives from the study of quantum logic. In quantum logic, the distributive law is taken to fail due to the uncertainty principle, and it is replaced with the "modular law," or in the case of infinite-dimensional Hilbert spaces, the "orthomodular law. The set of closed subspaces of an infinite-dimensional separable Hilbert space form an orthomodular lattice.

7.6 Non-separable Hilbert space

Every infinite-dimensional separable Hilbert space owns a unique non-separable companion Hilbert space that embeds its separable partner. The non-separable Hilbert space allows operators that maintain eigenspaces that in every dimension and every spatial direction contain closed sets of rational and irrational eigenvalues. These eigenspaces behave as dynamic sticky continuums.

Gelfand triple and ***Rigged Hilbert space*** are other names for the general non-separable Hilbert spaces.

In the non-separable Hilbert space, for operators with continuum eigenspaces, the reverse bra-ket method turns from a summation into an integration.

$$\left\langle \vec{g} \mid F \mid \vec{h} \right\rangle \equiv \int\iiint \left\{ \left\langle \vec{g} \middle| \vec{q} \right\rangle F\left(q\right) \left\langle \vec{q} \middle| \vec{h} \right\rangle \right\} dV d\tau \qquad (7.6.1)$$

Here we omitted the enumerating subscripts that were used in the countable base of the separable Hilbert space.

The shorthand for the operator F is now

$$F \equiv \left| \vec{q} \right\rangle F\left(q\right) \left\langle \vec{q} \right| \qquad (7.6.2)$$

For eigenvectors $\left| q \right\rangle$, the function $F\left(q\right)$ defines as

$$F\left(q\right) = \left\langle \vec{q} \mid F\vec{q} \right\rangle = \int\iiint \left\{ \left\langle \vec{q} \mid \vec{q}' \right\rangle F(q') \left\langle \vec{q}' \mid \vec{q} \right\rangle \right\} dV' d\tau' \qquad (7.6.3)$$

The reference operator $\mathcal{R}$ that provides the continuum natural parameter space as its eigenspace follows from

$$\langle \vec{g} \mid \mathcal{R}\vec{h} \rangle \equiv \int\iiint \left\{ \langle \vec{g} | \vec{q} \rangle q \langle \vec{q} | \vec{h} \rangle \right\} dV d\tau \tag{7.6.4}$$

The corresponding shorthand is

$$\mathcal{R} \equiv |\vec{q}\rangle q \langle \vec{q}| \tag{7.6.5}$$

The reference operator is a special kind of defined operator. Via the quaternionic functions that specify defined operators, the claim becomes clear that every infinite-dimensional separable Hilbert space owns a unique non-separable companion Hilbert space that can be considered to embed its separable companion.

The reverse bracket method combines Hilbert space operator technology with quaternionic function theory and indirectly with quaternionic differential and integral technology.

7.7 Quaternionic function space

Each quaternionic separable Hilbert space owns a reference operator that manages an eigenspace that is formed by the version of the quaternionic number system that this Hilbert space applies to specify the values of the inner product of its vector pairs. This eigenspace is the natural parameter space of this Hilbert space.

The eigenvectors of the reference operator constitute an orthonormal base of the Hilbert space. The reference operator is a natural operator. A category of normal operators can be defined that share the eigenvectors of the reference operator and use the target values that belong to the original eigenvalues as the new eigenvalues of the defined operator. These operators are natural operators. According to this reasoning is every quaternionic separable Hilbert space a quaternionic function space. In that function space, the eigenvectors of the reference operator represent Dirac delta distributions.

7.8 Converting quaternionic Hilbert space to complex-number-based Hilbert space

In its eigenspace, the reference operator provides a private parameter space, which settles the private geometric symmetry and the geometric center of the Hilbert space. The private parameter space turns the Hilbert space into a corresponding function space. The eigenvectors of the reference operator form an orthogonal base for the Hilbert space. This allows a special trick that abstracts a complex-number-based Hilbert space from a quaternionic Hilbert space. A complex-number-based Hilbert space can be abstracted from a quaternionic Hilbert space by taking all eigenvectors of its reference operator that belong to the same spatial direction together with the real number eigenvectors and use these as an orthogonal base of the new complex-number-based Hilbert space. This shows that complex-number-based Hilbert spaces can be considered subspaces of quaternionic Hilbert spaces.

7.8.1 Position space and change space

If the members of the real axis are interpreted as instants of time, then the spatial parts of the quaternions form spatial positions in a dynamic ***position space***. The dynamic position space corresponds to the eigenspace of the natural reference operator. Thus, another name of the natural reference operator is the dynamic position operator.

Another orthonormal base of the Hilbert space forms another function space. An orthonormal base exists in which each member can be written as a linear combination of all base vectors of the position space such that all superposition coefficients have the same norm. We call the resulting space a ***change space***. The eigenvectors of the change operator correspond to the parameter space of the change space. This is not a natural parameter space, and the change operator is not a natural operator. Any dynamic function that is defined in the position space corresponds with a function in the change space. That function is the Fourier transform of the original function that is defined in the dynamic position space. The existence of the Fourier transform leads to the uncertainty principle for spatial kinematic data.

Integrating in position space in a selected spatial direction results in the full compression of that dimension in change space.

7.8.2 Fourier transform

Fourier transforms are easier described in a complex-number-based Hilbert space. The complex-number-based Hilbert space results from selecting all base vectors that belong to the same spatial direction in the dynamic position space of the quaternionic Hilbert space and construct a new complex-number-based Hilbert space from the selected orthonormal base.

The Fourier transform in this complex-number-based Hilbert space is given by the relation between $f(x)$ and $\tilde{f}(\xi_n)$ in the sum

$$f(x) = \sum_{n=-\infty}^{\infty} \left\{ \tilde{f}(\xi_n) e^{2\pi i \xi_n x} (\xi_{n+1} - \xi_n) \right\} \tag{7.8.1}$$

In the limit where $\Delta\xi = (\xi_{n+1} - \xi_n) \rightarrow 0$ the sum becomes an integral

$$f(x) = \int_{-\infty}^{\infty} \left\{ \tilde{f}(\xi) e^{2\pi i \xi x} \right\} d\xi \tag{7.8.2}$$

The reverse Fourier transform runs as

$$\tilde{f}(\xi) = \int_{-\infty}^{\infty} \left\{ f(x) e^{-2\pi i \xi x} \right\} dx \tag{7.8.3}$$

In these formulas, the symbol i represents a normalized spatial number part of a complex number. i corresponds to the spatial direction that was selected for constructing the complex-number-based Hilbert space.

The function $e^{2\pi i p x}$ is an eigenfunction of the operator $i\dfrac{\partial}{\partial x}$ which is recognizable as part of the change operator (8.2.3).

$$i\frac{\partial}{\partial x}e^{2\pi ipx} = 2\pi pe^{2\pi ipx} \tag{7.8.4}$$

The eigenvalue p represents the eigenfunction and the eigenvector p in the change space. In the same sense, the function $e^{-2\pi ipx}$ is an eigenfunction of the position operator $-i\frac{\partial}{\partial p}$ and corresponds with the eigenvalue x of that operator.

$$-i\frac{\partial}{\partial p}e^{-2\pi ipx} = 2\pi xe^{-2\pi ipx} \tag{7.8.5}$$

The eigenvalue x represents the eigenfunction and the eigenvector x in the position space.

The Fourier transform of a Dirac delta function is

$$\tilde{\delta}(\xi) = \int_{-\infty}^{\infty}\left\{\delta(x)e^{-2\pi i\xi x}\right\}dx = 1 \tag{7.8.6}$$

The inverse transform tells

$$\delta(x) = \int_{-\infty}^{\infty}\left\{1\cdot e^{2\pi i\xi x}\right\}d\xi \tag{7.8.7}$$

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ip(x-a)}dp \tag{7.8.8}$$

$$e^{2\pi ipa} = \int_{-\infty}^{\infty}\delta(x-a)e^{2\pi ipx}dx \tag{7.8.9}$$

The operator $P_x = i\frac{\partial}{\partial x}$ is often called the momentum operator for the spatial direction $\vec{i}$ of the coordinate x. $\vec{P}$ differs from the classical momentum that is defined as the product of velocity $\vec{v}$ and mass m.

7.8.3 Uncertainty principle

The uncertainty principle states

$$\left(\int_{-\infty}^{\infty}(x-x_0)^2\left|f(x)\right|^2 dx\right)\left(\int_{-\infty}^{\infty}(\xi-\xi_0)^2\left|\tilde{f}(\xi)\right|^2 d\xi\right)\geq\frac{1}{16\pi^2} \quad (7.8.10)$$

For a Gaussian distribution, the equality sign holds. The Fourier transform of a Gaussian distribution is again a Gaussian distribution that has a different standard deviation.

If $f(x)$ spreads, then $\tilde{f}(\xi)$ shrinks and vice versa.

In this way, the characteristic function of a stochastic process can control the spread of the location density distribution of the produced location swarm.

8 Field equations

Field equations are quaternionic functions or quaternionic differential and integral equations that describe the behavior of the continuum part of quaternionic fields. In the context of this document, these quaternionic fields are eigenspaces of natural operators that reside in non-separable quaternionic Hilbert spaces. These eigenspaces can contain separable subspaces. The stickiness of the field goes together with differentiability.

8.1 Quaternions

We will use a vector cap to indicate the spatial part and we will indicate the scalar part with suffix $_r$. This differs from the earlier notation that uses boldface for the spatial part of the quaternion.

Thus, the number a will be represented by the sum $a=a_r+\vec{a}$. This means that the product $c=ab$ of two numbers a and b will split into several terms

$$c=c_r+\vec{c}=ab=\left(a_r+\vec{a}\right)\left(b_r+\vec{b}\right)=a_r b_r+a_r\vec{b}+\vec{a}b_r+\vec{a}\vec{b} \quad (8.1.1)$$

The product d of two spatial numbers $\vec{a}$ and $\vec{b}$ results in a real scalar part and a new spatial part

$$d = d_r + \vec{d} = \vec{a}\vec{b} \quad (8.1.2)$$

$d_r = -\langle \vec{a}, \vec{b} \rangle$ is the inner product of $\vec{a}$ and $\vec{b}$

$\vec{d} = \vec{a} \times \vec{b}$ is the outer product of $\vec{a}$ and $\vec{b}$

The spatial vector $\vec{d}$ is independent of $\vec{a}$ and independent of $\vec{b}$. This means that $\langle \vec{a}, \vec{d} \rangle = 0$ and $\langle \vec{b}, \vec{d} \rangle = 0$

For the inner product and the norm $\|\vec{a}\|$ holds $\langle \vec{a}, \vec{a} \rangle = \|\vec{a}\|^2$

Only three mutually independent spatial number parts can be involved in the outer product.

These formulas still do not determine the sign of the outer product. Apart from that sign, the outer product is fixed.

Quaternionic multiplication obeys the equation

$$\begin{aligned} c = c_r + \vec{c} = ab &= \left(a_r + \vec{a}\right)\left(b_r + \vec{b}\right) \\ &= a_r b_r - \langle \vec{a}, \vec{b} \rangle + a_r \vec{b} + \vec{a} b_r \pm \vec{a} \times \vec{b} \end{aligned} \quad (8.1.3)$$

The $\pm$ sign indicates the freedom of choice of the handedness of the product rule that exists when selecting a version of the quaternionic number system. The version must be selected before it can be used in calculations.

