A Non-linear Generalisation of Quantum Mechanics

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Abstract

A new definition of quantum-mechanical momentum is proposed which yields novel nonlinear generalisations of Schrödinger and Klein-Gordon equations.

Contents

1 Introduction

It is the received wisdom that Planck-Einstein-de Broglie law[1] $p^\nu = \hbar k^\nu$ belongs to the era of 'old quantum mechanics' and that in the realm of quantum mechanics the right (and more fundamental) perspective is to solve the Schrödinger equation for any case at hand. Although from an instrumentalist point of view this perspective has been quite successful, in this paper we advocate another perspective which will prove to be more fruitful with regard to the foundational questions of quantum mechanics. Our perspective is that *quantum mechanics is basically all about* $p^\nu = \hbar k^\nu$. To adopt such perspective we need to first scrutinise our understanding of its essential ingredient k^ν. By any rigorous mathematical definition[2] it is

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[1] The metric signature $(+,-,-,-)$ is used everywhere in this paper. Greek indices run over 0 to 3 (four dimensions) while Latin indices run over 1 to 3 (three dimensions of the Euclidean space).

[2] For example, an entity ϕ which satisfies the wave equation

$$\frac{\partial^2\phi}{\partial t^2} = v^2\nabla^2\phi,$$

required that a wave be defined as a field[3] on spacetime which satisfies a certain equation, without any explicit reference to its four-wavevector. On the other hand, according to our perspective $p^\nu = \hbar k^\nu$ is a fundamental law of nature and appearance of k^ν in such a law suggests that we must enforce all waves to acquire a mathematically well-defined four-wavevector. Consequently we must find a definition for the four-wavevector of a wave ψ in terms of the ψ itself, we shall call this process *harmonisation*.

2 Harmonisation

Considering the simplest case of a complex harmonic wave[4],

$$\psi(x^\mu) = e^{-ik_\mu x^\mu}$$

if we apply the gradient operator to both sides we have,

$$\partial_\mu \psi = -ik_\mu \psi$$

we realise that there are two possibilites for harmonisation:

1. *Operatorial* approach, $\hat{k}_\mu := i\partial_\mu$
2. *Logarithmic* approach[5], $k_\mu := i\frac{\partial_\mu \psi}{\psi} = i\partial_\mu(\log \psi)$

The logarithmic approach will be proved to yield a non-linear generalisation which reduces to the Schrödinger equation if $\nabla \cdot \mathbf{k} = 0$. Note that the role of the special form of the wavefunction considered here is only a mere *test function*; it can be objected that as we have started from this *particular* form of wavefunction our results would cease generality. This objection exists and deserves reflexion, and it can be as well raised about orthodox quantum mechanics itself. So our position is that 'if it works with orthodox quantum mechanics, there is no reason why it must not with a generalisation'.

3 Non-linear theory

We apply the Planck-Einstein-de Broglie law $p^\mu = \hbar k^\mu$ to the logarithmic approach to get

$$\mathbf{p} = -i\hbar\nabla(\log \psi) \quad \text{and} \quad E = i\hbar\frac{\partial}{\partial t}(\log \psi) \tag{1}$$

where v is the speed of propagation of the wave. Or, an entity ϕ which satisfies the Schrödinger equation.

[3] We only consider scalar fields in this paper. Sufficient conditions of smoothness are also assumed implicitly.

[4] Note that this is not the most general harmonic wave one can write. Moreover notice that in these definitions only forward-in-time waves are considered. It is not clear whether this preference of time direction affects the theory in a decisive manner.

[5] By log the principal values of the complex logarithm function is meant. Equivalently $\psi \neq 0$.

Substituting (1) in the law of conservation of energy[6]

$$E = \frac{\mathbf{p} \cdot \mathbf{p}}{2m} + V,$$

yields

$$i\hbar \frac{\partial}{\partial t} \log \psi = -\frac{\hbar^2}{2m} (\nabla \log \psi)^2 + V \tag{2}$$

To get the Schrödinger equation, notice that (2) is equivalent to

$$\boxed{i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{|\nabla \psi|^2}{\psi} + V\psi} \tag{3}$$

which differs from the Schrödinger equation only by the term

$$\frac{|\nabla \psi|^2}{\psi},$$

which we now show only in the *special case* where **k** is a solenoidal field, is equal to the corresponding term in Schrödinger equation.

