Magnetic symmetry of geometrical optics

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We propose that there exist magnetic symmetry in the geometrical optics. What are the consequences of the magnetic symmetry existence to the formulation of refractive index and its related curvature?

I. INTRODUCTION

Dirac proposed, due to symmetrical reasoning in the Maxwell's theory of electromagnetism, there exist magnetic monopole which has magnetic charge[1,2]. Inspired with Dirac idea of monopole, the magnetic symmetry was formulated as the $SU(2)$ *local* gauge theory, non-Abelian theory of the $(4+n)$-dimensions of unified space[3]. The $(4+n)$-dimensions of unified space is actually the $(3+1)$-dimensions of external space, i.e. the $(3+1)$-dimensions of curved spacetime, plus the n-dimensions of (curved) internal space[4].

To the best of our knowledge, the geometrical optics (the eikonal equation) is formulated without including magnetic symmetry[5–11]. The eikonal equation can be derived from the Maxwell equations[9]. Because of Maxwell's theory is $U(1)$ *local* gauge theory, Abelian theory, and the eikonal equation can be derived from the Maxwell's theory, *we argue that the geometrical optics is also* $U(1)$ *local gauge theory, Abelian theory and there exist magnetic symmetry in the geometrical optics.*

The formulation of the $SU(2)$ local gauge theory, non-Abelian theory of the $(4+n)$-dimensions of unified space[3] looks like the $SU(N)$ Yang-Mills theory[12] where the choice of the simple Lie group $G = U(1)$ reduces the Yang-Mills theory to Maxwell's theory[13].

Because of the Abelian $U(1)$ local gauge theory can be generalized to non-Abelian $SU(2)$ local gauge theory[14], it has the consequence that we can obtain the magnetic symmetry in the Abelian $U(1)$ local gauge theory from the magnetic symmetry of the non-Abelian $SU(2)$ local gauge theory.

We will apply the $U(1)$ gauge potential of the magnetic symmetry to the geometrical optics, especially in the formulation of eikonal equation, the refractive index and the curvature relation.

II. POTENTIAL AND FIELD STRENGTH TENSOR

In the $(3+1)$-dimensions of spacetime, for the geometrical optics approximation (short wavelength, $\lambda \to 0$[11]), *the four-vector potential*, A_α, and the field strength tensor, $F_{\alpha\beta}$, can be represented respectively as[6,7]

$$A_\alpha = a_\alpha \, e^{i\psi} \tag{1}$$

$$F_{\alpha\beta} = \nabla_\alpha A_\beta - \nabla_\beta A_\alpha \tag{2}$$

where $\psi(x, y, z, t)$ is *the phase, the eikonal*, a large quantity which is "almost linear" *in the coordinates and time*[10], *an angle as a function of spacetime.* The time derivative of ψ gives the frequency of the wave, $\partial\psi/\partial t = -\omega$ and the space derivatives of ψ gives the wave vector, $\vec{\nabla}\psi = \vec{k}$[10] which shows the direction of the ray through any point in space[10]. a_α is a slowly varying amplitude, a slowly varying function of the coordinates and time[10], $a_\alpha(x, y, z, t)$. ∇_α denotes a spacetime covariant derivative[5].

The equation of ray propagation in a medium with the refractive index, n, is[9,10]

$$|\vec{\nabla}\psi_1| = n \tag{3}$$

where $\psi_1(x, y, z)$ is a function of the coordinates only[10]. It is also called *the eikonal*[10] or *the optical length* (i.e. *a real scalar function of position*)[9]. So, the equation (3) is called *the eikonal equation*[9,10].

In case of a steady monochromatic wave, the frequency is constant and the time dependence of the eikonal, ψ, is given by term $-\omega t$, then the eikonal in eq.(3), ψ_1, can be expressed as[10]

$$\psi_1 = \frac{c}{\omega}\psi + ct \tag{4}$$

Here, ψ is same as ψ in eq.(1).

We assume that the geometrical optics "lives" in the $(4+n)$-dimensions of unified space, so the indices α, β in eqs.(1), (2) run from 1 to $4+n$. The eikonal, $\psi(x, y, z, t)$, as a function of coordinates and time then "lives" in the $(4+n)$-dimensions of unified space so it has a consequence that the eikonal, $\psi_1(x, y, z)$, should "lives" in the $(4+n)$-dimensions of unified space. *We denote* ψ_1 *with* ψ_μ *which describes that* ψ_1 *is actually a function of space, but it lives in the* $(4+n)$*-dimensions of unified space.* We can imagine it simply as e.g. a one-dimensional string which "lives" on the surface of the two dimensional space of paper. The gradient operator, $\vec{\nabla}$, in eq.(3) transforms to the four-gradient, ∂ (the covariant four-gradient, ∂_ν, or the contravariant four-gradient, ∂^ν)[15]. So, eqs.(3), (4) become

$$n_{\mu\nu} = |\partial_\nu \psi_\mu| \tag{5}$$

$$\psi_\mu = \frac{c}{\omega}\psi + ct \tag{6}$$

where μ, ν as α, β run from 1 to $4+n$.