Two quaternions that are each other’s inverse can rotate the spatial part of another quaternion.

$$c = ab / a \quad (8.1.4)$$

The construct rotates the spatial part of b that is perpendicular to $\vec{a}$ over an angle that is twice the angular phase θ of $a = \|a\| e^{\vec{i}\theta}$ where $\vec{i} = \vec{a} / \|\vec{a}\|$.

Cartesian quaternionic functions apply a quaternionic parameter space that is sequenced by a Cartesian coordinate system. In the parameter space, the real scalar parts of quaternions are often interpreted as instances of (proper) time, and the spatial parts are often interpreted as spatial locations. The real scalar parts of quaternionic functions represent dynamic scalar fields. The spatial parts of quaternionic functions represent dynamic vector fields.

8.2 Quaternionic differential calculus

The differential change can be expressed in terms of a linear combination of partial differentials. Now the total differential change df of field f equals

$$df = \frac{\partial f}{\partial \tau} d\tau + \frac{\partial f}{\partial x} \vec{i} dx + \frac{\partial f}{\partial y} \vec{j} dy + \frac{\partial f}{\partial z} \vec{k} dz \qquad (8.2.1)$$

In this equation, the partial differentials $\frac{\partial f}{\partial \tau}, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial y}$ behave like quaternionic differential operators.

The quaternionic nabla ∇ assumes the ***special condition*** that partial differentials direct along the axes of the Cartesian coordinate system in a natural parameter space of a non-separable Hilbert space. Thus,

$$\nabla = \sum_{i=0}^{4} \vec{e}_i \frac{\partial}{\partial x_i} = \frac{\partial}{\partial \tau} + \vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \qquad (8.2.2)$$

This will be applied in the next section by splitting both the quaternionic nabla and the function in a scalar part and a vector part.

The first-order partial differential equations divide the first-order change of a quaternionic field into five different parts that each represent a new field. We will represent the quaternionic field change operator by a quaternionic nabla operator. This operator behaves like a quaternionic multiplier.

The first order partial differential follows from

$$\nabla = \left\{ \frac{\partial}{\partial \tau}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} = \nabla_r + \vec{\nabla} \tag{8.2.3}$$

The spatial nabla $\vec{\nabla}$ is well-known as the del operator and is treated in detail in Wikipedia [5]. The partial derivatives in the change operator only use parameters that are taken from the natural parameter space.

$$\begin{aligned} \phi = \nabla \psi &= \left(\frac{\partial}{\partial \tau} + \vec{\nabla} \right) (\psi_r + \vec{\psi}) \\ &= \nabla_r \psi_r - \langle \vec{\nabla}, \vec{\psi} \rangle + \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} \end{aligned} \tag{8.2.4}$$

In a selected version of the quaternionic number system, only the corresponding version of the quaternionic nabla is active. In a selected Hilbert space, this version is always and everywhere the same.

The differential $\nabla \psi$ describes the change of field ψ. The five separate terms in the first-order partial differential have a separate physical meaning. All basic fields feature this decomposition. The terms may represent new fields.

$$\phi_r = \nabla_r \psi_r - \langle \vec{\nabla}, \vec{\psi} \rangle \tag{8.2.5}$$

$$\vec{\phi} = \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} \tag{8.2.6}$$

$\vec{\nabla} f$ is the gradient of f.

$\left\langle \vec{\nabla}, \vec{f} \right\rangle$ is the divergence of $\vec{f}$.

$\vec{\nabla} \times \vec{f}$ is the curl of $\vec{f}$.

$$\left(\vec{\nabla}, \vec{\nabla}\right)\psi = \Delta\psi = \nabla^2\psi \tag{8.2.7}$$

$$\left(\vec{\nabla}, \vec{\nabla} \times \vec{\psi}\right) = 0 \tag{8.2.8}$$

$$\vec{\nabla} \times \left(\vec{\nabla}\psi_r\right) = 0 \tag{8.2.9}$$

$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{\psi}\right) = \vec{\nabla}\left(\vec{\nabla}, \vec{\psi}\right) - \left(\vec{\nabla}, \vec{\nabla}\right)\vec{\psi} \tag{8.2.10}$$

Sometimes parts of the change get new symbols

$$\vec{E} = -\nabla_r\vec{\psi} - \vec{\nabla}\psi_r \tag{8.2.11}$$

$$\vec{B} = \vec{\nabla} \times \vec{\psi} \tag{8.2.12}$$

The formula (8.2.4) does not leave room for gauges. In Maxwell equations, the equation (8.2.5) is a gauge.

$$\left(\vec{\nabla}, \vec{B}\right) = 0 \tag{8.2.13}$$

$$\vec{\nabla} \times \vec{E} = -\nabla_r\vec{\nabla} \times \vec{\psi} - \vec{\nabla} \times \vec{\nabla}\psi_r = -\nabla_r\vec{B} \tag{8.2.14}$$

$$\left(\vec{\nabla}, \vec{E}\right) = -\nabla_r\left(\vec{\nabla}, \vec{\psi}\right) - \left(\vec{\nabla}, \vec{\nabla}\right)\psi_r \tag{8.2.15}$$

The conjugate of the quaternionic nabla operator defines another type of field change.

$$\nabla^* = \nabla_r - \vec{\nabla} \tag{8.2.16}$$

$$\zeta = \nabla^* \phi = \left(\frac{\partial}{\partial \tau} - \vec{\nabla}\right)\left(\phi_r + \vec{\phi}\right)$$
$$= \nabla_r \phi_r + \left\langle \vec{\nabla}, \vec{\phi} \right\rangle + \nabla_r \vec{\phi} - \vec{\nabla} \phi_r \mp \vec{\nabla} \times \vec{\phi} \qquad (8.2.17)$$

All dynamic quaternionic fields obey the same first-order partial differential equations (8.2.4) and (8.2.17).

$$\nabla^\dagger = \nabla^* = \nabla_r - \vec{\nabla} = \nabla_r + \vec{\nabla}^\dagger = \nabla_r + \vec{\nabla}^* \qquad (8.2.18)$$

In the Hilbert space, the quaternionic nabla is a normal operator.

$$\nabla^\dagger \nabla = \nabla \nabla^\dagger = \nabla^* \nabla = \nabla \nabla^* = \nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \qquad (8.2.19)$$

Are normal operators who are also Hermitian.

The separate operators $\nabla_r \nabla_r$ and $\left\langle \vec{\nabla}, \vec{\nabla} \right\rangle$ are also Hermitian operators. They can also be combined as $\square = \nabla_r \nabla_r - \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle$. This is the d'Alembert operator. The solutions of $\nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle = 0$ and $\nabla_r \nabla_r - \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle = 0$ differ. These two equations offer different solutions and for that reason, they deliver different dynamic behavior of the field. The equations control the behavior of the embedding field that physicists call their universe. This dynamic field exists everywhere in the reach of the parameter space of the function. Both equations also control the behavior of the symmetry-related fields. The homogeneous d'Alembert equation is known as the wave equation and offers waves and wave packages as its solutions. Both equations offer shock fronts as solutions but only the operators in (8.2.19) deliver shock fronts that feature a spin or polarization vector. Integration over the time domain turns both equations in the Poisson equation and removes the spin or polarization vector. Shock fronts require a corresponding actuator and occur only in odd numbers of participating dimensions. Spherical shock fronts require an isotropic actuator.

8.3 Continuity equations

Continuity equations are partial quaternionic differential equations.

8.3.1 Field excitations

The dynamic changes of the field are interpreted as field excitations or as field deformations or field expansions.

The field excitations that will be discussed here are solutions of mentioned second-order partial differential equations.

One of the second-order partial differential equations results from combining the two first-order partial differential equations $\phi = \nabla \psi$ and $\zeta = \nabla^* \phi$.

$$\begin{aligned} \zeta &= \nabla^* \varphi = \nabla^* \nabla \psi = \nabla \nabla^* \psi = \left(\nabla_r + \vec{\nabla}\right)\left(\nabla_r - \vec{\nabla}\right)\left(\psi_r + \vec{\psi}\right) \\ &= \left(\nabla_r \nabla_r + \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle\right)\psi \end{aligned} \tag{8.3.1}$$

All other terms vanish. $\left\langle \vec{\nabla}, \vec{\nabla} \right\rangle$ is known as the Laplace operator.

Integration over the time domain results in the Poisson equation

$$\rho = \left\langle \vec{\nabla}, \vec{\nabla} \right\rangle \psi \tag{8.3.2}$$

Under isotropic conditions, a very special solution of the Poisson equation is the Green's function $\frac{1}{4\pi \left|\vec{q} - \vec{q}'\right|}$ of the affected field [33].

This solution is the spatial Dirac $\delta\left(\vec{q}\right)$ pulse response of the field under strict isotropic conditions.

$$\nabla \frac{1}{\left|\vec{q}-\overrightarrow{q'}\right|} = -\frac{\left(\vec{q}-\overrightarrow{q'}\right)}{\left|\vec{q}-\overrightarrow{q'}\right|^3} \tag{8.3.3}$$

$$\begin{aligned} \left\langle \vec{\nabla},\vec{\nabla}\right\rangle \frac{1}{\left|\vec{q}-\overrightarrow{q'}\right|} &\equiv \left\langle \vec{\nabla},\vec{\nabla}\frac{1}{\left|\vec{q}-\overrightarrow{q'}\right|}\right\rangle \\ &= -\left\langle \vec{\nabla},\frac{\left(\vec{q}-\overrightarrow{q'}\right)}{\left|\vec{q}-\overrightarrow{q'}\right|^3}\right\rangle = 4\pi\delta\left(\vec{q}-\overrightarrow{q'}\right) \end{aligned} \tag{8.3.4}$$

This solution corresponds with an ongoing source or sink that exists in the field.

Change can take place in one spatial dimension or combined in two or three spatial dimensions.

Under isotropic conditions, the dynamic spherical pulse response of the field is a solution of a special form of the equation (8.3.1)

$$\left(\nabla_r\nabla_r + \left\langle \vec{\nabla},\vec{\nabla}\right\rangle\right)\psi = 4\pi\delta\left(\vec{q}-\overrightarrow{q'}\right)\theta\left(\tau \pm \tau'\right) \tag{8.3.5}$$

Here $\theta(\tau)$ is a step function and $\delta(\vec{q})$ is a Dirac pulse response. For the spherical pulse response, the pulse must be isotropic.

After the instant τ', the equation turns into a homogeneous equation.

A remarkably simple solution is the shock front in one dimension along the line $\vec{q}-\overrightarrow{q'}$.

$$\psi = f\left(\left|\vec{q}-\overrightarrow{q'}\right| \pm c\left(\tau-\tau'\right)\vec{n}\right) \tag{8.3.6}$$

Here $\vec{n}$ is a normed spatial quaternion. This spatial quaternion has an arbitrary direction that does not vary in time. Here, the normalized vector $\vec{n}$ can be interpreted as the polarization of the solution [41].

