Magnetic symmetry of geometrical optics

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We propose that there exist a magnetic monopole as a consequence of the magnetic symmetry of the geometrical optics in the $(4+d)$-dimensions of unified space. The magnetic symmetry is a consequence of the extra internal symmetry. The magnetic monopole is formulated using the generalized gauge theory made of an additional Killing vector field, which is internal and commute with the already existing fields. The extra internal symmetry restricts the internal metric and the gauge potential. The restricted gauge potential is made of the unrestricted electric part and the restricted magnetic part. As the consequence of the restricted gauge potential, the refractive index in the eikonal equation and in the refractive index-curvature relation are decomposed. The related curvature is treated as Abelian because the geometrical optics is an Abelian physical system.

Keywords: *unified space, extra internal symmetry, magnetic symmetry, gauge potential, magnetic monopole, geometrical optics, refractive index, eikonal equation, Fermat's principle, curvature, Abelian.*

Dirac proposed, due to symmetrical reasoning in the Maxwell's theory of electromagnetism, there exist magnetic symmetry appears as magnetic monopole which has magnetic charge[1,2]. Inspired by Dirac idea of the magnetic monopole, *the magnetic symmetry* was formulated as the $SU(2)$ *local* gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space[3]. The $(4+d)$-dimensions of unified space is the $(3+1)$-dimensions of external space-time, i.e. the $(3+1)$-dimensions of curved space-time, plus the d-dimensions of internal (isometry group, G) space[4].

The formulation of the $SU(2)$ local gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space[3], roughly speaking, looks like the $SU(N)$ Yang-Mills theory[5] where the choice of the simple Lie group $G=U(1)$ reduces the Yang-Mills theory to the Maxwell's theory[6]. *The role of magnetic symmetry of the $SU(2)$ local gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space, is to restrict the gauge potential*[3].

To the best of our knowledge, the geometrical optics (the eikonal equation) is formulated without including the magnetic symmetry[7–13]. The eikonal equation can be derived from the Maxwell equations[8]. Because of the Maxwell's theory is $U(1)$ *local*[14] gauge theory, Abelian theory, and the eikonal equation can be derived from the Maxwell equations, *we argue that the geometrical optics is also the $U(1)$ local gauge theory, Abelian theory, there exist the magnetic symmetry and the magnetic monopole in the geometrical optics.*

The gauge theory can be viewed as the general theory of relativity in a higher-dimension of unified space which consists of the $(3+1)$-dimensions of external space-time plus d-dimensions of internal (isometry group, G) space. If the metric tensor, $g_{\mu\nu}(\mu,\nu=1,2,..,4+d)$, in this $(4+d)$-dimensions of unified space has an d-dimensions of isometry group whose Killing vector fields span the internal space, *the general theory of relativity becomes the gauge theory of the isometry group in curved space-time*[3].

Let us choose the d Killing vector fields, ξ_i $(i=1,2,..,d)$ to satisfy the canonical commutation relations of the isometry group, G[3]

$$[\xi_i,\xi_j]=f_{ij}^{\ \ k}\ \xi_k \tag{1}$$

By definition, these Killing vector fields must also satisfy[3]

$$\pounds_{\xi_i}\ g_{\mu\nu}=0,\quad (i=1,2,..,d) \tag{2}$$

where $\pounds_{\xi_i}$ is the Lie derivative along the direction of ξ_i. The existence of the Killing vector fields means the existence of *an isometric mapping*, i.e. a mapping of the space-time onto itself. It means that the space-time have *an intrinsic symmetry*[15] (2). We will call, from now on, an intrinsic symmetry as *an internal symmetry*.

If we assume that the d Killing vector fields, ξ_i, are *orthonormal* to each other with respect to the metric tensor, $g_{\mu\nu}$, so that *the internal metric*, ϕ_{ik} $(i,k=1,2,..,d)$ of the d-dimensions of internal space[3]

$$\phi_{ik}=\xi_i^\mu\ \xi_k^\nu\ g_{\mu\nu} \tag{3}$$

becomes *the Cartan-Killing form*. In general, if the d Killing vector fields are *not kept to be orthonormal*, so that the internal metric (3) is left *arbitrary*, then we have, in addition to the gauge fields, the internal gravitons, ϕ_{ik}, as non-trivial dynamical fields in the theory. This theory is called *the generalized gauge theory*[3].