In case of the Maxwell's theory in the $(3+1)$-dimensions of spacetime, the four-vector potential, A^ρ, and the field strength tensor, $F^{\rho\tau}$, can be represented respectively as[16]

$$A^{\rho} = \left(\frac{V}{c}, A_x, A_y, A_z\right) \tag{7}$$

$$F^{\rho\tau} = \partial_{\rho} A^{\tau} - \partial_{\tau} A^{\rho} \tag{8}$$

where $\partial_{\rho} = \partial/\partial x_{\rho}$ is the four-gradient, i.e. the differentiation with respect to the covariant vector, x_{ρ}. We see that the formulations of the field strength tensor in the geometrical optics (2) and in the Maxwell's theory (8) look similar.

III. $SU(2)$ GAUGE POTENTIAL AND FIELD STRENGTH IN UNIFIED SPACE

Magnetic symmetry of the non-Abelian $SU(2)$ local gauge theory in the $(4+n)$-dimensions of unified space is formulated in relations with the symmetry of unified metric[3]. The Killing vector fields must satisfy[3]

$$\pounds_m \, g_{\mu\nu} = 0 \tag{9}$$

where m is *vector field*, $\pounds_m$ is the Lie derivative along the direction of m, $g_{\mu\nu}(\mu, \nu = 1, 2, .., 4+n)$ is metric in the $(4+n)$-dimensions of unified space[3].

The eq.(9) has consequence that

$$D_{\mu}\hat{m} = \partial_{\mu}\hat{m} + g\vec{B}_{\mu} \times \hat{m} = 0 \tag{10}$$

where $\hat{m}$ is *the multiplet* and $\vec{B}_{\mu}$ is *the gauge potential* of the n-dimensional isometry group[3]

$$\vec{B}_{\mu} = A_{\mu}\hat{m} - \frac{1}{g}\hat{m} \times \partial_{\mu}\hat{m} \tag{11}$$

where A_{μ} is *the unrestricted electric (scalar) potential*, i.e. the part which is *not* restricted by the condition (10) and $\hat{m}$ is *the restricted magnetic (vector) potential*, i.e. the part which is *completely determined* by the magnetic symmetry or the condition (10).

The corresponding field strength, $\vec{G}_{\mu\nu}$, to the gauge potential (11) is[3]

$$\vec{G}_{\mu\nu} = \partial_{\mu}\vec{B}_{\nu} - \partial_{\nu}\vec{B}_{\mu} + g\vec{B}_{\mu} \times \vec{B}_{\nu} \tag{12}$$

We see that the formulation of the field strength tensor in the non-Abelian $SU(2)$ local gauge theory (12) is different from eqs.(8) and (2), because of there exist the third term of (12) i.e. $g\vec{B}_{\mu} \times \vec{B}_{\nu}$.

IV. $U(1)$ GAUGE POTENTIAL AND FIELD STRENGTH IN UNIFIED SPACE

In an analogue that the choice of $G = U(1)$ reduces $SU(2)$ Yang-Mills theory to the $U(1)$ Maxwell's theory, from eqs.(11), (12) we obtain that the $U(1)$ gauge potential and the related $U(1)$ field strength tensor can be represented respectively as

$$\vec{B}_{\mu}^{U(1)} = A_{\mu}^{U(1)} \; \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_{\mu}\hat{m}^{U(1)} \tag{13}$$

$$\vec{G}_{\mu\nu}^{\;U(1)} = \partial_{\mu}\vec{B}_{\nu}^{\;U(1)} - \partial_{\nu}\vec{B}_{\mu}^{\;U(1)} \tag{14}$$

where $A_{\mu}^{U(1)}$ is the unrestricted electric (scalar) potential of the $U(1)$ gauge potential and $\hat{m}^{U(1)}$ is *the multiplet* of the n-dimensional $U(1)$ group.

Because, $\vec{G}_{\mu\nu}^{\;U(1)}$ in (14), $F^{\rho\tau}$ in (8) and $F_{\alpha\beta}$ in (2) are in principle same i.e. *the fields*, so we can replace $F^{\rho\tau}$ or $F_{\alpha\beta}$ with $\vec{G}_{\mu\nu}^{\;U(1)}$[17]. In other words, we can replace the four-vector potentials, A^{ρ} or A_{α}, with the $U(1)$ gauge potential, $\vec{B}_{\mu}^{\;U(1)}$. If we replace A_{α} with $\vec{B}_{\mu}^{U(1)}$ (by considering the harmonic form of notation) then eq.(1) becomes

$$\vec{B}_{\mu}^{\;U(1)} = a_{\mu} \; e^{i\psi} \tag{15}$$

This is the $U(1)$ gauge potential of the geometrical optics in the $(4+n)$-dimensions of unified space.