In isotropic conditions, we better switch to spherical coordinates. Then the equation gets the form

$$\left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial r^2}+2\frac{\partial}{r\partial r}\right)\psi = \left(\frac{\partial^2}{\partial\tau^2}+\frac{\partial^2}{\partial r^2}\right)(\psi r)=0 \tag{8.3.7}$$

The second line describes the second-order change of ψr in one dimension along the radius r. That solution is described above. A solution of this equation is

$$\psi r = f\left(r \pm c\tau\vec{n}\right) \tag{8.3.8}$$

The solution of (8.3.7) is described by

$$\psi = \frac{f\left(\left|\vec{q}-\vec{q'}\right| \pm c\left(\tau-\tau'\right)\vec{n}\right)}{\left|\vec{q}-\vec{q'}\right|} \tag{8.3.9}$$

The normalized vector $\vec{n}$ can be interpreted as the spin of the solution. The spherical pulse response acts either as an expanding or as a contracting spherical shock front. Over time this pulse response integrates into the Green's function. This means that the isotropic pulse injects the volume of the Green's function into the field. Subsequently, the front spreads this volume over the field. The contracting shock front collects the volume of the Green's function and sucks it out of the field.

The $\pm$ sign in the equation (8.3.5) selects between injection and subtraction.

Shock fronts only occur in one and three dimensions. A pulse response can also occur in two dimensions, but in that case, the pulse response is a complicated vibration that looks like the result of a throw of a stone in the middle of a pond.

Equations (8.3.1) and (8.3.2) show that the operators $\frac{\partial^2}{\partial \tau^2}$ and $\langle \vec{\nabla}, \vec{\nabla} \rangle$ are valid second-order partial differential operators. These operators combine in the quaternionic equivalent of the wave equation [6].

$$\varphi = \left(\frac{\partial^2}{\partial \tau^2} - \langle \vec{\nabla}, \vec{\nabla} \rangle \right) \psi = \Box \psi \tag{8.3.10}$$

This equation also offers one-dimensional and three-dimensional shock fronts as its solutions.

$$\psi = \frac{f\left(\left| \vec{q} - \vec{q}' \right| \pm c(\tau - \tau') \right)}{\left| \vec{q} - \vec{q}' \right|} \tag{8.3.11}$$

$$\psi = f\left(\left| \vec{q} - \vec{q}' \right| \pm c(\tau - \tau') \right) \tag{8.3.12}$$

These pulse responses do not contain the normed vector $\vec{n}$. Apart from pulse responses, the wave equation offers waves as its solutions.

If locally the field can be split into a time-dependent part $T(\tau)$ and a location-dependent part $A(\vec{q})$, then the homogeneous version of the wave equation can be transformed into the Helmholtz equation [7].

$$\frac{\partial^2 \psi}{\partial \tau^2} = \langle \vec{\nabla}, \vec{\nabla} \rangle \psi = -\omega^2 \psi \tag{8.3.13}$$

$$\psi(\vec{q},\tau) = A(\vec{q})T(\tau) \tag{8.3.14}$$

$$\frac{1}{T}\frac{\partial^2 T}{\partial \tau^2} = \frac{1}{A}\left\langle \vec{\nabla},\vec{\nabla}\right\rangle A = -\omega^2 \tag{8.3.15}$$

$$\left\langle \vec{\nabla},\vec{\nabla}\right\rangle A + \omega^2 A = 0 \tag{8.3.16}$$

$$\frac{\partial^2 T}{\partial \tau^2} + \omega^2 T = 0 \tag{8.3.17}$$

ω acts as quantum coupling between(8.3.16) and (8.3.17).

The time-dependent part $T(\tau)$ depends on initial conditions, or it indicates the switch of the oscillation mode. The switch of the oscillation mode means that temporarily the oscillation is stopped and instead an object is emitted or absorbed that compensates for the difference in potential energy. The location-dependent part of the field $A(\vec{q})$ describes the possible oscillation modes of the field and depends on boundary conditions. The oscillations have a binding effect. They keep moving objects within a bounded region.

For three-dimensional isotropic spherical conditions, the solutions have the form

$$A(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left\{\left(a_{lm}\, j_l(kr)\right) + b_{lm} Y_l^m(\theta,\varphi)\right\} \tag{8.3.18}$$

Here j_l and y_l are the spherical Bessel functions, and Y_l^m are the spherical harmonics [13][14]. These solutions play a role in the spectra of atomic modules.

Planar and spherical waves are the simpler wave solutions of the equation (8.3.13)

$$\psi(\vec{q},\tau) = \exp\left\{\vec{n}\left(\left\langle \vec{k},\vec{q}-\vec{q}_0\right\rangle - \omega\tau + \varphi\right)\right\} \tag{8.3.19}$$

$$\psi(\vec{q},\tau) = \frac{\exp\left\{\vec{n}\left(\left\langle\vec{k},\vec{q}-\vec{q}_0\right\rangle - \omega\tau + \varphi\right)\right\}}{\left|\vec{q}-\vec{q}_0\right|} \qquad (8.3.20)$$

A more general solution is a superposition of these basic types.

Two quite similar homogeneous second-order partial differential equations exist. They are the homogeneous versions of equations (8.3.5) and (8.3.10). The equation (8.3.5) has spherical shock-front solutions with a spin vector that behaves like the spin of elementary particles. Obviously, the field only reacts dynamically when it gets triggered by corresponding actuators. Pulses may cause shock fronts that after the trigger keep traveling. Oscillations of type (8.3.19) and (8.3.20) must be triggered by periodic actuators.

The inhomogeneous pulse activated equations are

$$\left(\nabla_r\nabla_r \pm \left\langle\vec{\nabla},\vec{\nabla}\right\rangle\right)\psi = 4\pi\delta\left(\vec{q}-\overrightarrow{q'}\right)\theta(\tau \pm \tau') \qquad (8.3.21)$$

Without the interaction with actuators, all vibrations and deformations of the field vanish until the affected field locally resembles a flat field. Only an ongoing stream of actuators can generate a more persistently deformed field. This is provided by an ongoing embedding of the actuators into the eigenspaces of operators that archive the dynamic fields.

8.4 Isotropic conditions

The two shock-front solutions show an interesting property of the Laplace operator. In isotropic conditions, the Poisson equation can be rewritten as

$$\varphi = \left\langle\vec{\nabla},\vec{\nabla}\right\rangle\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}\right)\psi = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\psi) \qquad (8.4.1)$$

The product $\phi = (r\psi)$ is a solution of a one-dimensional equation in which r plays the variable.

The same thing holds for all differential equations that contain the Laplace operator $\left\langle \vec{\nabla}, \vec{\nabla} \right\rangle$

So, spherical solutions of the second-order differential equations ξ / r can be obtained from the solutions ξ of one-dimensional second-order differential equations by dividing ξ with the distance r to the center.

It looks as if in isotropic conditions the quaternionic differential calculus can be scaled down to complex-number-based differential calculus. This already works at local scales. If on larger scales the isotropic condition is violated, then the coordinates of the complex-number-based abstraction must be adapted to the possibly deformed Cartesian coordinates of the quaternionic platform. This makes sense in the presence of moderate deformations of the quaternionic field. After adaptation, the map of each complex-number-based coordinate line becomes a geodesic.

These tricks are possible because complex-number-based Hilbert spaces can be considered subspaces of quaternionic Hilbert spaces.

8.5 Conversion to antiparticle

The switch from quaternionic Hilbert space to complex-number-based Hilbert space will have repercussions for the selection operator that generates the footprint of the Hilbert space. Instead of a three-dimensional spatial footprint, the selection mechanism will produce a one-dimensional spatial footprint. The three-dimensional hopping path transfers to a one-dimensional string of separate landing locations. The spherical pulse responses that act as spherical shock fronts become one-dimensional pulse responses that act as shock fronts. If they are evenly distributed in time, then they become evenly distributed in space. This means that the complex-number-based footprint will represent what we know as a photon. Photons obey the Planck-Einstein relation. E=h v.

Switching back to the quaternion-based Hilbert space offers the opportunity to switch to the situation in which the symmetry of the floating Hilbert space is converted to the antisymmetric version of the number system. Emission or absorption of the photon takes the duration of a full generation cycle of the hop landing location swarm. Observers can only perceive pair production or pair annihilation events.

8.6 Enclosure balance equations

Enclosure balance equations are quaternionic integral equations that describe the balance between the inside, the border, and the outside of an enclosure.

These integral balance equations base on replacing the del operator $\vec{\nabla}$ with a normed vector $\vec{n}$. The vector $\vec{n}$ is oriented outward and perpendicular to a local part of the closed boundary of the enclosed region.

$$\vec{\nabla}\psi \Leftrightarrow \vec{n}\psi \tag{8.6.1}$$

This approach turns part of the differential continuity equation into a corresponding integral balance equation.

$$\iiint \vec{\nabla}\psi dV = \oiint \vec{n}\psi dS \tag{8.6.2}$$

$\vec{n}\ dS$ plays the role of a differential surface. $\vec{n}$ is perpendicular to that surface.

This result separates into three parts

$$\vec{\nabla}\psi = -\left\langle \vec{\nabla},\vec{\psi} \right\rangle + \vec{\nabla}\psi_r \pm \vec{\nabla}\times\vec{\psi} \Leftrightarrow \vec{n}\psi$$
$$= -\left\langle \vec{n},\vec{\psi} \right\rangle + \vec{n}\psi_r \pm \vec{n}\times\vec{\psi} \tag{8.6.3}$$

The first part concerns the gradient of the scalar part of the field

$$\vec{\nabla}\psi_r \Leftrightarrow \vec{n}\psi_r \tag{8.6.4}$$

$$\iiint \vec{\nabla}\psi_r dV = \oint\!\!\oint \vec{n}\psi_r dS \tag{8.6.5}$$

The divergence is treated in an integral balance equation that is known as the Gauss theorem. It is also known as the divergence theorem [15].

$$\left\langle \vec{\nabla},\vec{\psi} \right\rangle \Leftrightarrow \left\langle \vec{n},\vec{\psi} \right\rangle \tag{8.6.6}$$

$$\iiint \left\langle \vec{\nabla},\vec{\psi} \right\rangle dV = \oint\!\!\oint \left\langle \vec{n},\vec{\psi} \right\rangle dS \tag{8.6.7}$$

The curl is treated in a corresponding integrated balance equation

$$\vec{\nabla}\times\vec{\psi} \Leftrightarrow \vec{n}\times\vec{\psi} \tag{8.6.8}$$

$$\iiint \vec{\nabla}\times\vec{\psi} dV = \oint\!\!\oint \vec{n}\times\vec{\psi} dS \tag{8.6.9}$$

Equation (8.6.7) and equation (8.6.9) can be combined in the extended theorem

$$\iiint \vec{\nabla}\vec{\psi} dV = \oint\!\!\oint \vec{n}\ \vec{\psi} dS \tag{8.6.10}$$

The method also applies to other partial differential equations. For example

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\psi}\right) = \vec{\nabla}\left\langle \vec{\nabla},\vec{\psi} \right\rangle - \left\langle \vec{\nabla},\vec{\nabla} \right\rangle\vec{\psi} \Leftrightarrow \vec{\nabla}\times\left(\vec{\nabla}\times\vec{\psi}\right)$$
$$= \vec{n}\left\langle \vec{n},\vec{\psi} \right\rangle - \left\langle \vec{n},\vec{n} \right\rangle\vec{\psi} \tag{8.6.11}$$

$$\iiint_V \left\{ \vec{\nabla}\times\left(\vec{\nabla}\times\vec{\psi}\right) \right\} dV = \oint\!\!\oint_S \left\{ \vec{\nabla}\left\langle \vec{\nabla},\vec{\psi} \right\rangle \right\} dS - \oint\!\!\oint_S \left\{ \left\langle \vec{\nabla},\vec{\nabla} \right\rangle\vec{\psi} \right\} dS \tag{8.6.12}$$

One dimension less, a similar relation exists.

