Magnetic symmetry of geometrical optics

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We propose that there exist magnetic monopole as a consequence of magnetic symmetry of the geometrical optics in the $(4+d)$-dimensions of unified space.

Keywords: *magnetic symmetry, magnetic monopole, geometrical optics, gauge potential, refractive index, curvature, unified space.*

I. INTRODUCTION

Dirac proposed, due to symmetrical reasoning in the Maxwell's theory of electromagnetism, there exist magnetic symmetry appears as magnetic monopole which has magnetic charge[1,2]. Inspired by Dirac idea of magnetic monopole, the magnetic symmetry was formulated as the $SU(2)$ *local* gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space[3]. The $(4+d)$-dimensions of unified space is the $(3+1)$-dimensions of external space-time, i.e. the $(3+1)$-dimensions of curved space-time, plus the d-dimensions of internal (isometry group, G) space[4].

The formulation of the $SU(2)$ local gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space[3], roughly speaking, looks like the $SU(N)$ Yang-Mills theory[5] where the choice of the simple Lie group $G = U(1)$ reduces the Yang-Mills theory to the Maxwell's theory[6]. *The role of magnetic symmetry of the $SU(2)$ local gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space is to restrict the gauge potential*[3].

To the best of our knowledge, the geometrical optics (the eikonal equation) is formulated without including the magnetic symmetry[7–13]. The eikonal equation can be derived from the Maxwell equations[8]. Because of the Maxwell's theory is $U(1)$ *local*[14] gauge theory, Abelian theory, and the eikonal equation can be derived from the Maxwell equations, *we argue that the geometrical optics is also $U(1)$ local gauge theory, Abelian theory and there exist the magnetic symmetry in the geometrical optics.*

II. $SU(2)$ GAUGE POTENTIAL AND FIELD STRENGTH IN UNIFIED SPACE

The magnetic symmetry of the non-Abelian $SU(2)$ local gauge theory in the $(4+d)$-dimensions of unified space is formulated in relation with *the metric symmetry* in the $(4+d)$-dimensions of unified space[3]. The Killing vector fields must satisfy[3] the metric symmetry equation below

$$\pounds_m \; g_{\mu\nu} = 0 \tag{1}$$

where m is *vector field*, $\pounds_m$ is the Lie derivative along the direction of m, $g_{\mu\nu}(\mu,\nu = 1,2,..,4+d)$ is metric in the $(4+d)$-dimensions of unified space[3].

The eq.(1) has consequence that there exist *the magnetic symmetry*[3] which can be written as below

$$D_\mu \hat{m}^{SU(2)} = \partial_\mu \hat{m}^{SU(2)} + g\vec{B}_\mu^{SU(2)} \times \hat{m}^{SU(2)} = 0 \tag{2}$$

where $\hat{m}^{SU(2)}$ is *the multiplet* in the d-dimensions of internal space and $\vec{B}_\mu^{SU(2)}$ is *the $SU(2)$ gauge potential* in the d-dimensions of internal space[3]

$$\vec{B}_\mu^{SU(2)} = A_\mu^{SU(2)} \; \hat{m}^{SU(2)} - \frac{1}{g}\hat{m}^{SU(2)} \times \partial_\mu \hat{m}^{SU(2)} \tag{3}$$

where $A_\mu^{SU(2)}$ is *the unrestricted electric potential (a scalar)*, i.e. the part which is *not restricted (not determined)* by the magnetic symmetry and $\hat{m}^{SU(2)}$ is *the restricted magnetic potential (a vector)*, i.e. the part which is *completely determined* by the magnetic symmetry[3].

The corresponding field strength, $\vec{G}_{\mu\nu}^{SU(2)}$, of the $SU(2)$ gauge potential (3) is[3]

$$\begin{aligned} &\vec{G}_{\mu\nu}^{SU(2)} \\ &= \; \partial_\mu \vec{B}_\nu^{SU(2)} - \partial_\nu \vec{B}_\mu^{SU(2)} + g\vec{B}_\mu^{SU(2)} \times \vec{B}_\nu^{SU(2)} \end{aligned} \tag{4}$$

We see that the third term on the right hand side of eq.(4) is the non-Abelian (non-commutative) term, non-linear term, the main difference compare to the field strength of the Maxwell's theory which is Abelian.