The magnetic monopole is formulated using the generalized gauge theory which has *an extra internal symmetry made of some additional Killing vector fields*, which are *internal* and which *commute* with the already existing fields, ξ_i. Let us assume that there exists only one such *vector field* denoted by m. By assumption[3] that m commutes with the already existing fields, ξ_i, we obtain

$$[m,\xi_i]=0,\quad (i=1,2,..,d) \tag{4}$$

and in analogy with the definition (2)

$$\pounds_m\ g_{\mu\nu}=0 \tag{5}$$

where $\mathcal{L}_m$ is the Lie derivative along the direction of m. The existence of the additional Killing vector field, m, means the existence of *an extra internal symmetry*[15] (5).

Since m is assumed to be internal[3], we have

$$m = m^i \ \xi_i \tag{6}$$

The commutation relation (4) tells us that *the multiplet*, $\hat{m}$, made of *the components*, m^i

$$\hat{m} = \begin{pmatrix} m^1 \\ m^2 \\ . \\ . \\ m^d \end{pmatrix} \tag{7}$$

The extra internal symmetry (5) restrict the internal metric (3) as well as, we will see, the gauge potential[3]. *The extra internal symmetry (5) has consequence that there exist the magnetic symmetry*[3], which can be written mathematically, as below

$$D_\mu \hat{m} = \partial_\mu \hat{m} + g\vec{B}_\mu \times \hat{m} = 0 \tag{8}$$

where $\vec{B}_\mu$ is *the gauge potential* of the isometry group, G. What about the value of the multiplet, $\hat{m}$? Let us assume for simplicity that the internal metric, ϕ_{ik}, is of the Cartan-Killing form. It means that, as we mention above, we assume that the d Killing vector fields, ξ_i, are orthonormal to each other with respect to the metric tensor, $g_{\mu\nu}$. With this simplification, the magnetic symmetry (8) implies, among others, that the multiplet, $\hat{m}$, must have *a constant length*[3]

$$\hat{m}^2 = \text{constant} \tag{9}$$

which we can choose to be *the unit* without loss of generality.

Previously, we mention that the role of magnetic symmetry is to restrict the gauge potential. To see how the magnetic symmetry (8) restricts the gauge potential, $\vec{B}_\mu$, let us consider, for simplicity, the case when *the isometry group is* $SU(2)$. In case of $SU(2)$, the magnetic symmetry (8) can be solved exactly for $\vec{B}_\mu$ as follow[3]

$$\vec{B}_\mu^{SU(2)} = A_\mu^{SU(2)} \ \hat{m}^{SU(2)} - \frac{1}{g}\hat{m}^{SU(2)} \times \partial_\mu \hat{m}^{SU(2)} \tag{10}$$

where $A_\mu^{SU(2)}$ is the (Abelian) component of $\vec{B}_\mu^{SU(2)}$ which is *not restricted* by the magnetic symmetry (8). We see that the gauge potential (10) is made of two parts, i.e. the unrestricted part, A_μ, and the other part which is completely determined by the magnetic symmetry. We will call the unrestricted part, A_μ, *electric* and the other part which is *completely restricted* by the magnetic symmetry, *magnetic*[3]. The restricted gauge potential, $\vec{B}_\mu^{SU(2)}$, with $A_\mu^{SU(2)} = 0$ and $\hat{m}^{SU(2)} = \hat{r}^{SU(2)}$ describes precisely *the Wu-Yang magnetic monopole*[16]. Here, $\hat{r}^{SU(2)}$ as $\hat{m}^{SU(2)}$ must have a constant length which we can choose to be unit without loss of generality.

The corresponding field strength, $\vec{G}_{\mu\nu}^{SU(2)}$, corresponding to the $SU(2)$ gauge potential (10) is[3]

$$\begin{aligned} &\vec{G}_{\mu\nu}^{SU(2)} \\ &= \partial_\mu \vec{B}_\nu^{SU(2)} - \partial_\nu \vec{B}_\mu^{SU(2)} + g\vec{B}_\mu^{SU(2)} \times \vec{B}_\nu^{SU(2)} \end{aligned} \tag{11}$$

We see that the third term on the right hand side of eq.(11) is *the non-Abelian (non-commutative) term what produces the non-linear term* in the equation[17]. This term is the main difference compare to the field strength of the Maxwell's theory which is Abelian.