$$\iint_S \left(\left\langle \vec{\nabla} \times \vec{a}, \vec{n} \right\rangle\right) dS = \oint_C \left\langle \vec{a}, d\vec{l} \right\rangle \tag{8.6.13}$$

This is known as the Stokes theorem[16]

The curl can be presented as a line integral

$$\left\langle \vec{\nabla} \times \vec{\psi}, \vec{n} \right\rangle \equiv \lim_{A \to 0} \left(\frac{1}{A} \oint_C \langle \vec{\psi}, d\vec{r} \rangle \right) \tag{8.6.14}$$

8.7 Derivation of physical laws

The quaternionic equivalents of Ampère's law are [19]

$$\vec{J} \equiv \vec{\nabla} \times \vec{B} = \nabla_r \vec{E} \Leftrightarrow \vec{J} \equiv \vec{n} \times \vec{B} = \nabla_r \vec{E} \tag{8.7.1}$$

$$\iint_S \left\langle \vec{\nabla} \times \vec{B}, \vec{n} \right\rangle dS = \oint_C \left\langle \vec{B}, d\vec{l} \right\rangle = \iint_S \left\langle \vec{J} + \nabla_r \vec{E}, \vec{n} \right\rangle dS \tag{8.7.2}$$

The quaternionic equivalents of Faraday's law are [20]:

$$\nabla_r \vec{B} = \vec{\nabla} \times (\nabla_r \vec{\psi}) = -\vec{\nabla} \times \vec{E} \Leftrightarrow \nabla_r \vec{B} = \vec{n} \times (\nabla_r \vec{\psi}) = -\vec{\nabla} \times \vec{E} \tag{8.7.3}$$

$$\oint_c \left\langle \vec{E}, d\vec{l} \right\rangle = \iint_S \left\langle \vec{\nabla} \times \vec{E}, \vec{n} \right\rangle dS = -\iint_S \left\langle \nabla_r \vec{B}, \vec{n} \right\rangle dS \tag{8.7.4}$$

$$\vec{J} = \vec{\nabla} \times \left(\vec{B} - \vec{E} \right) = \vec{\nabla} \times \vec{\varphi} - \nabla_r \vec{\varphi} = \vec{v} \rho \tag{8.7.5}$$

$$\iint_S \left\langle \vec{\nabla} \times \vec{\varphi}, \vec{n} \right\rangle dS = \oint_C \left(\left\langle \vec{\varphi}, d\vec{l} \right\rangle \right) = \iint_S \langle \vec{v} \rho + \nabla_r \vec{\varphi}, \vec{n} \rangle dS \tag{8.7.6}$$

The equations (8.7.4) and (8.7.6) enable the derivation of the ***Lorentz force*** [21].

$$\vec{\nabla} \times \vec{E} = -\nabla_r \vec{B} \tag{8.7.7}$$

$$\frac{d}{d\tau}\iint_{S}\left\langle \vec{B},\vec{n}\right\rangle dS=\iint_{S(\tau_0)}\left\langle \dot{\vec{B}}\left(\tau_0\right),\vec{n}\right\rangle ds+\frac{d}{d\tau}\iint_{S(\tau)}\left\langle \vec{B}\left(\tau_0\right),\vec{n}\right\rangle ds \quad (8.7.8)$$

The Leibniz integral equation states [22]

$$\frac{d}{dt}\iint_{S(\tau)}\left\langle \vec{X}\left(\tau_0\right),\vec{n}\right\rangle dS$$
$$=\iint_{S(\tau_0)}\left\langle \dot{\vec{X}}\left(\tau_0\right)+\left\langle \vec{\nabla},\vec{X}\left(\tau_0\right)\right\rangle \vec{v}\left(\tau_0\right),\vec{n}\right\rangle dS-\oint_{C(\tau_0)}\left\langle \vec{v}\left(\tau_0\right)\times\vec{X}\left(\tau_0\right),d\vec{l}\right\rangle$$

(8.7.9)

With $\vec{X}=\vec{B}$ and $\left\langle \vec{\nabla},\vec{B}\right\rangle =0$ follows

$$\frac{d\Phi_B}{d\tau}=$$
$$\frac{d}{d\tau}\iint_{S(\tau)}\left\langle \dot{\vec{B}}\left(\tau\right),\vec{n}\right\rangle dS=\iint_{S(\tau_0)}\left\langle \vec{B}\left(\tau_0\right),\vec{n}\right\rangle dS-\oint_{C(\tau_0)}\left\langle \vec{v}\left(\tau_0\right)\times\vec{B}\left(\tau_0\right),d\vec{l}\right\rangle$$
$$=-\oint_{C(\tau_0)}\left\langle E\left(\tau_0\right),d\vec{l}\right\rangle-\oint_{C(\tau_0)}\left\langle \vec{v}\left(\tau_0\right)\times\vec{B}\left(\tau_0\right),d\vec{l}\right\rangle$$

(8.7.10)

The electromotive force (EMF) ε equals [23]

$$\varepsilon=\oint_{C(\tau_0)}\left\langle \frac{\vec{F}\left(\tau_0\right)}{q},d\vec{l}\right\rangle =-\left.\frac{d\Phi_B}{d\tau}\right|_{\tau=\tau_0}$$
$$=\oint_{C(\tau_0)}\left\langle \vec{E}\left(\tau_0\right),d\vec{l}\right\rangle+\oint_{C(\tau_0)}\left\langle \vec{v}\left(\tau_0\right)\times\vec{B}\left(\tau_0\right),d\vec{l}\right\rangle \quad (8.7.11)$$

$$\vec{F}=q\vec{E}+q\vec{v}\times\vec{B} \quad (8.7.12)$$

9 Systems of Hilbert spaces

Only a subtle difference exists between an elemental vector space and the Hilbert space that maps this vector space onto itself. The inner product is the most important difference. The fact that the Hilbert space is complete is another difference. Several properties of Hilbert spaces are the consequence of these differences. An important property is the private natural parameter space of the Hilbert space that provides its geometric symmetry and its geometric center. An important restriction is that Hilbert spaces can only cope with number systems that are associative division rings. This excludes octonions and biquaternions. Each Hilbert space selects a version of an associative division ring that is determined by the coordinate systems, which sequence the elements of the chosen number system. This takes the restrictions to a deeper level. The selected version determines the geometric symmetry and the geometric center of the Hilbert space.

These restrictions still leave the possibility that in a system of Hilbert spaces all members share the same underlying elemental vector space. In this system, one of the members acts as the background platform. All other members float with their geometric center over the parameter space of the background platform. If the background platform features infinite dimensions, then its non-separable companion also becomes part of the background platform. The resulting system of Hilbert spaces will be called the ***Hilbert repository***. The Hilbert repository distributes its storage capability over its participating members. The floating members act as read-only storage that is filled at the birth of the considered Hilbert space and not changed after that instant.

9.1 Hilbert repository

Sharing the same underlying vector space imposes new restrictions and enables new capabilities. The restrictions enforce that not all possible Hilbert spaces can be a member of the Hilbert repository. The coordinate systems of the selected versions of the number systems must have their Cartesian coordinate axes in parallel. This limits the allowed symmetries to a small set.

This restriction is not obvious and currently known mathematics does not yet deliver this hard requirement. The existence of this restriction is derived from the Standard Model of particle physics. The Standard Model reflects the knowledge of particle physicists that is derived from measurements. In the Standard Model, the set of elementary fermions show great similarity with the set of floating separate quaternionic Hilbert systems that populate the Hilbert repository. Elementary fermion types appear to correspond with the differences between the symmetries of the allowed floating separable Hilbert spaces and the symmetry of the background platform.

The differences between the symmetries of the floating platforms and the background platform generate sources and sinks that locate at the geometric centers of the floating platforms. The sources and sinks correspond to symmetry-related charges that may be zero or can have one of a restricted set of values. Non-zero symmetry-related charges generate corresponding symmetry-related fields.

10 Dynamics in the Hilbert repository

10.1 Embedding in the background platform

The differences in the symmetry between the platforms only become apparent when a floating platform is embedded into the background platform or more specific when eigenvalues of a dedicated selection operator are mapped to corresponding eigenvectors in the background platform. A special operator in the non-separable Hilbert space of the background platform acts as the embedding field for discrete eigenvalues that originate from the eigenspace of the selection operator that resides in the floating platform. The eigenspace of the selection operator is filled in advance by a stochastic preselection process. The selector of the stochastic preselection process hops around in the eigenspace of the reference operator such that after sequencing the timestamps an ongoing hopping path results that recurrently regenerates hop landing location swarm that can be described by a stable location density distribution. The Fourier transform of this location density distribution equals the characteristic function of the stochastic selection mechanism. The hop landing location swarm generates the footprint of the floating platform in the eigenspace of the operator that manages the embedding field in the background platform. The coverage of the embedding field lets the field act as a sticky medium. The sticky medium resists the embedding of objects that break the symmetry of the embedding field. It appears that only isotropic symmetry breaks can deform the embedding field. The sticky medium reacts to the deformation by moving the deformation in all directions away from the embedding location until it vanishes at infinity. Differential calculus shows that the sticky medium reacts with a spherical pulse response that behaves as a spherical shock front that diminishes its amplitude with increasing distance from the location of the pulse. The pulse responses can superpose and join into a more persistent and more smoothed local deformation. This occurs when large amounts of nearby point-like actuators cooperate during a long enough time interval.

Without these streaming processes, not many dynamics would occur in the embedding field. The hop landing locations were created before the start of running time a stochastic process that filled the eigenspaces of the selection operator. We will use the name footprint operator for the selection operator of the floating platforms, and we will use the name universe field for the field that embeds the footprints of the floating platforms. They are archived in quaternionic storage bins that contain a timestamp and a three-dimensional spatial number. After sequencing the timestamps, the eigenspace of the footprint operator contains an ongoing hopping path that recurrently regenerates a coherent hop landing location swarm that is described by a stable location density distribution. The ongoing embedding process maps the hopping path into the embedding field.

10.2 Footprint

An ongoing embedding of a stream of symmetry-disturbing eigenvalues will cause a persistent deformation of the embedding field. The eigenspace of the footprint operator can archive a cord of quaternionic storage bins that contain the timestamps and the landing locations that will be embedded. After sequencing the timestamps, the archive shows an ongoing hopping path that translates into an ongoing embedding process. This embedding process runs during the running episode of the Hilbert repository and acts as an ***imaging process*** in which the image quality is characterized by an Optical Transfer Function [25][26]. This function is the Fourier transfer of the Point Spread Function. The Point Spread Function can be interpreted as a hop landing location density distribution. Its Fourier transform is the Optical Transfer Function of the embedding of the footprint of the considered object.

10.2.1 Footprint mechanism

The mechanism that generates the content of the eigenspace of the footprint operator did its work in the creation episode of the Hilbert repository. The private natural parameter space of the Hilbert space already exists in this creation episode. The timestamps and the hopping locations of the hopping path were taken from this private parameter space. The footprint mechanism owns a characteristic function that ensures that the hopping path recurrently regenerates a hop landing location swarm that features a stable location density distribution which is the Fourier transform of the characteristic function of the footprint mechanism. The location density distribution equals the mentioned Point Spread Function, and the characteristic function equals the corresponding Optical Transfer Function [26].

The hopping path, the hop landing location swarm, the location density distribution, and the Point Spread Function reside in the position space of the Hilbert space. The location density distribution equals the Point Spread Function and describes the hop landing location swarm.

The Optical Transfer Function equals the characteristic function of the footprint mechanism, and both reside in the change space.