III. POTENTIAL AND FIELD STRENGTH OF GEOMETRICAL OPTICS

In the $(3+1)$-dimensions of space-time, for the geometrical optics approximation (short wavelength, $\lambda \to 0$[10]), *the four-vector potential*, $\vec{B}_\mu$, and the related *field strength tensor*, $\vec{G}_{\mu\nu}$, can be represented respectively as[12,13,15]

$$\vec{B}_\mu = a_\mu \; e^{i\psi} \tag{5}$$

$$\vec{G}_{\mu\nu} = \nabla_\mu \vec{B}_\nu - \nabla_\nu \vec{B}_\mu \tag{6}$$

where $\psi(x,y,z,t)$ is *phase* (*eikonal*) and amplitude, a_μ, is a slowly varying function of coordinates and time[9]. ∇_μ denotes a space-time covariant derivative[11]. We see from eq.(5), the amplitude, a_μ, has the same dimension as the displacement from equilibrium[16], the oscillating variable[17], the four-vector potential $\vec{B}_\mu$.

IV. EIKONAL EQUATION

In case of *a steady monochromatic wave*, the frequency[18] is constant and the time dependence of the eikonal, ψ, is given by a term $-\omega t$ where ω is a notation for (angular) frequency[9]. Let us introduce ψ_1, a function, which is also called *eikonal*[9]. The relation between ψ_1 and ψ can be expressed as[9]

$$\psi_1 = \frac{c}{\omega}\psi + ct \tag{7}$$

where the eikonal, ψ_1, is a function of coordinates only[9]. In the 3-dimensions of space it is denoted by $\psi_1(x, y, z)$.

The equation of ray propagation in a transparent medium with the refractive index as a scalar, n, is[8,9]

$$|\vec{\nabla}\psi_1| = |\vec{n}| = n \tag{8}$$

where $\vec{\nabla}$ is a notation for gradient. Because ψ_1 is a function of coordinates only, then the refractive index is also a function of coordinates only. In the 3-dimensions of space, the refractive index is denoted by $n(x, y, z)$.

The equation (8) is called *the eikonal equation*[8,9], i.e. a type of the first order linear partial differential equation. The analysis of partial differential equation for steady state is very important for e.g. formulation of the Atiyah-Singer index theorem, an effort for finding the existence and uniqueness of solutions to linear partial differential equations of elliptic type on closed manifold[19]. Probably, we could apply the magnetic symmetry of geometrical optics to the Atiyah-Singer index theorem. Progress work was reported[20].

V. GEOMETRICAL OPTICS IN UNIFIED SPACE

Because the eikonal, ψ_1, is a function of coordinates only (it is time-independent), so $\psi_1(x, y, z)$ becomes the $(3 + d)$-dimensions eikonal which lives in the $(4 + d)$-dimensions of unified space. The gradient operator, $\vec{\nabla}$, in eq.(8) transforms to the covariant four-gradient, ∂_ν. So, eq. (8) becomes

$$|\partial_\nu\psi_1| = |\vec{n}_\nu| = n \tag{9}$$

where ν runs from 1 to $4+d$ by considering that the time components of ψ_1 and n are zero.

We see from eq.(9), the refractive index is *a scalar, a real number*. But, the refractive index is not simply a scalar[21]. In *linear optics*, the refractive index is described by *a second rank tensor* which describes that the electric field component along one axis may be affected by the electric field component along another axis[21]. In other words, the second rank tensor of refractive index describes *anisotropic linear optics* and the zeroth rank tensor (scalar) of refractive index describes *isotropic linear optics*[22].

Our works[23,24] show that *the second rank tensor of refractive index is a consequence of the fourth rank totally covariant tensor of (linear or Abelian) Riemann-Christoffel curvature*. Naturally, it means that the fourth rank totally covariant tensor of Abelian Riemann-Christoffel curvature describes the anisotropic linear optics. We will work with the scalar of refractive index related with the second rank tensor of Abelian Ricci curvature which describes isotropic linear optics.