In the $(3+1)$-dimensions of space-time, for the geometrical optics approximation (short wavelength, $\lambda \to 0$[10]), *the four-vector potential*, $\vec{B}_\mu$, and the related *field strength*, $\vec{G}_{\mu\nu}$, can be represented respectively as[12,13,18]

$$\vec{B}_\mu = a_\mu \ e^{i\psi} \tag{12}$$

and

$$\vec{G}_{\mu\nu} = \nabla_\mu \vec{B}_\nu - \nabla_\nu \vec{B}_\mu \tag{13}$$

where $\psi(x, y, z, t)$ is *phase* (*eikonal*) and amplitude, a_μ, is a slowly varying function of coordinates and time[9]. ∇_μ denotes a space-time covariant derivative[11]. We see from eq.(12), the amplitude, a_μ, has the same dimension as the displacement from equilibrium[19], the oscillating variable[20], the four-vector potential $\vec{B}_\mu$.

In case of *a steady monochromatic wave*, the frequency[21] is constant and the time dependence of the eikonal, ψ, is given by a term $-\omega t$ where ω is a notation for (angular) frequency[9]. Let us introduce ψ_1, a function, which is also called *eikonal*[9]. The relation between ψ_1 and ψ can be expressed as[9]

$$\psi_1 = \frac{c}{\omega}\psi + ct \tag{14}$$

where the eikonal, ψ_1, is a function of coordinates only[9]. In the 3-dimensions of space it is denoted by $\psi_1(x, y, z)$.

The equation of ray propagation in a transparent medium can be written in relation with the refractive index, n, as below[8,9]

$$|\vec{\nabla}\psi_1| = |\vec{n}| = n \tag{15}$$

where n is *a scalar*, $\vec{\nabla}$ is a notation for gradient. Because ψ_1 is a function of coordinates only, then the refractive index is also a function of coordinates only. More precisely, the refractive index is *a smooth continuous function* of the position[22]. In the 3-dimensions of space, the refractive index is denoted by $n(x, y, z)$.

The equation (15) is called *the eikonal equation*[8,9], i.e. *a type of the first order linear partial differential equation*. The analysis of partial differential equation for steady state is very important e.g. for formulating the Atiyah-Singer index theorem, an effort for finding the existence and uniqueness of solutions to linear partial differential equations of elliptic type on closed manifold[23]. Because, as we will see, the refractive index is related with the curvature and we can apply the magnetic symmetry to the formulation of the refractive index, so probably, we could apply the magnetic symmetry of geometrical optics to the curvature part of the Atiyah-Singer index theorem. Progress work was reported[24].

Let us formulate the eikonal, ψ_1, in the $(4+d)$-dimensions of unified space. Because the eikonal, ψ_1, is a function of coordinates only (it is time-independent), so $\psi_1(x,y,z)$ becomes the $(3+d)$-dimensions eikonal which lives in the $(4+d)$-dimensions of unified space. The gradient operator, $\vec{\nabla}$, in eq.(15) transforms to the covariant four-gradient, ∂_ν. So, eq. (15) becomes

$$|\partial_\nu \psi_1| = |\vec{n}_\nu| = n \tag{16}$$

where ν runs from 1 to $4+d$ by considering that the time components of ψ_1 and n are zero.

We see from eq.(16), the refractive index is *a scalar, a real number. The zeroth rank tensor (scalar)* of refractive index describes *isotropic linear optics*[25]. But, the refractive index can be not simply a scalar[26]. The refractive index can also be expressed as *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[26]. The second rank tensor of refractive index describes *anisotropic linear optics*[25].

Our works[27,28] show that *the second rank tensor of refractive index is a consequence of the fourth rank totally covariant tensor of Riemann-Christoffel curvature*, $R_{\mu\nu\rho\sigma}$. Naturally, it means that the fourth rank totally covariant tensor of Abelian Riemann-Christoffel curvature describes the anisotropic linear optics. *In this article, we will work with the refractive index as a scalar related with the second rank tensor of Abelian Ricci curvature where we apply the magnetic symmetry to the refractive index.* This work could be extended to the second rank tensor of refractive index related with the fourth rank totally covariant tensor of Abelian Riemann-Christoffel curvature.