V. THE REFRACTIVE INDEX-CURVATURE IN UNIFIED SPACE

In general relativity, light rays follow *null geodesics*[18], i.e. the line-element of the "world" of space-time, ds, *vanishes*[19]. The null geodesics are *the tracks of rays of light*[19]. Mathematically, the tracks of rays of light are expressed in *the Fermat's principle*.

To simplify the problem, the Fermat's principle is formulated in case of a static gravitational field[11], isotropic and spherically symmetric metric[20]. The Fermat's principle is

$$\delta \int_{r_1}^{r_2} n \; dr = 0 \tag{16}$$

The relation between refractive index and curvature can be derived from the Fermat's principle (16) as below[10]

$$\frac{1}{R} = \vec{N} \cdot \frac{\vec{\nabla} n}{n} \tag{17}$$

where $\vec{N}$ is the unit vector along the principal normal, R is the radius of curvature and n is *the refractive index*. We see from eq.(17), the rays are therefore bent in the direction of increasing refractive index[21].

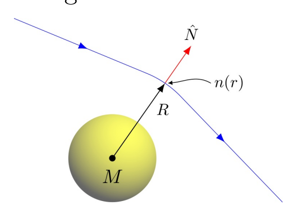

Fig. 1 The illustration of eq.(17).

In the geometry of (3+1)-dimensions of curved space-time, eq.(17) can be written as[22,23]

$$R_{\mu\nu\rho\sigma} = gN_\sigma\ \partial_\rho \ln n_{\mu\nu} \tag{18}$$

where $R_{\mu\nu\rho\sigma}$ is the Riemann-Christoffel curvature tensor.

We assume that the form of eq.(18) does not change if we extent the formulation to the $(4+n)$-dimensions of unified space. So, we can say that eq.(18) is the relation between refractive index and curvature in the $(4+n)$-dimensions of unified space.

VI. THE GAUGE POTENTIAL-REFRACTIVE INDEX-CURVATURE IN UNIFIED SPACE

We see from eq.(15)

$$e^{i\psi} = \vec{B}_\mu^{U(1)} a_\mu^{-1} \tag{19}$$

Using Euler's formula, eq.(19) can be written as

$$\cos\psi + i\sin\psi = \vec{B}_\mu^{U(1)} a_\mu^{-1} \tag{20}$$

To simplify the problem, we only take the real part of (20), then eq.(20) becomes

$$\cos\psi = \vec{B}_\mu^{U(1)} a_\mu^{-1} \tag{21}$$

$$\psi = \arccos \vec{B}_\mu^{U(1)} a_\mu^{-1} \tag{22}$$

Substituting eqs.(22), (13) into eq.(6), we obtain

$$\psi_\mu = \frac{c}{\omega} \arccos\left(A_\mu^{U(1)}\ \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \right) a_\mu^{-1} + ct \tag{23}$$

Substituting eq.(23) into eq.(5), we obtain

$$\left|\partial_\nu \left\{ \frac{c}{\omega} \arccos\left(A_\mu^{U(1)} \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \right) a_\mu^{-1} + ct \right\}\right| = n_{\mu\nu} \tag{24}$$

Substituting eq.(24) into (18), we obtain

$$gN_\sigma\ \partial_\rho \ln \left|\partial_\nu \left\{ \frac{c}{\omega} \arccos \left(A_\mu^{U(1)} \hat{m}^{U(1)} - \frac{1}{g}\hat{m}^{U(1)} \times \partial_\mu \hat{m}^{U(1)} \right) a_\mu^{-1} + ct \right\}\right| = R_{\mu\nu\rho\sigma} \tag{25}$$

VII. DISCUSSION AND CONCLUSION

We see from eqs.(24), (25) that there exist the magnetic symmetry (magnetic monopole) represented by $\hat{m}^{U(1)}$ in the geometrical optics, especially in the eikonal equation (24), the refractive index and the Riemann-Christoffel curvature tensor relation (25) where both are formulated in the $(4+n)$-dimensions of unified space.

$A_\mu^{U(1)}$ and $\hat{m}^{U(1)}$ contribute to the refractive index and in turn to the refractive index and the Riemann-Christoffel curvature tensor relation. In other words, the formulations of the refractive index, the refractive index and the Riemann-Christoffel curvature tensor relation consist of the unrestricted electric (scalar) potential of the $U(1)$ gauge potential and the multiplet or the restricted magnetic (vector) potential of the n-dimensional $U(1)$ group.

