# The octets of the natural numbers

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**Abstract.** "Every natural number, with the exception of 0 and 1, can be written in a unique way as a linear combination of consecutive powers of 2, with the coefficients of the linear combination being -1 or +1". From this theorem, four fundamental properties of odd numbers are implied: the conjugate of an odd number, the L/R symmetry, the transpose of an odd number and the octets of odd numbers. Using these properties we obtain a categorization of the composite odd numbers and a factorization algorithm.

Keywords: Number theory, Conjugate of odd number, Symmetry L/R, Transpose of odd number, Odd number octet, Factorization algorithm.

## 1. Introduction

In this article, we start by proving the theorem: "Every odd number $\Pi$, with the exception of 1, can be uniquely written in the form $\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$, $\nu \in \mathbb{N}$, $\nu + 1 = \left[ \frac{\ln \Pi}{\ln 2} \right]$, $\beta_i = \pm 1, i = 0,1,2,...,\nu-1$ where $\left[ \frac{\ln \Pi}{\ln 2} \right]$ is the integer part of $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$" (theorem 2.1). It's easy to show that a theorem of this kind doesn't hold for any other natural number different than 2.

The main difference between our theorem and the known arithmetic systems (binary, decimal etc.) is that the coefficients of the linear combination can take the negative value. These negative values highlight two properties of odd numbers, the "conjugate" of an odd number (definition 2.1) and the "L/R symmetry" (definition 3.1).

Another property of odd numbers is that of the "transpose" of an odd number (definition 4.1). The transpose of an odd number can be defined in any arithmetic system and its value depends on the system used. However, the main mathematical object to which we come to, "the odd number octet", is defined from a combination of conjugates and transposes of odd numbers (equation (5.1)). Thus, the odd number octet emerges only by using theorem 2.1. Using these properties we obtain a categorization of the composite odd numbers and a factorization algorithm.

## 2. Odd numbers as linear combinations of consecutive powers of 2

We prove the following theorem:

**Theorem 2.1.** *Every odd number* $\Pi$*, with the exception of* 1*, can be uniquely written in the form*

$$\Pi = \Pi\left(\nu, \beta_i\right) = 2^{\nu+1} + 2^{\nu} \pm 2^{\nu-1} \pm 2^{\nu-2} \pm ... \pm 2^1 \pm 2^0 = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i$$

$$\nu \in \mathbb{N}, \nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right], \beta_i = \pm 1, i = 0,1,2,...,\nu - 1 \tag{2.1}$$

*where* $\left[\frac{\ln \Pi}{\ln 2}\right]$ *is the integer part of* $\frac{\ln \Pi}{\ln 2} \in \mathbb{R}$.

*Proof.* For $\Pi = 3$ we have $\nu + 1 = \left[\frac{\ln 3}{\ln 2}\right] \Rightarrow \nu = 0$ and from equation (2.1) we obtain

$$\Pi = 3 = 2^1 + 2^0.$$

We now examine the case where $\nu \in \mathbb{N}^* = \{1,2,3,...\}$. The lowest value that the odd number $\Pi$ of equation (2.1) can obtain is

$$\Pi_{\min} = \Pi\left(\nu\right) = 2^{\nu+1} + 2^{\nu} - 2^{\nu-1} - 2^{\nu-2} - ... - 2^1 - 1$$

$$\Pi_{\min} = \Pi\left(\nu\right) = 2^{\nu+1} + 1. \tag{2.2}$$

The largest value that the odd number $\Pi$ of equation (2.1) can obtain is

$$\Pi_{\max} = \Pi\left(\nu\right) = 2^{\nu+1} + 2^{\nu} + 2^{\nu-1} + 2^{\nu-2} + ... + 2^1 + 1$$

$$\Pi_{\max} = \Pi\left(\nu\right) = 2^{\nu+2} - 1. \tag{2.3}$$

Thus, for the odd numbers $\Pi = \Pi\left(\nu, \beta_i\right)$ of equation (2.1) the following inequality holds

$$\Pi_{\min} = 2^{\nu+1} + 1 \le \Pi\left(\nu, \beta_i\right) \le 2^{\nu+2} - 1 = \Pi_{\max}. \tag{2.4}$$