In analogy with $SU(N)$ Yang-Mills theory, the choice of the simple Lie group, $G = U(1)$ reduces the $SU(2)$ local gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space to the Maxwell's theory, $U(1)$ local gauge theory, Abelian theory in the $(4+d)$-dimensions of unified space. We obtain from eqs.(10), (11) that *the restricted* $U(1)$ *gauge potential* and its related field strength can be represented respectively as

$$\vec{B}_\mu^{U(1)} = A_\mu^{U(1)}\ \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \tag{17}$$

and

$$\vec{G}_{\mu\nu}^{\ U(1)} = \partial_\mu \vec{B}_\nu^{\ U(1)} - \partial_\nu \vec{B}_\mu^{\ U(1)} \tag{18}$$

where $A_\mu^{U(1)}$ is the electric potential part of the $U(1)$ gauge potential which is not restricted by the magnetic symmetry (8) and $\hat{m}^{U(1)}$ is the multiplet or magnetic potential part of the $U(1)$ gauge potential which is completely determined by the magnetic symmetry (8). Both, $A_\mu^{U(1)}$ and $\hat{m}^{U(1)}$, are the d-dimensional physical objects which live in the $(4+d)$-dimensions of unified space. In analogy with the Wu-Yang monopole, if we treat the restricted gauge potential, $\vec{B}_\mu^{U(1)}$ (17), with $A_\mu^{U(1)} = 0$ and $\hat{m}^{U(1)} = \hat{r}^{U(1)}$ then the restricted gauge potential, $\vec{B}_\mu^{U(1)}$, describes *the magnetic monopole of the* $U(1)$ *Maxwell's theory*.

As we mention previously that the eikonal equation of the geometrical optics can be derived from the Maxwell equations, it has a consequence that the field strength of the geometrical optics, $\vec{G}_{\mu\nu}$ (13), and the field strength of the Maxwell's theory, $\vec{G}_{\mu\nu}^{\ U(1)}$ (18), in principle are *same* i.e. *both are fields*[29]. In turn, it has a consequence that we can replace $\vec{G}_{\mu\nu}$ with $\vec{G}_{\mu\nu}^{\ U(1)}$. In other words, we can replace the four-vector potential, $\vec{B}_\mu$ (12), with the restricted $U(1)$ gauge potential, $\vec{B}_\mu^{\ U(1)}$ (17). If we replace $\vec{B}_\mu$ with $\vec{B}_\mu^{U(1)}$ then eq.(12) becomes

$$\vec{B}_\mu^{\ U(1)} = a_\mu\ e^{i\psi} \tag{19}$$

Eq.(19) expresses $U(1)$ *gauge potential of the geometrical optics* in the $(4+d)$-dimensions of unified space. This $U(1)$ gauge potential of the geometrical optics (19) is the same as the restricted $U(1)$ gauge potential of the Maxwell's theory in the $(4+d)$-dimensions of unified space (17). The related field strength of the geometrical optics has the same form as the field strength of the Maxwell's theory (18) where the gauge potential is given by (19).

Eq.(19) can be written as

$$e^{i\psi} = \vec{B}_\mu^{U(1)}\ a_\mu^{-1} \tag{20}$$

Using Euler's formula, eq.(20) can be written as

$$\cos\psi + i\sin\psi = \vec{B}_\mu^{U(1)}\ a_\mu^{-1} \tag{21}$$

Eq.(21) shows us that $\vec{B}_\mu^{U(1)} a_\mu^{-1}$ is *a complex function*. To simplify the problem, we take the real part[30] of (21) only, we obtain

$$\cos\psi = \vec{B}_\mu^{U(1)}\ a_\mu^{-1} \tag{22}$$

where ψ in eq.(22), i.e. phase (eikonal or "gauge") is *an angle*. This angle has value

$$\psi = \arccos\left(\vec{B}_\mu^{U(1)}\ a_\mu^{-1}\right) \tag{23}$$

Substituting eqs.(23), (17) into eq.(14), we obtain

$$\frac{c}{\omega}\arccos\left[\left(A_\mu^{U(1)}\ \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)}\right) a_\mu^{-1}\right] + ct = \psi_1 \tag{24}$$