Nothing is said about the distribution of the timestamps. In imaging processes, the distribution of discrete objects in the imaging beam can often be characterized as the result of a combination of a Poisson process and a binomial process, where the binomial process is implemented by a spatial point spread function. In that case, the Poisson process handles the distribution of the timestamps.

10.2.2 Footprint characteristics

The footprint generates a nearly constant stream of potential point-like actuators in the form of a swarm that features a constant location density distribution. The actuators that originate from the same floating separable Hilbert space have a constant symmetry. Some of these actuator symmetries can disturb the symmetry of the embedding field and therefore they can generate pulse responses that at least temporarily deform this field. A symmetry disturbance that generates a spherical pulse response must represent an isotropic difference between the two symmetries. A sufficiently constant and sufficiently dense and coherent stream of such actuators can generate a persistent deformation.

10.3 Stickiness

The first-order partial differential equation indicates what happens when a field resists change.

In that case, the terms in the equation try to compensate each other.

$$\phi = \phi_r + \vec{\phi} = \nabla \psi = \nabla_r \psi_r - \left\langle \vec{\nabla}, \vec{\psi} \right\rangle + \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} = 0? \tag{10.3.1}$$

The scalar part and the vector part are treated separately.

$$\phi_r = \nabla_r \psi_r - \left\langle \vec{\nabla}, \vec{\psi} \right\rangle = 0? \tag{10.3.2}$$

$$\vec{\phi} = \nabla_r \vec{\psi} + \vec{\nabla} \psi_r \pm \vec{\nabla} \times \vec{\psi} = 0? \tag{10.3.3}$$

For example, if the curl equals zero, then

$$\nabla_r \vec{\psi} = -\vec{\nabla} \psi_r \tag{10.3.4}$$

will set the vector part of the change to zero. In this way, vector change parts can compensate for scalar change parts.

The Green's function, the shock fronts, and the oscillations also demonstrate the stickiness of dynamic quaternionic fields. Discrete sets of quaternions do not show this stickiness.

The stickiness of the field tends to flatten the field and it resists new deformations of the field.

10.3.1 Potential

In physics, potential energy is the energy held by an object because of its position relative to other objects.

The gravitational potential at a location is equal to the work (energy transferred) per unit mass that would be needed to move an object to that location from a fixed reference location [29][30][31][32][34].

The spherical shock fronts integrate over time into the Green's function of the field. Thus, the shock front injects the content of the Green's function into the affected field. All spherical shock fronts spread the contents of the front over the full field.

We consider the gravitational potential to be zero at infinity. Thus, if infinity is selected as a reference location, then the gravitational potential at a considered location is equal to the work (energy transferred) per unit mass that would be needed to move an object from infinity to that location. Thus, the potential at a location represents the reverse action of the combined spherical shock fronts that act at that location.

10.3.2 Center of deformation

The deformation potential $V(r)$ describes the effect of a local response to an isotropic point-like actuator and reflects the work that must be done by an agent to bring a unit amount of the injected stuff from infinity back to the considered location.

$$V(r) = m_p G / r \tag{10.3.5}$$

Here m_p represents the mass that corresponds to the full pulse response. G takes care for adaptation to physical units. r is the distance to the location of the pulse.

A stream of footprint actuators recurrently regenerates a coherent swarm of embedding locations in the dynamic universe field. That swarm generates a potential

$$V(r) = MG / r \tag{10.3.6}$$

Here M represents the mass that corresponds to the considered swarm of pulse responses. r is the distance to the center of the deformation. This formula is valid at sufficiently large values of r such that the whole swarm can be considered as a point-like object.

In a coherent swarm of massive objects $p_i, i = 1,2,3,\ldots n$, each with static mass m_i at locations r_i, the center of mass $\vec{R}$ follows from [28]

$$\sum_{i=1}^{n} m_i \left(\vec{r}_i - \vec{R}\right) = \vec{0} \qquad (10.3.7)$$

Thus

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{n} m_i \vec{r}_i \qquad (10.3.8)$$

Where

$$M = \sum_{i=1}^{n} m_i \qquad (10.3.9)$$

In the following, we will consider an ensemble of massive objects that own a center of mass $\vec{R}$ and a fixed combined mass M as a single massive object that locates at $\vec{R}$. The separate masses m_i may differ because, at the instant of summation, the corresponding deformation might have partly faded away.

$\vec{R}$ can be a dynamic location. In that case, the ensemble must move as one unit. The problem with the treatise in this paragraph is that in physical reality, point-like objects that possess a static mass do not exist. Only pulse responses that temporarily deform the field exist. Except for black holes, these pulse responses constitute all massive objects that exist in the universe.

10.4 Pulse location density distribution

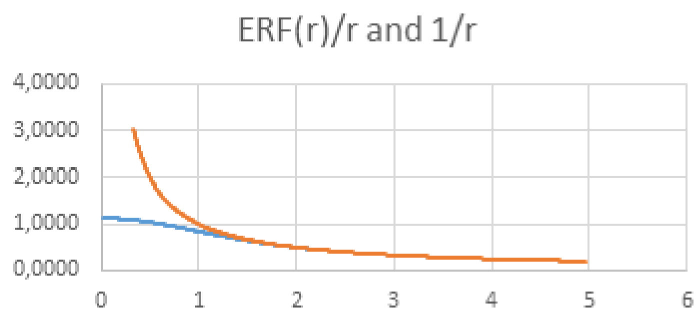
It is false to treat a pulse location density distribution as a set of point-like masses as is done in formulas (10.3.7) and (10.3.8). Instead, the gravitational potential follows from the convolution of the location density distribution and the Green's function. This calculation is still not correct, because the exact result depends on the fact that the deformation that is due to a pulse response quickly fades away and the result also depends on the density of the distribution. If these effects can be ignored, then the resulting gravitational potential of a Gaussian density distribution would be given by [35]

$$g(r) \approx GM \frac{ERF(r)}{r} \tag{10.4.1}$$

Where $ERF(r)$ is the well-known error function. Here the gravitational potential is a perfectly smooth function that at some distance from the center equals the approximated gravitational potential that was described above in the equation (10.3.6). As indicated above, the convolution only offers an approximation because this computation does not account for the influence of the density of the swarm and it does not compensate for the fact that the deformation by the individual pulse responses quickly fades away. Thus, the exact result depends on the duration of the recurrence cycle of the swarm.

In the example, we apply a normalized location density distribution, but the actual location density distribution might have a higher amplitude.

This might explain why some elementary module types exist in multiple generations. These generations appear to have their own mass. For example, elementary fermions exist in three generations. The two more massive generations usually get the name muon or tau generation.



This might also explain why different first-generation elementary particle types show different masses. Due to the convolution, and the coherence of the location density distribution, the blue curve does not show any sign of the singularity that is contained in the red curve, which shows the Green's function.

In physical reality, no point-like static mass object exists. The most important lesson of this investigation is that far from the gravitational center of the distribution the deformation of the field is characterized by the here shown simplified form of the gravitation potential

$$\phi(r) \approx \frac{GM}{r} \tag{10.4.2}$$

Warning: This simplified form shares its shape with the Green's function of the deformed field. This does not mean that the Green's function owns a mass that equals $M_G = \frac{1}{G}$. The functions only share the form of their tail.

10.5 Rest mass

The weakness in the definition of the gravitation potential is the definition of the unit of mass and the fact that shock fronts move with a fixed finite speed. Thus, the definition of the gravitation potential only works properly if the geometric center location of the swarm of injected spherical pulses is at rest in the affected embedding field. The consequence is that the mass that follows from the definition of the gravitation potential is the ***rest mass*** of the considered swarm. We will call the mass that is corrected for the motion of the observer relative to the observed scene the ***inertial mass***.

10.6 Observer

The inspected location is the location of a hypothetical test object that owns an amount of mass. It can represent an elementary particle or a conglomerate of such particles. This location is the target location in the embedding field. The embedding field is supposed to be deformed by the embedded objects.

Observers can access information that is retrieved from storage locations that for them have a historic timestamp. That information is transferred to them via the dynamic universe field. This dynamic field embeds both the observer and the observed event. The dynamic geometric data of point-like objects are archived in Euclidean format as a combination of a timestamp and a three-dimensional spatial location. The embedding field affects the format of the transferred information. The observers perceive in spacetime format. A hyperbolic Lorentz transform converts the Euclidean coordinates of the background parameter space into the spacetime coordinates that are perceived by the observer.

10.6.1 Lorentz transform

In dynamic fields, shock fronts move with speed c. In the quaternionic setting, this speed is unity.

$$x^2 + y^2 + z^2 = c^2\tau^2 \qquad (10.6.1)$$

In flat dynamic fields, swarms of triggers of spherical pulse responses move with lower speed v.

For the geometric centers of these swarms still holds:

$$x^2 + y^2 + z^2 - c^2\tau^2 = x'^2 + y'^2 + z'^2 - c^2\tau'^2 \tag{10.6.2}$$

If the locations $\{x, y, z\}$ and $\{x', y', z'\}$ move with uniform relative speed v, then

$$ct' = ct\cosh(\omega) - x\sinh(\omega) \tag{10.6.3}$$

$$x' = x\cosh(\omega) - ct\sinh(\omega) \tag{10.6.4}$$

$$\cosh(\omega) = \frac{\exp(\omega) + \exp(-\omega)}{2} = \frac{c}{\sqrt{c^2 - v^2}} \tag{10.6.5}$$

$$\sinh(\omega) = \frac{\exp(\omega) - \exp(-\omega)}{2} = \frac{v}{\sqrt{c^2 - v^2}} \tag{10.6.6}$$

$$\cosh(\omega)^2 - \sinh(\omega)^2 = 1 \tag{10.6.7}$$

This is a hyperbolic transformation that relates two coordinate systems, which is known as a Lorentz boost [8].

This transformation can concern two platforms P and P' on which swarms reside and that move with uniform relative speed.

However, it can also concern the storage location P that contains a timestamp t and spatial location $\{x, y, z\}$ and platform P' that has coordinate time t and location $\{x', y', z'\}$.

In this way, the hyperbolic transform relates two platforms that move with uniform relative speed. One of them may be a floating Hilbert space on which the observer resides. Or it may be a cluster of such platforms that cling together and move as one unit. The other may be the background platform on which the embedding process produces the image of the footprint.

The Lorentz transform converts a Euclidean coordinate system consisting of a location $\{x, y, z\}$ and proper timestamps τ into the perceived coordinate system that consists of the spacetime coordinates $\{x', y', z', ct'\}$ in which t' plays the role of coordinate time. The uniform velocity v causes time dilation $\Delta t' = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}}$ and length contraction

$\Delta L' = \Delta L\sqrt{1-\frac{v^2}{c^2}}$

10.6.2 Minkowski metric

Spacetime is ruled by the Minkowski metric [9].

In flat field conditions, proper time τ is defined by

$$\tau = \pm\frac{\sqrt{c^2t^2 - x^2 - y^2 - z^2}}{c} \tag{10.6.8}$$

And in deformed fields, still

$$ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 \tag{10.6.9}$$

Here ds is the spacetime interval and $d\tau$ is the proper time interval. dt is the coordinate time interval

10.6.3 Schwarzschild metric

Polar coordinates convert the Minkowski metric to the Schwarzschild metric [10]. The proper time interval $d\tau$ obeys

$$c^2 d\tau^2 = \left(1-\frac{r_s}{r}\right)c^2 dt^2 - \left(1-\frac{r_s}{r}\right)^{-1} dr^2 - r^2\left(d\theta^2 + \sin^2 d\varphi^2\right) \tag{10.6.10}$$

Under pure isotropic conditions, the last term on the right side vanishes.