What does it mean physically that the refractive index and the curvature are decomposed into the unrestricted electric (scalar) potential of the $U(1)$ gauge potential and the restricted magnetic (vector) potential of the n-dimensional $U(1)$ group?

Related with the gravitational lensing problem (when the light passes near a massive mass object, the curvature of space-time due to such a massive mass object will deflect the light path), what is the consequence of the decomposition of the refractive index to the angle of deflection of the light path?

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[1] P.A.M. Dirac, *Quantised Singularities in the Electromagnetic Field*, Proc. Roy. Soc. **A 133** 60 1931.

[2] P.A.M. Dirac, *The Theory of Magnetic Poles*, Physical Review Volume 74, Number 7 October 1, 1948.

[3] Y.M. Cho, *Restricted gauge theory*, Physical Review D, Volume 21, Number 4, 15 February 1980.

[4] The internal space is an abstract space where the magnetic symmetry "lives". We can not "see" this internal space due to the symmetry we are assumed (Y.M. Cho, Pong Soo Jang, *Unified Geometry of Internal Space with Spacetime*, June 1975).

[5] A.B. Balakin, A.E. Zayats, *Non-minimal Wu-Yang monopole*, https://arxiv.org/pdf/gr-qc/0612019.pdf, 2006.

[6] Alexander B. Balakin, Alexei E. Zayats, *Non-minimal Einstein-Maxwell theory: the Fresnel equation and the Petrov classification of a trace-free susceptibility tensor*, https://arxiv.org/pdf/1710.08013.pdf, 2018.

[7] A.B. Balakin, A.E. Zayats, *Ray Optics in the Field of a Nonminimal Dirac Monopole*, Gravitation and Cosmology, 2008, Vol.14, No.1, pp.86-94.

[8] Kazuo Ota Cottrell, Jong Ping Hsu, *Gauge independence of the eikonal equation in Yang-Mills gravity*, Eur. Phys. J. Plus (2015) **130**: 147.

[9] Max Born, Emil Wolf, *Principles of Optics*, Pergamon Press, 1993.

[10] L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, 1984.

[11] L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields*, Fourth Revised English Edition, Butterworth-Heinemann, 1987.

[12] Cho restricted gauge theory is a self consistent subset of a non-Abelian $SU(2)$ gauge theory which tries to describe the *infrared regime* of the Yang-Mills gauge theory (Sedigheh Deldar, Ahmad

Mohamadnejad, *Quark Confinement in Restricted $SU(2)$ Gauge Theory*, https://arxiv.org/pdf/1208.2165.pdf, 2012).

[13]David Tong, *Gauge Theory*, http://www.damtp.cam.ac.uk/user/tong/gaugetheory.html, 2018.

[14]In mathematics, the special unitary group of degree N, denoted $SU(N)$, is the Lie group of $N \times N$ unitary matrices with determinant 1. The more general unitary matrices may have complex determinants with absolute value 1, rather than real 1 in the special case. The special unitary group is a subgroup of the unitary group $U(N)$, consisting of all $N \times N$ unitary matrices (Wikipedia, *Special unitary group*).

[15]Wikipedia, *Four-gradient*. Access date: 22 Apr 2021.

[16]David J. Griffiths, *Introduction to Electrodynamics*, Cambridge University Press, 2017.

[17]Y.M. Cho, *Private communication*.

[18]Rajaram Nityananda, Joseph Samuel, *Fermat's principle in general relativity*, Physical Review D, Volume 43, Number 10, 15 May 1992.

[19]E.T. Whittaker, *Note on the law that light-rays are the null geodesies of a gravitational field*, 1927.

[20]Soma Mitra, Somenath Chakrabarty, *Fermat's Principle in Curved Spacetime, No Emission from Schwarzschild Black Holes as Total Internal Reflection and Black Hole Unruh Effect*, arXiv:1512.03885v1 [gr-qc] 12 Dec 2015.

[21]Light changes its direction of propagation when it encounters an inhomogeneity in the medium. The curvature of the path is used to quantify this change of direction. This curvature is defined as the ratio of the change in the direction of propagation to the length measured along the curved path (K. Iizuka, *Engineering Optics*, p.108).

[22]Miftachul Hadi, *Linear and non-linear refractive indices in Riemannian and topological spaces*, https://osf.io/69pwy, 2020.

[23]Miftachul Hadi, Utama Alan Deta, Andri Sofyan Husein, *Linear and non-linear refractive indices in curved space*, Journal of Physics: Conference Series **1796** (2021) 012125.