The number $N\left(\Pi\left(\nu, \beta_i\right)\right)$ of odd numbers in the closed interval $\left[2^{\nu+1} + 1, 2^{\nu+2} - 1\right]$ is

$$N\left(\Pi\left(\nu, \beta_i\right)\right) = \frac{\Pi_{\max} - \Pi_{\min}}{2} + 1 = \frac{\left(2^{\nu+2} - 1\right) - \left(2^{\nu+1} + 1\right)}{2} + 1$$

$$N\left(\Pi\left(\nu, \beta_i\right)\right) = 2^{\nu}. \tag{2.5}$$

The integers $\beta_i, i = 0,1,2,...,\nu - 1$ in equation (2.1) can only take two values, $\beta_i = -1 \vee \beta_i = +1$, thus equation (2.1) gives exactly $2^{\nu} = N\left(\Pi\left(\nu, \beta_i\right)\right)$ odd numbers. Therefore, for every $\nu \in \mathbb{N}^*$ equation (2.1) gives all odd numbers in the interval $\Omega_{\nu} = \left[2^{\nu+1}, 2^{\nu+2}\right]$.

From inequality (2.4) we obtain

$$2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1$$

$$2^{\nu+1} < 2^{\nu+1} + 1 \le \Pi \le 2^{\nu+2} - 1 < 2^{\nu+2}$$

$$2^{\nu+1} < \Pi < 2^{\nu+2}$$

$$(\nu+1)\ln 2 < \ln \Pi < (\nu+2)\ln 2$$

from which we get

$$\frac{\ln \Pi}{\ln 2} - 1 < \nu + 1 < \frac{\ln \Pi}{\ln 2}$$

and finally

$$\nu + 1 = \left[ \frac{\ln \Pi}{\ln 2} \right]. \tag{2.6}$$

For $\Pi = 1$ we have $\nu + 1 = \left[ \frac{\ln 1}{\ln 2} \right] \Rightarrow \nu = -1 \notin \mathbb{N}$.

We prove now that every odd number $\Pi \neq 1$ can be uniquely written in the form of equation (2.1). We write the odd $\Pi$ as

$$\begin{aligned} &\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \beta_i 2^i \\ &\beta_i = \pm 1, i = 0,1,2,...,\nu-1 \\ &\nu + 1 = \left[ \frac{\ln \Pi}{\ln 2} \right] \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} &\Pi = 2^{\nu+1} + 2^{\nu} + \sum_{i=0}^{\nu-1} \gamma_i 2^i \\ &\gamma_i = \pm 1, i = 0,1,2,...,\nu-1. \\ &\nu + 1 = \left[ \frac{\ln \Pi}{\ln 2} \right] \end{aligned} \tag{2.8}$$

From equations (2.7), (2.8) we get

$$\begin{aligned} &(\beta_0 - \gamma_0)\cdot 2^0 + (\beta_1 - \gamma_1)\cdot 2^1 + (\beta_2 - \gamma_2)\cdot 2^2 + ... + (\beta_{\nu-1} - \gamma_{\nu-1})\cdot 2^{\nu-1} = 0 \\ &\beta_i = \pm 1, i = 0,1,2,...,\nu-1 \\ &\gamma_i = \pm 1, i = 0,1,2,...,\nu-1 \\ &i \in \{0,1,2,...,\nu-1\} \end{aligned} \quad . \tag{2.9}$$

If, in equation (2.9), there are $i \in \{0,1,2,...,\nu-1\}$ such that $\beta_i \neq \gamma_i$ and let $k$ be the smallest of them, dividing by $2^{k+1}$ we get an odd number equal to an even number. So, it follows that $\beta_i = \gamma_i \forall i = 0,1,2,...,\nu-1$. □

In order to write an odd number $\Pi \neq 1,3$ in the form of equation (2.1) we initially define the $\nu \in \mathbb{N}^*$ from equation (2.6). Then, we calculate the sum

$2^{\nu+1} + 2^{\nu}$.