VI. $U(1)$ GAUGE POTENTIAL AND FIELD STRENGTH IN UNIFIED SPACE

Analog with $SU(N)$ Yang-Mills theory, the choice of $G = U(1)$ reduces the $SU(2)$ local gauge theory, non-Abelian theory in the $(4+d)$-dimensions of unified space to the $U(1)$ Maxwell's theory. We obtain from eqs.(3), (4) that *the restricted* $U(1)$ *gauge potential* and its related field strength of the Maxwell's theory can be represented respectively as

$$\vec{B}_\mu^{U(1)} = A_\mu^{U(1)}\ \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu\hat{m}^{U(1)} \tag{10}$$

$$\vec{G}_{\mu\nu}^{\ U(1)} = \partial_\mu\vec{B}_\nu^{\ U(1)} - \partial_\nu\vec{B}_\mu^{\ U(1)} \tag{11}$$

where $A_\mu^{U(1)}$ is the d-dimensions unrestricted electric potential of the $U(1)$ gauge potential and $\hat{m}^{U(1)}$ is the d-dimensions multiplet or restricted magnetic potential of the $U(1)$ gauge potential. *The second term on the right hand side of eq.(10) expresses the magnetic monopole.*

Because the field strength of Maxwell's theory, $\vec{G}_{\mu\nu}^{\ U(1)}$ in eq.(11), and the field strength of geometrical optics, $\vec{G}_{\mu\nu}$ in eq.(6), in principle are same i.e. *both are fields*[25], we can replace $\vec{G}_{\mu\nu}$ with $\vec{G}_{\mu\nu}^{\ U(1)}$. In other words, we can replace the four-vector potential, $\vec{B}_\mu$ in eq.(5), with the $U(1)$ gauge potential, $\vec{B}_\mu^{\ U(1)}$ in eq.(10). If we replace $\vec{B}_\mu$ with $\vec{B}_\mu^{U(1)}$ then eq.(5) becomes

$$\vec{B}_\mu^{\ U(1)} = a_\mu\ e^{i\psi} \tag{12}$$

Eq.(12) expresses the form of *the restricted* $U(1)$ *gauge potential of the geometrical optics* in the $(4 + d)$-dimensions of unified space.

Eq.(12) can be written as

$$e^{i\psi} = \vec{B}_\mu^{U(1)}\ a_\mu^{-1} \tag{13}$$

Using Euler's formula, eq.(13) can be written as

$$\cos\psi + i\sin\psi = \vec{B}_\mu^{U(1)}\ a_\mu^{-1} \tag{14}$$

Eq.(14) shows us that $\vec{B}_\mu^{U(1)} a_\mu^{-1}$ is *a complex function*. To simplify the problem, we take the real part of (14) only, we obtain

$$\cos\psi = \vec{B}_\mu^{U(1)}\ a_\mu^{-1} \tag{15}$$

where ψ in eq.(15), i.e. phase (eikonal or "gauge") is *an angle*. This angle has value

$$\psi = \arccos\left(\vec{B}_{\mu}^{U(1)}\ a_{\mu}^{-1}\right) \tag{16}$$

Substituting eqs.(16), (10) into eq.(7), we obtain

$$\frac{c}{\omega}\arccos\left[\left(A_{\mu}^{U(1)}\ \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_{\mu}\hat{m}^{U(1)}\right) a_{\mu}^{-1}\right] + ct = \psi_1 \tag{17}$$

If we substitute eq.(17) into eq.(9), then the eikonal equation (9) becomes

$$\left|\partial_{\nu}\left\{\frac{c}{\omega}\arccos\left[\left(A_{\mu}^{U(1)}\hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_{\mu}\hat{m}^{U(1)}\right) a_{\mu}^{-1}\right] + ct\right\}\right| = n \tag{18}$$

where n is a dimensionless quantity, a scalar, a real number, i.e. a function of $(3+d)$-coordinates which lives in the $(4+d)$-dimensions of unified space, $A_{\mu}^{U(1)}$ is the unrestricted electric potential, i.e. the part which is *not* restricted by the magnetic symmetry (2) and $\hat{m}^{U(1)}$ is the restricted magnetic potential, i.e. the part which is *completely determined* by the magnetic symmetry (2). *Eq.(18) shows us that the refractive index is decomposed. There exist the magnetic monopole as a consequence of the magnetic symmetry of the geometrical optics where the magnetic symmetry is a consequence of the metric symmetry of the $(4+d)$-dimensions of unified space.*