If we substitute eq.(24) into eq.(16), then the eikonal equation (16) becomes

$$\left|\partial_\nu \left\{\frac{c}{\omega}\arccos\left[\left(A_\mu^{U(1)} \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)}\right) a_\mu^{-1}\right] + ct\right\}\right| = n \tag{25}$$

where n is a dimensionless quantity, a scalar, a real number, i.e. a function of $(3+d)$-coordinates which lives in the $(4+d)$-dimensions of unified space. *Eq.(25) shows us that the refractive index is decomposed into the unrestricted electric part and the restricted magnetic part by the magnetic symmetry (8). This magnetic symmetry (8) is a consequence of the extra internal symmetry of the $(4+d)$-dimensions of unified space (5). In analogy with the Maxwell's theory, if we treat the restricted gauge potential, $\vec{B}_\mu^{U(1)}$ (17), with $A_\mu^{U(1)}=0$ and $\hat{m}^{U(1)}=\hat{r}^{U(1)}$ then the restricted gauge potential, $\vec{B}_\mu^{U(1)}$, describes the magnetic monopole of the $U(1)$ geometrical optics* as below

$$\vec{B}_\mu^{U(1)} = -\,\frac{1}{g}\;\hat{r}^{U(1)} \times \partial_\mu \hat{r}^{U(1)} \tag{26}$$

The equation of ray propagation in *a steady state* can also be derived from *the Fermat's principle*, $\delta\psi_1 = 0$[9,31]. We obtain the refractive index-curvature relation as below[9,32]

$$\frac{1}{R} = \hat{N} \cdot \frac{\vec{\nabla} n}{n} \tag{27}$$

where n here is *a scalar*, the same as n in the eikonal equation (15), (16), $\hat{N}$ is the unit vector along the principal normal, R is the radius of curvature. $1/R$ is the curvature of 1-dimension of space, $\kappa(R)=1/R$. We see that eq.(27) is *a type of the first order non-linear partial differential equation.* The nonlinearity is due to the form of $n^{-1}\,\vec{\nabla}n$. *Physically, eq.(27) says that the rays are bent in the direction of increasing the refractive index*[9,33].

The dimension of curvature of eq.(27) can be extended to any arbitrary number of dimensions[34]. In the $(4+d)$-dimensions of unified space, eq.(27) can be written as

$$\frac{R_{\mu\nu}}{g} = N_\nu\;\partial_\mu \ln n \tag{28}$$

where $R_{\mu\nu}$ is the second rank tensor of Ricci curvature[15,35], a function of the metric tensor and $g = |(\det g_{\mu\nu})|$, is a scalar, a real number. Why do we formulate the curvature in eq.(28) as the second rank tensor of Ricci curvature? It is because of the related refractive index in eq.(28) is a scalar.

The second rank tensor of Ricci curvature, $R_{\mu\nu}$, can be obtained by contraction from the totally covariant fourth rank tensor of Riemann-Christoffel curvature, $R_{\mu\nu\rho\sigma}$,[15] which is *non-linear*, so *the second rank tensor of Ricci curvature is also non-linear.* The fourth rank tensor of Riemann-Christoffel curvature, $R^\gamma{}_{\nu\rho\sigma}$, is *a commutation relation of covariant derivative operator*, $\nabla_\sigma\nabla_\rho - \nabla_\rho\nabla_\sigma$[15] and $R_{\mu\nu\rho\sigma}$ can be obtained from $R^\gamma{}_{\nu\rho\sigma}$, through relation $R_{\mu\nu\rho\sigma} = g_{\mu\gamma}\;R^\gamma{}_{\nu\rho\sigma}$. Here, the metric tensor, $g_{\mu\gamma}$, is a type of function which takes as input *a pair of tangent vectors*[36]. We see that *nonlinearity is linked with noncommutativity. Noncommutativity does produce nonlinearity*[17]. *Non-commutative is other name of non-Abelian.* Noncommutativity or non-Abelian analysis will be useful when we consider the relation between the curvature and the field strength (the gauge potential). In turn, the relation of curvature and refractive index.