If $2^{\nu+1}+2^{\nu}<\Pi$ we add $2^{\nu-1}$, whereas if $2^{\nu+1}+2^{\nu}>\Pi$ then we subtract it. By repeating the process exactly $\nu$ times we write the odd number $\Pi$ in the form of equation (2.1). The number $\nu$ of steps needed in order to write the odd number $\Pi$ in the form of equation (2.1) is extremely low compared to the magnitude of the odd number $\Pi$, as derived from inequality (2.4).

**Example 2.1.** For the odd number $\Pi=23$ we obtain from equation (2.6)

$$\nu+1=\left[\frac{\ln 23}{\ln 2}\right]\Rightarrow \nu=3.$$

Then, we have

$2^{\nu+1}+2^{\nu}=2^4+2^3=24>23$ (thus $2^2$ is subtracted)

$2^4+2^3-2^2=20<23$ (thus $2^1$ is added)

$2^4+2^3-2^2+2^1=22<23$ (thus $2^0=1$ is added)

$2^4+2^3-2^2+2^1+1=23.$

Fermat numbers $F_s$ can be written directly in the form of equation (2.1), since they are of the form $\Pi_{\min}$,

$$F_s=2^{2^s}+1=\Pi_{\min}\left(2^s-1\right)=2^{2^s}+2^{2^s-1}-2^{2^s-2}-2^{2^s-3}-\ldots-2^1-1 \quad (2.10)$$
$$s\in\mathbb{N}$$

Mersenne numbers $M_p$ can be written directly in the form of equation (2.1), since they are of the form $\Pi_{\max}$,

$$M_p=2^p-1=\Pi_{\max}\left(p-2\right)=2^{p-1}+2^{p-2}+2^{p-3}+\ldots+2^1+1 \quad (2.11)$$
$$p=\textit{prime}$$

We now give the following definition:

**Definition 2.1.** *We define as the conjugate of the odd* $\Pi\geq 3$,

$$\Pi=\Pi\left(\nu,\beta_i\right)=2^{\nu+1}+2^{\nu}+\sum_{i=0}^{\nu-1}\beta_i 2^i$$
$$\beta_i=\pm 1, i=0,1,2,\ldots,\nu-1 \quad (2.12)$$
$$\nu\in\mathbb{N}^*$$

*the odd* $\Pi^*$,

$$\Pi^*=\Pi^*\left(\nu,\gamma_j\right)=2^{\nu+1}+2^{\nu}+\sum_{j=0}^{\nu-1}\gamma_j 2^j$$
$$\gamma_i=\pm 1, j=0,1,2,\ldots,\nu-1 \quad (2.13)$$
$$\nu\in\mathbb{N}^*$$

*for which it is*

$\gamma_k = -\beta_k \forall k = 0,1,2,...,\nu-1$. (2.14)

For conjugate odds, the following corollary holds:

**Corollary 2.1.** *For the conjugate odds* $\Pi = \Pi(\nu, \beta_i) \geq 3$ *and* $\Pi^* = \Pi^*(\nu, \gamma_i)$ *the following hold:*

1. $\left(\Pi^*\right)^* = \Pi$. (2.15)

2. $\Pi^* = 3 \cdot 2^{\nu+1} - \Pi$. (2.16)

3. $\Pi$ *is divisible by* $3$ *if and only if* $\Pi^*$ *is divisible by* $3$.

4. *Two conjugate odd numbers cannot have common factors greater than 3.*

5. *Conjugates* $\Pi$ *and* $\Pi^*$ *are equidistant from the middle* $3 \cdot 2^{\nu}$ *of the interval* $\Omega_\nu = \left[2^{\nu+1}, 2^{\nu+2}\right]$.