VII. THE REFRACTIVE INDEX-CURVATURE IN UNIFIED SPACE

The equation of ray propagation in *a steady state* can also be derived from *Fermat's principle*[9]. We obtain the refractive index-curvature relation as below[9]

$$\frac{1}{R} = \hat{N} \cdot \frac{\vec{\nabla} n}{n} \tag{19}$$

where $\hat{N}$ is the unit vector along the principal normal, R is the radius of curvature. Eq.(19) is the equation of curvature of the 1-dimension of space, $\kappa(R) = 1/R$, i.e. a function of R variable. We see that eq.(19) is *a type of non-linear partial differential equation* because there exist the form of $n^{-1}\ \vec{\nabla} n$. *Physically, eq.(19) says that the rays are therefore bent in the direction of increasing refractive index*[9,26].

The dimension of curvature of eq.(19) can be extended to an arbitrary number of dimensions[27]. In the $(4+d)$-dimensions of unified space, for a scalar of the refractive index, n, eq.(19) can be written as[23,24,28]

$$\frac{R_{\mu\nu}}{g} = N_{\nu}\ \partial_{\mu} \ln n \tag{20}$$

where $R_{\mu\nu}$ is *the second rank tensor of Ricci curvature*[29,30], *a function* and $g = |(\det\ g_{\mu\gamma})|$, is *a scalar, a real number.*

Because the second rank tensor of Ricci curvature, $R_{\mu\nu}$, can be obtained by contraction from the totally covariant fourth rank tensor of Riemann-Christoffel curvature, $R_{\mu\nu\rho\sigma}$,[29] which is *non-linear*, so the second rank tensor of Ricci curvature is also non-linear. The fourth rank tensor of Riemann-Christoffel curvature, $R^{\gamma}{}_{\nu\rho\sigma}$, is a commutator of covariant derivative operator, $\nabla_{\sigma}\nabla_{\rho} - \nabla_{\rho}\nabla_{\sigma}$[29] and $R_{\mu\nu\rho\sigma}$ can be obtained from $R^{\gamma}{}_{\nu\rho\sigma}$, through relation $R_{\mu\nu\rho\sigma} = g_{\mu\gamma}\ R^{\gamma}{}_{\nu\rho\sigma}$. So, is there a relation between nonlinearity and noncommutativity? Nonlinearity is linked with noncommutativity. *Noncommutativity does produce nonlinearity*[31]. Non-commutative is other name of non-Abelian. The commutativity analysis will be useful when we consider the relation between the curvature and the field strength (the gauge potential).

Because n in eq.(20) is a scalar, we can substitute this scalar with n in eq.(18). We obtain

$$\frac{R_{\mu\nu}}{g} = N_{\nu}\ \partial_{\mu} \ln\left|\partial_{\nu}\left[\frac{c}{\omega}\arccos\left(\frac{\vec{B}_{\mu}^{U(1)}}{a_{\mu}}\right) + ct\right]\right| \tag{21}$$

where the restricted $U(1)$ gauge potential, $\vec{B}_{\mu}^{U(1)}$, is given in eq.(10). We see from eq.(21), *the curvature, which is shown by the second rank tensor of Ricci curvature, is related naturally with the gauge potential.* It is in a harmony that *the curvature and the field strength are identical*[32].

Because the related field strength of the restricted $U(1)$ gauge potential in the geometrical optics is Abelian and the curvature is related naturally with the gauge potential, as shown in eq.(21), so *the second rank tensor of Ricci curvature is Abelian.* It has a consequence that the non-linear terms, i.e. the third and the fourth terms, on the right hand side of the Ricci curvature tensor equation below[29] vanish

$$R_{\mu\nu} = \frac{\partial\Gamma^{\rho}{}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}{}_{\mu\rho}}{\partial x^{\nu}} + \Gamma^{\sigma}{}_{\mu\nu}\ \Gamma^{\rho}{}_{\rho\sigma} - \Gamma^{\sigma}{}_{\mu\rho}\ \Gamma^{\rho}{}_{\nu\sigma} \tag{22}$$

So we have *the second rank tensor of Abelian Ricci curvature* as below

$$R_{\mu\nu} = \frac{\partial\Gamma^{\rho}{}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}{}_{\mu\rho}}{\partial x^{\nu}} \tag{23}$$

where $\Gamma^{\rho}{}_{\mu\nu}$ is the Christoffel symbol of the second kind[29].