According to mainstream physics, in the environment of a black hole, the symbol r_s stands for the Schwarzschild radius [11].

$$r_s = \frac{2GM}{c^2} \qquad (10.6.11)$$

The variable r equals the distance to the center of mass of the massive object with mass M .

The Hilbert Book model finds a different value for the boundary of a spherical black hole. That radius is a factor of two smaller.

10.6.4 Event horizon

The gravitational potential energy $U(r)$

$$U(r) = \frac{mMG}{r} \qquad (10.6.12)$$

at the event horizon $r = r_{eh}$ of a black hole is supposed to be equal to the mass-energy equivalent of an object that has unit mass $m = 1$ and is brought by an agent from infinity to that event horizon. Dark energy objects are energy packages in the form of one-dimensional shock fronts that are a candidate for this role. Photons are strings of equidistant samples of these energy packages. The energy equivalent of the unit mass objects is

$$E = mc^2 = \frac{mMG}{r_{eh}} \qquad (10.6.13)$$

Or with $m = 1$

$$r_{eh} = \frac{MG}{c^2} \qquad (10.6.14)$$

At the event horizon, all energy of the dark energy object is consumed to compensate for the gravitational potential energy at that location. No field excitation and in particular no shock front can pass the event horizon.

10.7 Inertial mass

The Lorentz transform also gives the transform of the rest mass to the mass that is relevant when the embedding field moves relative to the floating platform of the observed object with uniform speed $\vec{v}$.

In that case, the inertial mass M relates to the test mass M0 as

$$M = \gamma M_0 = \frac{M_0}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{10.7.1}$$

This indicates that the formula (10.3.6) for the gravitational potential at distance r must be changed to

$$V(r) = \frac{M_0 G}{r\sqrt{1 - \frac{v^2}{c^2}}} \tag{10.7.2}$$

10.8 Inertia

The relation between inertia and mass is complicated [36][37]. We apply an artificial field that resists its changing. The condition that for each type of massive object, the gravitational potential is a static function, and the condition that in free space, the massive object moves uniformly, establish that inertia rules the dynamics of the situation. These conditions define an artificial quaternionic field that resists change. The scalar part of the artificial field is represented by the gravitational potential, and the uniform speed of the massive object represents the imaginary (vector) part of the field.

The first-order change of the quaternionic field can be divided into five separate partial changes. Some of these parts can compensate for each other.

Mathematically, the statement that in the first approximation nothing in the field ξ changes indicates that locally, the first-order partial differential $\nabla\xi$ will be equal to zero.

$$\zeta = \nabla\xi = \nabla_r\xi_r - \left\langle \vec{\nabla}, \vec{\xi} \right\rangle + \vec{\nabla}\xi_r + \nabla_r\vec{\xi} \pm \vec{\nabla} \times \vec{\xi} = 0 \tag{10.8.1}$$

Thus

$$\zeta_r = \nabla_r\xi_r - \left\langle \vec{\nabla}, \vec{\xi} \right\rangle = 0 \tag{10.8.2}$$

$$\vec{\zeta} = \vec{\nabla}\xi_r + \nabla_r\vec{\xi} \pm \vec{\nabla} \times \vec{\xi} = 0 \tag{10.8.3}$$

These formulas can be interpreted independently. For example, according to the equation (10.8.2), the variation in time of ξ_r can compensate the divergence of $\vec{\xi}$. The terms that are still eligible for change must together be equal to zero. For our purpose, the curl $\vec{\nabla} \times \vec{\xi}$ of the vector field $\vec{\xi}$ is expected to be zero. The resulting terms of the equation (10.8.3) are

$$\nabla_r\vec{\xi} + \vec{\nabla}\xi_r = 0 \tag{10.8.4}$$

In the following text plays $\vec{\xi}$ the role of the vector field and ξ_r plays the role of the scalar gravitational potential of the considered object. For elementary modules, this special field concerns the effect of the hop landing location swarm that resides on the floating platform on its image in the embedding field. It reflects the activity of the stochastic process and the uniform movement in the free space of the floating platform over the background platform. It is characterized by a mass value and by the uniform velocity of the floating platform with respect to the background platform. The real part conforms to the deformation that the stochastic process causes. The imaginary part conforms to the speed of movement of the floating platform. The main characteristic of this field is that it tries to keep its overall change zero. The author calls ξ the ***conservation field***.

At a large distance r, we approximate this potential by using the formula

$$\zeta_r(r) \approx \frac{GM}{r} \tag{10.8.5}$$

Here M is the inertial mass of the object that causes the deformation. The new artificial field $\xi = \left\{\frac{GM}{r}, \vec{v}\right\}$ considers a uniformly moving mass as a normal situation. It is a combination of scalar potential $\frac{GM}{r}$ and speed $\vec{v}$. This speed of movement is the relative speed between the floating platform and the background platform. At rest this speed is uniform.

If this object accelerates, then the new field $\left\{\frac{GM}{r}, \vec{v}\right\}$ tries to counteract the change $\dot{\vec{v}}$ of the vector field $\vec{v}$ by compensating this with an equivalent change of the scalar part $\frac{GM}{r}$ of the new field ξ. According to the equation (10.8.4), this equivalent change is the gradient of the real part of the field.

$$\vec{a} = \dot{\vec{v}} = -\vec{\nabla}\left(\frac{GM}{r}\right) = \frac{GM\,\vec{r}}{|\vec{r}|^3} \tag{10.8.6}$$

This generated vector field acts on masses that appear in its realm.

Thus, if two uniformly moving masses m and M exist in each other's neighborhood, then any disturbance of the situation will cause the gravitational force

$$\vec{F}\left(\vec{r}_1-\vec{r}_2\right)=m_0\vec{a}=\frac{Gm_0M\left(\vec{r}_1-\vec{r}_2\right)}{\left|\vec{r}_1-\vec{r}_2\right|^3}=\gamma\frac{Gm_0M_0\left(\vec{r}_1-\vec{r}_2\right)}{\left|\vec{r}_1-\vec{r}_2\right|^3} \qquad (10.8.7)$$

Here $M=\gamma M_0$ is the inertial mass of the object that causes the deformation. m_0 is the rest mass of the observer.

The inertial mass M relates to its rest mass M_0 as

$$M=\gamma M_0=\frac{M_0}{\sqrt{1-\frac{v^2}{c^2}}} \qquad (10.8.8)$$

This formula holds for all elementary particles except for quarks.

The problem with quarks is that these particles do not provide an isotropic symmetry difference. They must first combine into hadrons to be able to generate an isotropic symmetry difference. This phenomenon is known as ***color confinement***.

10.9 Momentum

In the formula (10.8.7) that relates mass to force the factor γ that corrects for the relative speed can be attached to m_0 or to M_0

$$\vec{F}\left(\vec{r}_1-\vec{r}_2\right)=\gamma\frac{Gm_0M_0\left(\vec{r}_1-\vec{r}_2\right)}{\left|\vec{r}_1-\vec{r}_2\right|^3} \qquad (10.9.1)$$

The force relates to the temporal change of the momentum vector $\vec{P}$ of the observer

$$\vec{F}=\dot{\vec{P}}=\frac{d\vec{P}}{dt} \qquad (10.9.2)$$

The momentum vector $\vec{P}$ is part of a quaternionic momentum P. The momentum depends on the relative speed of the moving object that causes the deformation which defines the mass. The speed is determined relative to the field that embeds the object and that gets deformed by the investigated object. For free elementary particles, the speed equals the floating speed of the platform on which the particle resides.

$$P = P_r + \vec{P} \tag{10.9.3}$$

$$\|P\|^2 = P_r^2 + \|\vec{P}\|^2 \tag{10.9.4}$$

$$\vec{P} = \gamma m_0 \vec{v} \tag{10.9.5}$$

$$\|\vec{P}\|^2 = \gamma^2 m_0^2 \|\vec{v}\|^2 \tag{10.9.6}$$

$$\|P\|^2 = \gamma^2 m_0^2 c^2 = P_r^2 + \gamma^2 m_0^2 \|\vec{v}\|^2 \tag{10.9.7}$$

$$\|P\| = \gamma m_0 c = E / c \tag{10.9.8}$$

$$E = \gamma m_0 c^2 \tag{10.9.9}$$

$$\begin{aligned} P_r^2 &= \gamma^2 m_0^2 c^2 - \gamma^2 m_0^2 \|\vec{v}\|^2 \\ &= \gamma^2 m_0^2 \left(c^2 - \|\vec{v}\|^2\right) = \gamma^2 m_0^2 c^2 \left(1 - \left\|\frac{\vec{v}}{c}\right\|^2\right) = m_0^2 c^2 \end{aligned} \tag{10.9.10}$$

$$P_r = m_0 c = \frac{E}{\gamma c} \tag{10.9.11}$$

$$\|\vec{P}\| = \gamma m_0 \|\vec{v}\| \tag{10.9.12}$$

$$P = P_r + \vec{P} = m_0 c + \gamma m_0 \vec{v} = \frac{E}{\gamma c} + \gamma m_0 \vec{v} \tag{10.9.13}$$

If $\vec{v} = \vec{0}$ then $\vec{P} = \vec{0}$ and $\|P\| = P = P_r = m_0 c$

Here Einstein’s famous mass-energy equivalence is involved.

$$E = \gamma m_0 c^2 = mc^2 \tag{10.9.14}$$

The disturbance by the ongoing expansion of the embedding field suffices to put the gravitational force into action. The description also holds when the field ξ describes a conglomerate of platforms and M_2 represents the mass of the conglomerate.

The artificial field ξ represents the habits of the underlying model that ensures the constancy of the gravitational potential and the uniform floating of the considered massive objects in free space.

Inertia ensures that the third-order differential (the third-order change) of the deformed field is minimized. It does that by varying the speed of the platforms on which the massive objects reside.

Inertia bases mainly on the definition of mass that applies to the region outside the sphere where the gravitational potential behaves like the Green’s function of the field. There, the formula $\xi_r = \frac{GM}{r}$ applies. Further, it bases on the intention of modules to keep the gravitational potential inside the mentioned sphere constant. At least that holds when this potential is averaged over the regeneration period. In that case, the overall change $\nabla\zeta$ in the conservation field ζ equals zero. Next, the definition of the conservation field supposes that the swarm which causes the deformation moves as one unit. Further, the fact is used that the solutions of the homogeneous second-order partial differential equation can superpose in new solutions of that same equation.

The popular sketch in which the deformation of our living space is presented by smooth dips is obviously false. The story that is represented in this paper shows the deformations as local extensions of the field, which represents the universe. In both sketches, the deformations elongate the information path, but none of the sketches explain why two masses attract each other. The above explanation founds on the habit of the stochastic process to recurrently regenerate the same time average of the gravitational potential, even when that averaged potential moves uniformly. Without the described habit of the stochastic processes, inertia would not exist.

The applied artificial field also explains the gravitational attraction by black holes.

The artificial field that implements mass inertia also plays a role in other fields. Similar tricks can be used to explain the electrical force from the fact that the electrical field is produced by sources and sinks that can be described with the Green's function.