The refractive index, n, in eqs. (25), (28), is a scalar. So, by substituting n in eq.(25) into eq.(28), we obtain

$$\frac{R_{\mu\nu}}{g} = N_\nu\;\partial_\mu \ln\left|\partial_\nu\left[\frac{c}{\omega}\arccos\left(\frac{\vec{B}_\mu^{U(1)}}{a_\mu}\right)+ct\right]\right| \tag{29}$$

where the restricted $U(1)$ gauge potential, $\vec{B}_\mu^{U(1)}$, is given by eq.(17). We see from eq.(29), *the curvature, which is shown by the second rank tensor of Ricci curvature, is related naturally with the restricted gauge potential,* $\vec{B}_\mu^{U(1)}$. It is in a harmony that *the curvature and the field strength are identical*[37].

The related field strength of the restricted $U(1)$ gauge potential of the geometrical optics, as we mention previously, has the same form as the field strength of the Maxwell's theory which is Abelian, so *the field strength of the geometrical optics is also Abelian.* It means that *the geometrical optics is an Abelian physical system.* It has a consequence that the related curvature in the refractive index-curvature relation (29) should be Abelian. So, *the second rank tensor of Ricci curvature, $R_{\mu\nu}$ (29), is Abelian.* The Abelian Ricci curvature means that the non-linear terms, i.e. the third and the fourth terms, on the right hand side of the Ricci curvature tensor equation below[15] vanish

$$R_{\mu\nu} = \frac{\partial\Gamma^\rho{}_{\mu\nu}}{\partial x^\rho} - \frac{\partial\Gamma^\rho{}_{\mu\rho}}{\partial x^\nu} + \Gamma^\sigma{}_{\mu\nu}\;\Gamma^\rho{}_{\rho\sigma} - \Gamma^\sigma{}_{\mu\rho}\;\Gamma^\rho{}_{\nu\sigma} \tag{30}$$

So, we obtain *the second rank tensor of Abelian Ricci curvature*

$$R_{\mu\nu} = \frac{\partial\Gamma^\rho{}_{\mu\nu}}{\partial x^\rho} - \frac{\partial\Gamma^\rho{}_{\mu\rho}}{\partial x^\nu} \tag{31}$$

where $\Gamma^\rho{}_{\mu\nu}$ is the Christoffel symbol of the second kind[15].

Substituting eq.(31) into eq.(29), the refractive index-curvature relation (29) becomes

$$\begin{aligned}&\frac{1}{g}\left(\frac{\partial\Gamma^\rho{}_{\mu\nu}}{\partial x^\rho} - \frac{\partial\Gamma^\rho{}_{\mu\rho}}{\partial x^\nu}\right)\\ &= N_\nu\;\partial_\mu \ln\left|\partial_\nu\left[\frac{c}{\omega}\arccos\left(\frac{\vec{B}_\mu^{U(1)}}{a_\mu}\right)+ct\right]\right|\end{aligned} \tag{32}$$

where the left hand side of (32) is the Abelian Ricci curvature and $\vec{B}_\mu^{U(1)}$ is the $U(1)$ restricted gauge potential (17). We interpret that *the Abelian Ricci curvature in the left hand side of (32) is decomposed, as a consequence that the refractive index (the gauge potential) in the right hand side of (32) is decomposed.*

Symmetry is a fundamental principle in universe, including physics. An isometric mapping in geometry has a consequence that the space-time have an internal symmetry (2). If we extend the $(3+1)$-dimensions of space-time including the d-dimensions of internal (isometry group) space means that we generalize the general theory of relativity including the gauge theory, then we obtain the $(4+d)$-dimensions of unified space.

