Magnetic symmetry, curvature and Gauss-Bonnet-Chern theorem

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We reformulate Gauss-Bonnet-Chern theorem in relation with magnetic symmetry of geometrical optics. If Euler-Poincare characteristic is a topological invariant, should unrestricted electric potential of $U(1)$ gauge potential be a topological invariant?

I. GAUSS-BONNET-CHERN THEOREM AND CURVATURE

Related with the Riemannian-Christoffel curvature tensor, $R_{\mu\nu\rho\sigma}$, the Gauss-Bonnet-Chern theorem[1] can be written as[2]

$$\chi(M^{2n}) = (-1)^n \frac{1}{2^{2n}\pi^n n!} \int_{M^{2n}} \sum \epsilon_{\mu\nu} \sum R_{\mu\nu\rho\sigma}\ dx^\rho \ \wedge \ dx^\sigma \tag{1}$$

where $\chi(M^{2n})$ is the Euler-Poincare characteristic.

Eq.(1) relates the Riemannian geometry which is local geometry with the topological space which is global geometry. The Riemann-Christoffel curvature tensor is a local invariant and the Euler-Poincare characteristic is a global invariant.

II. MAGNETIC SYMMETRY AND CURVATURE

Refer to our previous work[3]

$$\left|\partial_\nu \left\{\frac{c}{\omega} \arccos\left(A_\mu^{U(1)}\hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)}\right) a_\mu^{-1} + ct\right\}\right| = n_{\mu\nu} \tag{2}$$

where $A_\mu^{U(1)}$ is the unrestricted electric (scalar) potential of the $U(1)$ gauge potential, $\hat{m}^{U(1)}$ is the restricted magnetic (vector) potential of the n-dimensional $U(1)$ group and $n_{\mu\nu}$ is the refractive index.

The relation of refractive index-curvature becomes[3]

$$gN_\sigma\ \partial_\rho \ln \left|\partial_\nu \left\{\frac{c}{\omega} \arccos \left(A_\mu^{U(1)}\hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)}\right) a_\mu^{-1} + ct\right\}\right| = \ R_{\mu\nu\rho\sigma} \tag{3}$$

Eqs.(2),(3) show that there exist the magnetic symmetry (magnetic monopole) represented by $\hat{m}^{U(1)}$ in the geometrical optics, especially in the refractive index (2) and the refractive index-Riemann-Christoffel curvature tensor relation (3) where both are formulated in the $(4+n)$-dimensions of unified space.

III. MAGNETIC SYMMETRY AND GAUSS-BONNET-CHERN THEOREM

Substituting eq.(3) into eq.(1), we obtain

$$(-1)^n \frac{1}{2^{2n}\pi^n n!} \int_{M^{2n}} \sum \epsilon_{\mu\nu} \sum gN_\sigma\ \partial_\rho \ln \left|\partial_\nu \left\{\frac{c}{\omega} \arccos \left(A_\mu^{U(1)}\hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)}\right) a_\mu^{-1} + ct\right\}\right|\ dx^\rho \ \wedge \ dx^\sigma = \chi(M^{2n}) \tag{4}$$

IV. DISCUSSION AND CONCLUSION

Because the Euler-Poincare characteristic is a topological invariant[3], so we think that the unrestricted electric potential of $U(1)$ gauge potential, $A_\mu^{U(1)}$, should be a topological invariant.

If the Gauss-Bonnet-Chern theorem is a special case of the Atiyah-Singer index theorem, what does the magnetic symmetry imply to the Atiyah-Singer index theorem?

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[1] For even-dimensional oriented compact Riemannian manifold, M^{2n}, the Gauss-Bonnet-Chern theorem is a special case of the Atiyah-Singer index theorem (Spalluci E. et al (2004), *Pfaffian.* In: Duplij S., Siegel W., Bagger J. (eds), Concise Encyclopedia of Supersymmetry, Springer, Dordrecht.).

[2] Miftachul Hadi, *Linear and non-linear refractive indices in Riemannian and topological spaces*, https://vixra.org/pdf/2105.0163v1.pdf, 2020.

[3] Miftachul Hadi, *Magnetic symmetry of geometrical optics*, https://vixra.org/pdf/2104.0188v1.pdf, 2021.