10.9.1 Forces

In the Hilbert repository, all symmetry-related charges are located at the geometric center of an elementary particle and all these particles own a footprint that for isotropic symmetry differences can deform the embedding field. In that case, the particle features mass and forces might be coupled to acceleration via

$$F = m\vec{a} \tag{10.9.15}$$

Or to momentum via

$$F = \dot{\vec{P}} \tag{10.9.16}$$

11 Symmetry restrictions

11.1 Using volume integrals to determine the symmetry-related charges

In its simplest form in which no discontinuities occur in the integration domain Ω, the generalized Stokes theorem runs as

$$\int_{\Omega} d\omega = \int_{\partial\Omega} \omega = \oint_{\Omega} \omega \tag{11.1.1}$$

We separate all point-like discontinuities from the domain Ω by encapsulating them in an extra boundary. Symmetry centers represent spherically shaped or cube-shaped closed parameter space regions H_n^x that float on a background parameter space $\mathfrak{R}$. The boundaries ∂H_n^x separate the regions from the domain H_n^x. The regions H_n^x are platforms for local discontinuities in basic fields. These fields are continuous in the domain $\Omega - H$.

$$H = \bigcup_{n=1}^{N} H_n^x \tag{11.1.2}$$

The symmetry centers $\mathfrak{S}_n^x$ are encapsulated in regions H_n^x, and the encapsulating boundary ∂H_n^x is not part of the disconnected boundary, which encapsulates all continuous parts of the quaternionic manifold ω that exists in the quaternionic model.

$$\int_{\Omega - H} d\omega = \int_{\partial\Omega \cup \partial H} \omega = \int_{\partial\Omega} \omega - \sum_{k=1}^{N} \int_{\partial H_n^x} \omega \tag{11.1.3}$$

In fact, it is sufficient that ∂H_n^x surrounds the current location of the elementary module. We will select a boundary, which has the shape of a small cube of which the sides run through a region of the parameter spaces where the manifolds are continuous.

If we take everywhere on the boundary the unit normal to point outward, then this reverses the direction of the normal on ∂H_n^x which negates the integral. Thus, in this formula, the contributions of boundaries $\{\partial H_n^x\}$ are subtracted from the contributions of the boundary $\partial\Omega$. This means that $\partial\Omega$ also surrounds the regions $\{\partial H_n^x\}$

This fact renders the integration sensitive to the ordering of the participating domains.

Domain Ω corresponds to part of the background parameter space $\mathfrak{R}$. As mentioned before the symmetry centers $\mathfrak{S}_n^x$ represent encapsulated regions $\left\{\partial H_n^x\right\}$ that float on the background parameter space $\mathfrak{R}$. The Cartesian axes of $\mathfrak{S}_n^x$ are parallel to the Cartesian axes of background parameter space $\mathfrak{R}$. Only the orderings along these axes may differ.

Further, the geometric center of the symmetry center $\mathfrak{S}_n^x$ is represented by a floating location on parameter space $\mathfrak{R}$.

The symmetry center $\mathfrak{S}_n^x$ is characterized by a private symmetry flavor. That symmetry flavor relates to the Cartesian ordering of this parameter space. With the orientation of the coordinate axes fixed, eight independent Cartesian orderings are possible.

The consequence of the differences in the symmetry flavor on the subtraction can best be comprehended when the encapsulation ∂H_n^x is performed by a ***cubic space form*** that is aligned along the Cartesian axes that act in the background parameter space. Now the six sides of the cube contribute differently to the effects of the encapsulation when the ordering of H_n^x differs from the Cartesian ordering of the reference parameter space $\mathfrak{R}$. Each discrepant axis ordering corresponds to one-third of the surface of the cube. This effect is represented by the ***geometric symmetry-related charge***, which includes the ***color charge*** of the symmetry center. It is easily comprehensible related to the algorithm which below is introduced for the computation of the geometric symmetry-related charge. Also, the relation to the color charge will be clear. ***Thus, this effect couples the ordering of the local parameter spaces to the geometric symmetry-related charge of the encapsulated elementary module.*** The differences with the ordering of the surrounding parameter space determine the value of the geometric symmetry-related charge of the object that resides inside the encapsulation!

11.2 Symmetry flavor

The Cartesian ordering of its private parameter space determines the symmetry flavor of the platform [17]. For that reason, this symmetry is compared with the reference symmetry, which is the symmetry of the background parameter space. Four arrows indicate the symmetry of the platform. The background is represented by:

↑↑↑↑

Now the geometric symmetry-related charge follows in two steps.

1. Count the difference of the spatial part of the geometric symmetry of the platform with the spatial part of the geometric symmetry of the background parameter space.
2. Switch the sign of the result for anti-particles.

Symmetrieversion					
Ordering x y z τ	Sequence	Handedness Right/Left	Color charge	Electric charge * 3	Symmetry type.
↑↑↑↑	0	R	N	+0	neutrino
↓↑↑↑	1	L	R	− 1	down quark
↑↓↑↑	2	L	G	− 1	down quark
↓↓↑↑	3	R	B	+2	up quark
↑↑↓↑	4	L	B	−1	down quark
↓↑↓↑	5	R	G	+2	up quark
↑↓↓↑	6	R	R	+2	up quark
↓↓↓↑	7	L	N	− 3	electron
↑↑↑↓	8	R	N	+3	positron
↓↑↑↓	9	L	R	− 2	anti-up quark
↑↓↑↓	10	L	G	− 2	anti-up quark
↓↓↑↓	11	R	B	+1	anti-down quark
↑↑↓↓	12	L	B	− 2	anti-up quark
↓↑↓↓	13	R	G	+1	anti-down quark
↑↓↓↓	14	R	R	+1	anti-down quark
↓↓↓↓	15	L	N	− 0	anti-neutrino

Probably, the neutrino and the antineutrino own an abnormal handedness.

The suggested particle names that indicate the symmetry type are borrowed from the Standard Model. In the table, compared to the standard model, some differences exist with the selection of the anti-predicate. All considered particles are elementary fermions. The freedom of choice in the polar coordinate system might determine the spin [18]. The azimuth range is 2π radians, and the polar angle range is π radians. Symmetry disturbance means a difference between the platform symmetry and the symmetry of the background. Neutrinos do not disturb symmetry. Instead, they probably may cause conflicts with the handedness of the multiplication rule.

In the Hilbert repository, only point-like charges occur that represent sources or sinks. These charges move with the geometrical center of the corresponding particle.

11.3 Potential of the electric field

The potential of an electromagnetic field is a quaternionic function.

$$\phi(r) = \phi_r(r) + \vec{\phi}(r) \tag{11.3.1}$$

The corresponding force is the Lorentz force.

$$\begin{aligned}\vec{F}(r) &= Q\left[-\vec{\nabla}\phi_r - \nabla_r\vec{\phi} + \vec{v}\times\left(\vec{\nabla}\times\vec{\phi}\right)\right]\\ &= Q\left[\vec{E} + \vec{v}\times\vec{B}\right]\end{aligned} \tag{11.3.2}$$

A stream of symmetry-related actuators that is represented by a source or sink and is characterized by a symmetry-related charge Q generates a scalar potential

$$\phi_r(r) = \frac{Q}{4\pi\varepsilon_0 r} \tag{11.3.3}$$

This means that its observation is affected by inertia in a way that is like the way that the observation of the gravitational potential is affected. This becomes noticeable in the electric force between two charges.

11.3.1 Coulomb force

The electric charge is coupled to the geometric center of a massive object.

Another electric charge is coupled to another massive object. The charges repel or attract the charges that are located at the other geometric center. Thus, a relative speed of the two geometric centers is changed into an acceleration.

With electromagnetic potentials, the force derives from the Lorentz force. If the magnetic potential $\vec{\xi}$ equals zero, then only part of the electric field results.

$$E = -\vec{\nabla}\xi_r = \frac{Q_1(\vec{r}_1 - \vec{r}_2)}{4\pi\varepsilon_0 |\vec{r}_1 - \vec{r}_2|^3} \tag{11.3.4}$$

Thus, if two uniformly moving charges Q_1 and Q_2 exist in each other's neighborhood, then any disturbance of the situation will cause the electrical force

$$\vec{F}(\vec{r}_1 - \vec{r}_2) = Q_2 E = \frac{\varepsilon Q_1 Q_2 (\vec{r}_1 - \vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|^3} \tag{11.3.5}$$

The force repels for two sources or two sinks and attracts for the combination of a source and a sink.

These formulas hold for all elementary particles including quarks.

12 Basic fields

12.1 Coupling of basic fields

Besides the fact that the geometric center of the elementary particles also forms the geometric center of the symmetry-related field of this particle the coupling of the symmetry of the particle to the Cartesian coordinate system of the particle couples the basic fields of the particle to the background field that acts as our universe. It tries to keep the Cartesian coordinate systems in parallel. This couples the curl of the particle's geometric symmetry-related field to the curl of the embedding background field. A non-zero curl might even couple to the otherwise undetermined direction of the spin vector in the spherical shock fronts. This couples the direction of spin to a non-zero magnetic field.

13 Conglomerates

The Hilbert repository suggests that apart from the quarks all elementary fermions are constituted by excitations of the dynamic field that represents our universe. These excitations are spherical pulse responses that act as spherical shock fronts that locally and temporarily deform this embedding field. The quarks can combine into hadrons and are then also capable of generating spherical shock fronts.

Further, the spherical shock fronts appear to constitute all discrete massive objects that exist in the universe. The exception to this rule is formed by encapsulated regions that contain countable sets of objects and therefore do not form a compact continuum. We call these regions black holes because no field excitations exist in these regions and no field excitations can enter or leave these regions. Still, the region can and will deform its continuous surround.

The above statement suggests that elementary fermions can constitute higher generations of fermions and can generate bosons. The notorious exception is formed by photons. Photons are constituted by chains of equidistant one-dimensional shock fronts.

The reason for this suggestion is that the footprint of elementary fermions is generated by stochastic processes that own a characteristic function, which is controlled from and specified in change space.

This opens the possibility to also define the conglomerates of elementary particles in change space. Each conglomerate is defined by a private stochastic process that owns a characteristic function, which is a dynamic superposition of the characteristic functions of the components of the conglomerate. The superposition coefficients act as displacement generators. In this way, these coefficients specify the internal positions of the components. These dynamic coefficients define internal oscillations.

In change space, the location in the configuration space has no significance. Thus, components of a composite can locate far from each other in configuration space. This is the reason that entanglement exists. Entanglement becomes noticeable when components obey exclusion principles.

13.1 Modular system

The definition of these conglomerates causes that apart from black holes and photons, the discrete objects that exist in our universe and embed in the dynamic universe field form an extensive modular system with the elementary fermions as the elementary modules and individual modular systems at the top of the hierarchies.

13.2 Module types

Module types form type communities. These communities have a much longer lifespan than individual modules. In the competition between module communities, the community that takes the best care for its members and that also takes care of the module communities on which it relies have the best chance of survival. This fact contrasts Darwin's statement about the survival of the fittest individual.

13.3 Atoms

Compound modules are composite modules for which the images of the geometric centers of the platforms of the components coincide in the background platform. The charges of the platforms of the elementary modules establish the binding of the corresponding platforms. Physicists and chemists call these compound modules atoms or atomic ions.