The magnetic monopole is formulated using the generalized gauge theory which has an extra internal symmetry (5) made of the extra internal vector field which is internal and commute with the already existing vector fields. The extra internal symmetry restricts the internal metric (as well as the gauge potential) so that the extra internal vector field has value as a constant length or we can choose to be the unit without loss of generality. The existence of extra internal symmetry has consequence that there exist the magnetic symmetry (8). The magnetic symmetry can be solved exactly, in case of $SU(2)$ isometry group, by the $SU(2)$ restricted gauge potential (10). The restricted gauge potential is made of two parts, i.e. the unrestricted electric part and the restricted magnetic part which is completely determined by the magnetic symmetry. If we set the unrestricted electric part as zero and we choose the restricted magnetic part $\hat{m}^{SU(2)} = \hat{r}^{SU(2)}$ then we obtain the Wu-Yang magnetic monopole.

In analogy with $SU(N)$ Yang-Mills theory, the choice of the simple Lie group, $G = U(1)$ reduces the $SU(2)$ local gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space to the Maxwell's theory, $U(1)$ local gauge theory, Abelian theory in the $(4+d)$-dimensions of unified space. In case of the $U(1)$ restricted gauge potential, if we set the unrestricted electric part as zero and we choose the restricted magnetic part $\hat{m}^{U(1)} = \hat{r}^{U(1)}$ then we obtain the magnetic monopole of $U(1)$ Maxwell's theory.

The eikonal equation of the geometrical optics can be derived from the Maxwell equations. It has the consequence that the field strength of the geometrical optics and the field strength of the Maxwell's theory, in principle are same, i.e. both are fields. So, we can replace the field strength of the geometrical optics with the field strength of the Maxwell's theory. In turn, the four-vector potential of the geometrical optics can be replaced with the restricted gauge potential. At present, the geometrical optics becomes an $U(1)$ local gauge theory, Abelian theory in the $(4+d)$-dimensions of unified space, the same as the Maxwell's theory. In case of the restricted gauge theory of the geometrical optics, if we set the unrestricted electric part as zero and we choose the restricted magnetic part $\hat{m}^{U(1)} = \hat{r}^{U(1)}$ then we obtain the magnetic monopole of $U(1)$ geometrical optics (26).

The refractive index in the eikonal equation is a dimensionless quantity, i.e. a scalar, a single real variable. The replacement of the four-vector potential of the geometrical optics with the restricted gauge potential has consequence that the refractive index is no longer single real variable but it is decomposed into the unrestricted electric part and the restricted magnetic part. What does it imply to our understanding of the refractive index?

We see in this article so far, the ray propagation in the geometrical optics is descibed by the eikonal equation with obey the Fermat's principle and the refractive index-curvature relation which can be derived from (as a consequence of) the Fermat's principle. Both equations are formulated in the steady state (time-independent) physical system. Could Fermat's principle apply for a non-steady state (time-dependent) physical system?

The curvature in the refractive index-curvature relation is formulated as the second rank tensor of Ricci curvature. Why? It is because of the related refractive index is a scalar, the zeroth rank tensor. Can the refractive index in the refractive index-curvature relation be e.g. the second rank tensor, instead of a scalar? What is the consequence to the curvature if the refractive index is e.g. a second rank tensor?

In case of the refractive index-curvature relation, because the geometrical optics is an Abelian physical system then the related curvature is also Abelian. The Abelian curvature means that the non-linear terms of the Ricci curvature equation vanish. The non-linear terms of the Ricci curvature equation vanish means that the direction of related tangent vectors is indistinguishable. So, the commutation relation of covariant derivative operator gives result zero, i.e. it is commute. We also interpret that the Abelian Ricci curvature is decomposed as a consequence that the refractive index (the gauge potential) is decomposed (32). Is the Abelian Ricci curvature itself also decomposed?

By considering that the gauge transformation is the transformation to the gauge potential and the curvature (actually, the connection on a principal fiber bundle is identical with the gauge potential[37]) is related with the gauge potential, so could we do "the gauge transformation" to the curvature (the connection on a principal fiber bundle)? What is the consequence of the restricted gauge potential (the gauge potential decomposition) to the connection on a principal fiber bundle (the curvature)? Is the connection on a principal fiber bundle also decomposed?

We see from eqs.(20), (21), the restricted gauge potential is formulated in relation with a complex function, $e^{i\psi}$. Without simplifying that we only take the real part, what is the consequence of this complex function to the restricted potential, also to the refractive index in the eikonal equation and to the curvature in the refractive index-curvature relation? Could the curvature have a complex value?