Substituting eq.(23) into eq.(21), the refractive index-curvature relation (21) becomes

$$\frac{1}{g}\left(\frac{\partial\Gamma^{\rho}{}_{\mu\nu}}{\partial x^{\rho}} - \frac{\partial\Gamma^{\rho}{}_{\mu\rho}}{\partial x^{\nu}}\right) = N_{\nu}\ \partial_{\mu} \ln\left|\partial_{\nu}\left[\frac{c}{\omega}\arccos\left(\frac{\vec{B}_{\mu}^{U(1)}}{a_{\mu}}\right) + ct\right]\right| \tag{24}$$

where the restricted $U(1)$ gauge potential, $\vec{B}_{\mu}^{U(1)}$, is given in eq.(10). We see from eq.(24), although the nonlinearity properties of the second rank tensor of Ricci curvature vanish, eq.(24) is still a non-linear equation. The nonlinearity of eq.(24) is because of there exist natural logarithm function there. *The Abelian curvature, i.e. the second rank tensor of Ricci curvature divided by g, is decomposed as a consequence that the refractive index is decomposed*. It consists of the unrestricted electric potential (a scalar), $A_\mu^{U(1)}$, i.e. the part which is not restricted by the magnetic symmetry (2) and the restricted magnetic potential (a vector), $\hat{m}^{U(1)}$, i.e. the part which is completely determined by the magnetic symmetry (2).

VIII. DISCUSSION AND CONCLUSION

Symmetry is a fundamental principle in physics. The existence of magnetic monopole is a consequence of the magnetic symmetry (2). This magnetic symmetry is a consequence of the metric symmetry (1) of the $(4+d)$-dimensions of unified space where the magnetic monopole lives in the d-dimensions of internal space, an abstract space.

In the geometrical optics, the existence of magnetic monopole is shown by the eikonal equation (18) and the refractive index-curvature relation (24) where both eqs.(18), (24) are formulated in the $(4+d)$-dimensions of unified space.

The refractive index as shown by the eikonal equation (18) is a dimensionless quantity, i.e. real variable. But, as a consequence that there exist the magnetic monopole, the refractive index is not single real variable. The refractive index is decomposed. It consists of the part which is not restricted by the magnetic symmetry (2) i.e. the part of electric potential, a scalar potential, $A_\mu^{U(1)}$, and the part which is completely determined by the magnetic symmetry (2), i.e. the part of magnetic potential, a vector potential, $\hat{m}^{U(1)}$.

As a consequence that the geometrical optics is an Abelian system, so the refractive index-curvature relation (24) is formulated in relation with the Abelian curvature. What we mean with Abelian curvature is the second rank tensor of Ricci curvature which loss the non-linear terms as shown in eq.(23) divided with g. The Abelian curvature is a function of space, because the related refractive index is a function of space. The Abelian curvature follows the natural logarithm function of the refractive index. What is the consequence of the decomposition of the refractive index to the Abelian curvature? Is the Abelian curvature also decomposed?

The Abelian curvature (24) is related naturally with the gauge potential. It is in a harmony that the curvature (the second rank tensor of Ricci curvature) and the field strength are identical. If charge is the most fundamental quantity in the gauge field theory and the metric tensor is the most fundamental quantity in the curvature geometry, is there relation between the charge and the metric tensor?

We see from eqs.(13), (14) that $e^{i\psi} = \cos\psi + i\sin\psi$ is a complex function. What is the consequence[33] if we are using this complex function to formulate of the eikonal equation and the refractive index-curvature relation?

By considering that the gauge field theory is related with the gauge transformation to the potential and the Abelian curvature (24) is related naturally with the gauge potential, so what is the consequence if we do "the gauge transformation" to the curvature, such as the metric tensor?

The eikonal equation and the refractive index-curvature relation describe the ray propagation in a steady state (time-independent). Both equations of ray propagation in a steady state can be derived from the Fermat's principle. Can Fermat's principle apply for non-steady state (time-dependent)?

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