In free compound modules, the geometric symmetry-related charges do not take part in the oscillations. The targets of the private stochastic processes of the elementary modules oscillate. This means that the hopping path of the elementary module folds around the oscillation path and the hop landing location swarm gets smeared along the oscillation path. The oscillation path is a solution to the Helmholtz equation. Each fermion must use a different oscillation mode. A change of the oscillation mode goes together with the emission or absorption of a photon. The center of emission coincides with the geometrical center of the compound module. During the emission or absorption, the oscillation mode, and the hopping path halt, such that the emitted photon does not lose its integrity. Since all photons share the same emission duration, that duration must coincide with the regeneration cycle of the hop landing location swarm. Absorption cannot be interpreted so easily. In fact, it can only be comprehended as a time-reversed emission act. Otherwise, the absorption would require an incredible aiming precision for the photon.

The type of stochastic process that controls the binding of components appears to be responsible for the absorption and emission of photons and the change of oscillation modes. If photons arrive with too low energy, then the energy is spent on the kinetic energy of the common platform. If photons arrive with too high energy, then the energy is distributed over the available oscillation modes, and the rest is spent on the kinetic energy of the common platform, or it escapes into free space. The process must somehow archive the modes of the components. It can apply the private platform of the components for that purpose. Most probably, the current value of the dynamic superposition coefficient is stored in the eigenspace of a special superposition operator.

13.4 Molecules

Molecules are conglomerates of compound modules that each keep their private geometrical center. However, electron oscillations are shared among the compound modules. Together with the geometric symmetry-related charges, this binds the compound modules into the molecule.

14 Two episodes

The footprint operator is already present at the time of the creation of the Hilbert repository and determines the behavior of the elementary particle throughout its existence. This fact is a great mystery. The humanly derived math does not yet offer an explanation. However, the existence of the footprint operator makes it possible to divide the model of physical reality into a preparatory episode in which there is no flowing time and an ongoing episode in which a continuing step-by-step embedding of the hop landing locations mimics the activities of the stochastic processes. The embedding process uses the stored and ordered time stamps to realize the corresponding hop landings. The range of running time is equal to the range of the archived time stamps. At the beginning of the running time, the field that represents our universe is still virginal and corresponds to the background parameter space. After the first footprints are completed, the relevant elementary particles can start to form composite objects.

14.1 In the beginning

Before the embedding processes that mimic the activity of the stochastic processes started their action, the content of the universe was empty. It was represented by a flat field that in its spatial part, was equal to the parameter space of the background platform. At the beginning instant, a huge number of these mimicked stochastic processes started their triggering of the dynamic field that represents the universe. The triggers may cause spherical pulse responses that act as spherical shock fronts. These spherical shock fronts temporarily deform the universe field. In that case, they will also persistently expand the universe. Thus, from that moment on, the universe started expanding. This did not happen at a single point. Instead, it happened at a huge number of locations that were distributed all over the spatial part of the parameter space of the quaternionic function that describes the dynamic universe field.

Close to the beginning of time, all distances were equal to the distances in the flat parameter space. Soon, these islands were uplifted with volume that was emitted at nearby locations. This flooding created growing distances between used locations. After some time, all parameter space locations were reached by the generated shock fronts. From that moment on the universe started acting as an everywhere expanded continuum that contained deformations which in advance were remarkably small. Where these deformations grew, the distances grew faster than in the environment. A more uniform expansion appears the rule and local deformations form the exception. Deformations make the information path longer and give the idea that time ticks slower in the deformed and expanded regions. This corresponds with the gravitational redshift of photons.

Composed modules only started to be generated after the presence of enough elementary modules. The generation of photons that reflected the signatures of atoms only started after the presence of these compound modules. However, the spurious one-dimensional shock fronts could be generated from the beginning.

This picture differs considerably from the popular scene of the big bang that started at a single location [12].

The expansion is the fastest in areas where spherical pulse responses are generated. For that reason, it is not surprising that the measured Hubble constant differs from place to place.

14.2 RTOS

The archival of dynamic geometric data that takes place in the creation episode is determining the life story of the elementary particles. The activity of the stochastic processes is mimicked by the ongoing embedding process that implements the dynamic geometric data as an ongoing hopping path that recurrently regenerates a coherent hop landing location swarm that has a stable location density distribution. This location density distribution is the Fourier transform of the characteristic function of the stochastic process that filled the eigenspace of the footprint operator that resides at the private platform of the elementary particle. This activity acts as a Real-Time Operating System. The recurrent regeneration of the hop landing location swarm implements an effective guard against deadlocks and race conditions.

15 Dark objects

Mainstream physics suggests the existence of two types of dark objects [39][40]. These are dark energy and dark matter. In contrast to mainstream physics, the Hilbert Book Model presents these two types of dark objects as field excitations that act as shock fronts. Together these special field excitations constitute, except for black holes, all discrete objects that exist in the universe.

Dark matter objects are spherical pulse responses that behave as spherical shock fronts. They constitute the footprints of elementary particles. Further, they populate as a veiling glare the universe in the neighborhood of large assemblies of conglomerates of elementary particles.

Dark energy objects are one-dimensional pulse responses and behave as one-dimensional shock fronts. They appear spread over the universe, but more specifically divided equidistantly in chains that constitute photons. Photons obey the Planck-Einstein relation $E = h\nu$ [24]. This means that the emission duration of photons is fixed and since all shock fronts move with speed c, at the instant of emission, all photons must feature the same length.

Dark energy objects may change the kinetic energy of the floating platforms. If floating platforms cling together, then the kinetic energy of the conglomerate is changed. The massive objects may absorb or emit dark energy objects. In contrast, will atoms absorb and emit photons. Photons are quantized patterns of dark energy objects.

15.1 Black holes

We introduce a ***discontinuum*** as the antonym of a continuum. The universe is a mixed field. It can contain a set of enclosed spatial regions that encapsulate a discontinuum. A discontinuum is a dense discrete set. A discontinuum is countable. In physics, the equivalent of a discontinuum is a black hole. The enclosing surface is a continuum with a lower dimension than the enclosed region. No field excitations exist inside the discontinuum. Thus, no field excitations can pass the enclosing surface. Since a discontinuum deforms the surrounding continuum, this enclosed region owns an amount of mass. Together with the spherical shock fronts and the elementary modules, the discontinuums are the only objects in the universe that own mass. The mass of spherical shock fronts is volatile. Only when gathered in coherent and dense ensembles these shock fronts can cause a persistent amount of mass. That happens in the footprint of elementary modules. It also happens in the halos of galaxies. So, black holes can only be perceived by their gravitational potential. However, outside the border of the black hole, many phenomena can occur that are caused by the activity of massive objects that are attracted by the enormous gravitation that the black hole generates. Elementary particles that hover with their platform over the encapsulated region can drop part of their footprint actuators into the black hole. In this way, black holes can steadily grow. This paper does not consider the join of black holes and it does not consider the birth of a black hole by squeezing one or more neutron stars.

16 The Standard Model of particle physics

The Standard Model of particle physics is a useful report on what we think we know about these particles. It says nothing about our universe and it also does not say anything about the electromagnetic field, but it reports on electric charges and color charges. The Standard Model pretends to treat the elementary particles, but it does not describe the structure and the behavior of these objects. The Standard Model does not treat dark matter and it does not treat dark energy. Some scientists consider some theories as a part of the Standard Model. This concerns Quantum Field Theory, Quantum Electrodynamics, and Quantum Chromodynamics. This puts these theories in an undeserved position because none of these theories is well-founded. Mathematicians have in vain tried to explain the strangely limited diversity of types of elementary particles and the mentioned theories do not explain this restricted diversity.

This indicates that a great chance exists that the currently available theories are on a wrong track. A more fundamental approach may exist that also explains the diversity of the particle types. That theory must explain why so few different particle types exist and why these particle types divide into categories, such as fermions and bosons, and why these particle types differ in mass, electric charge, and color charge.

The Hilbert repository does not resemble the structure of the full set of elementary particles that are listed in the Standard Model. Instead, the set of particle types in the Hilbert repository resembles closely the set of elementary fermions in the Standard Model. This might indicate that the bosons listed in the Standard Model are not elementary modules and it may indicate that some bosons that the Standard Model considers as elementary particles differ from the model of elementary particles that the Hilbert repository provides. This will certainly hold for photons. Other bosons will be other elementary particle types or they are conglomerates of elementary fermions. Also, muon-type and tau-type fermions may be conglomerates of elementary fermions.

A guide may be that only elementary fermions act as elementary modules and form the conglomerates that populate our universe.

17 Conclusions

The structure and the behavior of the Hilbert repository show an astonishing similarity with the structure and behavior of the set of elementary fermions in the Standard Model of particle physics.

The universe is a dynamic field that is archived in the background platform of the Hilbert repository. This dynamic field can be described by a quaternionic function. Quaternionic differential calculus describes the dynamics of this field. Apart from the wave equation exists another second-order partial differential equation.

Electric charges only appear at the geometric centers of the floating platforms on which elementary fermions reside.

The shortlist of electric charges and color charges in the Standard Model conforms with the shortlist of symmetry-related charges in the Hilbert repository.

Sources and sinks represent the symmetry-related charges.

Elementary fermions behave as elementary modules. Except for black holes they constitute all massive objects that exist in the universe.

Stochastic processes that own a characteristic function and can be considered as a combination of a Poisson process and a binomial process implement the wavefunction of elementary fermions. These processes produce an ongoing hopping path that recurrently regenerates a coherent hop landing location swarm that is described by a stable location density distribution. An ongoing embedding process images the hop landing locations on the dynamic universe field. If the hop landings deform the embedding field, then the generated spherical shock fronts blur the hop landing location swarm. The resulting deformation is described by the gravitational potential of the particle. That gravitational potential determines the mass of the particle.

Dark objects play an essential role in the dynamics of the universe field.

Dark matter objects are spherical pulse responses that behave as spherical shock fronts and integrate over time in the Green’s function of the field.

Dark matter objects constitute the footprints of elementary fermions.

Dark matter objects explain the origin of gravity.

Dark energy objects are one-dimensional pulse responses and behave as one-dimensional shock fronts. They appear spread over the universe, but more specifically they constitute photons divided equidistantly in chains. Photons obey the Planck-Einstein relation.

Black holes are considered encapsulated discontinuous regions that exist in a continuous surround. They become noticeable by their gravitational potential and by the phenomena that occur at their border.

The Hilbert repository supports both quantum physics and cosmology. This powerful structure enables the introduction of a creation episode in which time does not yet exist as a flowing progression indicator. At the beginning of flowing time, the universe is a virgin flat field that corresponds to a version of the quaternionic number system. Coordinate systems determine the symmetry of this version.

This document offers an alternative for the Higgs mechanism as an explanation for the origin of gravity. This opposes the addition of the Higgs particle to the set of elementary particles that are registered in the Standard Model. This document also differs in the way that photons are treated. Both deviations are due to the discovery of the importance that shock fronts mean for the interaction between fields and actuators. Established physics appears to ignore shock fronts. These objects play the most important part in what happens on the deepest levels of the structure that represents the skeleton of our living environment.

17.1 Existential questions

This paper considers a creation episode but does not treat the creation of the Hilbert repository itself. The Hilbert repository follows from extending the basic notion of a vector space. So, as soon as elemental vector spaces with all their ingredients exist, then the Hilbert repository will also exist. That does not guarantee that the eigenspaces of the footprint operators will be filled by a stochastic process that owns a characteristic function. The author could not yet find the reason why such stochastic processes already exist at the birth of the Hilbert repository and why the generated hopping path recurrently recreates a hop landing location swarm that has a stable location density distribution. In contrast, the private parameter space of Hilbert spaces is present at the birth of every Hilbert space. With that parameter space, every Hilbert space owns a geometric symmetry and a geometric center.

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