We see that the magnetic monopole of the geometrical optics can be obtained from the magnetic symmetry which is a consequence of the extra internal symmetry. In analogy with the magnetic monopole of the geometrical optics, could we obtain all other members of topological solitons using the same way?

In the refractive index-curvature relation, we see that the Abelian Ricci curvature is related naturally with the restricted gauge potential. It is in a harmony that the curvature and the field strength are identical. If charge is the fundamental quantity in the gauge field theory and the metric tensor is the fundamental quantity in the curvature geometry, is there relation between the charge and the metric tensor?

If the metric tensor can be simplified using the frame field or tetrad or vierbein, then what is the consequence of this simplification to the internal symmetry, the magnetic symmetry? In turn what is the consequence of this simplification to the curvature in the refractive index-curvature relation? Could the frame field or tetrad or vierbein be generalized?

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[3] Y.M. Cho, *Restricted gauge theory*, Physical Review D, Volume 21, Number 4, 15 February 1980.

[4] The internal space is an abstract space where the magnetic symmetry "lives". We can not "see" this internal space due to the symmetry we are assumed (Y.M. Cho, Pong Soo Jang, *Unified Geometry of Internal Space with Spacetime*, June 1975).

[5] Cho restricted gauge theory is a self consistent subset of a non-Abelian $SU(2)$ gauge theory which tries to describe the *infrared regime* of the Yang-Mills gauge theory (Sedigheh Deldar, Ahmad Mohamadnejad, *Quark Confinement in Restricted $SU(2)$ Gauge Theory*, https://arxiv.org/pdf/1208.2165.pdf, 2012).

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[17] Michael Atiyah, *Mathematics in the 20th Century*, Bull. London Math. Soc. **34** (2002) 115. DOI: 10.1112/S0024609301008566.

[18] We treat the four-vector potential, $\vec{B}_\mu$, is the same as wave field, ϕ, (any component of $\vec{E}$ or $\vec{H}$) given by a formula of the type $\phi = ae^{i\psi}$, where the amplitude a is a slowly varying function of coordinates and time and phase (eikonal), ψ, is a large quantity which is "almost linear" in coordinates and the time (L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984.). Both $\vec{B}_\mu$ and ϕ are solutions of the wave equation.

[19] H.J. Pain, *The Physics of Vibrations and Waves*, John Wiley and Sons Limited, 3rd Edition, 1983.

[20] Wikipedia, *Amplitude*.

[21] The time derivative of phase, ψ, gives the angular frequency of the wave, $\partial\psi/\partial t = -\omega$ and the space derivatives of ψ gives the wave vector, $\vec{\nabla}\psi = \vec{k}$, which shows the direction of the ray propagation through any point in space (L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984).

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[30] The refractive index is often described as a real value. However, in a lossy material, the attenuation of the electric field is described through an imaginary part of the refractive index (Karsten Rottwitt, Peter Tidemand-Lichtenberg, *Nonlinear Optics: Principles and Applications*, CRC Press, 2015).

[31] If we compare the Fermat's principle, $\delta\psi_1 = 0$, and the least action principle, $\delta S = 0$, we see that the eikonal, ψ_1, has the same role as the action, S.

[32] Soma Mitra, Somenath Chakrabarty, *Fermat's Principle in Curved Spacetime, No Emission from Schwarzschild Black Holes as Total Internal Reflection and Black Hole Unruh Effect*, https://arxiv.org/pdf/1512.03885.pdf, 2015.

[33] Light changes its direction of propagation when it encounters *an inhomogeneity* in the medium. The curvature of the path is used to quantify this change of direction. This curvature is defined as the ratio of the change in the direction of propagation to the length measured along the curved path (K. Iizuka, *Engineering Optics*, 2008, p.104).

[34] The Riemann-Christoffel curvature tensor and the subsequent formulas, such as Ricci curvature tensor, Ricci curvature scalar, are valid in spaces of arbitrary number of dimensions (Moshe Carmeli, *Classical Fields: General Relativity and Gauge Theory*, John Wiley and Sons, Inc., 1982.)

[35] Richard L. Faber, *Differential Geometry and Relativity Theory: An Introduction*, Marcel Dekker, Inc., 1983.

[36] Wikipedia, *Metric tensor*.

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