$$\nabla \cdot \mathbf{k} = 0 \tag{4}$$

$$\Longrightarrow \nabla \cdot \left(\frac{\nabla \psi}{\psi}\right) = 0$$

$$\Longrightarrow \frac{\psi \nabla^2 \psi - |\nabla \psi|^2}{\psi^2} = 0$$

for $\psi \neq 0$ therefore

$$\frac{|\nabla \psi|^2}{\psi} = \nabla^2 \psi \tag{5}$$

In other words, Schrödinger equation is a special case of a non-linear theory. We can now manifestly see how linearity arises from non-linearity, and how an eigenvalue problem which is the representative of quantum *discreteness* is only an approximation to a non-linear *but continuous* reality. In this light the superposition 'principle' is only an approximate feature of nature and has a limited domain of applicability.

To attain generality, therefore, we must consider $\nabla \cdot \mathbf{k} \neq 0$. If we recall from orthodox quantum mechanics[2] that via the conservation of probability current, Schrödinger equation is associated with an incompressible 'fluid' of probability (*without* source/sinks), $\nabla \cdot \mathbf{k} \neq 0$ means that the new equation (3) includes also generation/destruction of probabilities, and thus might be able to give an account of the continuity of application of a complete quantum theory, i.e. it might be able to include particles, perhaps as solitons.

This understanding is in harmony with our understanding of orthodox quantum mechanics in terms of *mutually exclusive* processes of unitary evolution versus reduction[7], for Schrödinger equation *by assumption* cannot describe particles and this shortcoming in orthodox quantum mechanics is overlooked by resorting to the superficial notion of the 'collapse' of

[6] Similar to the familiar derivation of the Schrödinger equation using conservation of energy.

[7] **U**-process and **R**-process à la Penrose.

the wavefunction, in which after each measurement probabilities are *reset* to evolve again according to Schrödinger equation.
The observation in which *an eigenvalue problem arises from a more general **nonlinear** problem* is worthy of emphasis. As a simple example consider how *Helmholtz equation*

$$\nabla^2 \phi = -k^2 \phi$$

can be an approximation to the following nonlinear equation

$$\nabla^2 \log \phi = -k^2,$$

For the LHS is,

$$\nabla^2 \log \phi = \nabla \cdot (\frac{\nabla \phi}{\phi}) = \frac{1}{\phi} \nabla^2 \phi - \frac{|\nabla \phi|^2}{\phi^2}$$

which must be equal to $-k^2$, therefore

$$\nabla^2 \phi - \frac{|\nabla \phi|^2}{\phi} = -k^2 \phi$$

If we use the approximation

$$\frac{|\nabla \phi|^2}{\phi} \approx 0$$

reduces to the Helmholtz equation. Therefore it is conceivable for eigenvalues of the theory of quantum mechanics to be approximations and/or special cases to more general nonlinear equations.

4 Non-linear generalisation of Klein-Gordon equation

The logarithmic definition of momentum can be readily substituted in $E^2 = p^2 c^2 + m^2 c^4$ to yield

$$\boxed{\frac{1}{c^2} (\frac{\partial \psi}{\partial t})^2 - |\nabla \psi|^2 + (\frac{mc}{\hbar})^2 \psi^2 = 0} \tag{6}$$

which can also be written as

$$-\frac{\langle \partial_\mu \psi, \partial^\mu \psi \rangle}{\psi^2} = (\frac{mc}{\hbar})^2$$

Similar to the case for non-relativistic equation (3), this equation as well is reduced to the Klein-Gordon equation by the (now-relativistic) condition

$$\partial_\mu p^\mu = \partial_\mu k^\mu = 0 \tag{7}$$

$$\Longrightarrow \frac{\psi \Box \psi - \langle \partial_\mu \psi, \partial^\mu \psi \rangle}{\psi^2} = 0$$

$$\Longrightarrow \frac{\langle \partial_\mu \psi, \partial^\mu \psi \rangle}{\psi^2} = \frac{\Box \psi}{\psi}$$

$$\frac{\Box \psi}{\psi} = -(\frac{mc}{\hbar})^2 \Longrightarrow \left(\Box + (\frac{mc}{\hbar})^2 \right) \psi = 0$$

References

[1] Dürr, Detlef - Teufel, Stefan (2009). ‘Bohmian Mechanics’. Springer. Page 145.

[2] Sakurai. ‘Modern Quantum Mechanics’.