*Proof.* 1. The 1 of the corollary is an immediate consequence of definition 4.1.

2. From equations (2.12), (2.13) and (2.14) we get

$\Pi + \Pi^* = \left(2^{\nu+1} + 2^{\nu}\right) + \left(2^{\nu+1} + 2^{\nu}\right)$

and equivalently

$\Pi + \Pi^* = 3 \cdot 2^{\nu+1}$.

3. If the odd $\Pi$ is divisible by $3$ then it is written in the form $\Pi = 3x, x = odd$ and from equation (4.17) we get $3x + \Pi^* = 3 \cdot 2^{\nu+1}$ and equivalently $\Pi^* = 3\left(2^{\nu+1} - x\right)$. Similarly we can prove the inverse.

4. If $\Pi = xy, \Pi^* = xz$, x, y, z odd numbers, from equation (2.16) we have $x\left(y+z\right) = 3 \cdot 2^{\nu+1}$ and consequently $x = 3$.

5. From equation (2.16) we obtain

$\Pi - 3 \cdot 2^{\nu} = 3 \cdot 2^{\nu} - \Pi^*$

$\left|\Pi - 3 \cdot 2^{\nu}\right| = \left|3 \cdot 2^{\nu} - \Pi^*\right|$. □

From corollary 2.1 we have that 3 is the only odd number which is equal to its conjugate; $3^* = 3 \cdot 2^{0+1} - 3 = 3$. For the $\Pi = 1$ we define

$1^* = 1$. (2.17)

Also, from equation

$\left(\Pi + X\right) + \left(\Pi^* - X\right) = 3 \cdot 2^{\nu+1}$ (2.18)

it follows that, if the odds $\Pi + X$ and $\Pi - X$, $X = even$ belong to the interval $\Omega_\nu = \left[2^{\nu+1}, 2^{\nu+2}\right]$, then they are conjugates

$$\left(\Pi + X\right)^* = \Pi^* - X \ . \tag{2.19}$$

It is easily proven that theorem (2.1) is also valid for even numbers that are not powers of 2. In order to write an even number $E$ that is not a power of 2 in the form of equation (2.1), initially it is consecutively divided by 2 and it takes of the form of equation

$$\begin{aligned} &E = 2^l \cdot \Pi \\ &\Pi = odd, \Pi \neq 1, l \in \mathbb{N}^* \end{aligned} . \tag{2.20}$$

Then, we write the odd number $\Pi$ in the form of equation (2.1).

**Example 2.2.** By consecutively dividing the even number 368 by 2 we obtain

$$E = 368 = 2^4 \cdot 23.$$

Then, we write the odd number $\Pi$ =23 in the form of equation (2.1),

$$23 = 2^4 + 2^3 - 2^2 + 2^1 + 1$$

and we get

$$368 = 2^4 \cdot \left(2^4 + 2^3 - 2^2 + 2^1 + 1\right)$$

$$368 = 2^8 + 2^7 - 2^6 + 2^5 + 2^4.$$

This equation gives the unique way in which the even number 368 can be written in the form of equation (2.1). For even numbers the lowest power of two in equation (2.1) is different from $1 = 2^0$.

## 3. The L/R symmetry

We now give the following definition:

**Definition 3.1.**1. *The odd number* $\Pi$ *in the equation* (2.1) *has symmetry* $L$ *when there exists an index L so that*

$$\begin{aligned} &\beta_L = +1 \\ &\beta_{L-1} = \beta_{L-2} = ... = \beta_1 = \beta_0 = -1. \\ &L \in \left\{1, 2, 3, ..., \nu - 1\right\} \end{aligned} \tag{3.1}$$

2. *The odd number* $\Pi$ *in the equation* (2.1) *has symmetry* $R$ *when there exists an index R so that*

$$\begin{aligned} &\beta_R = -1 \\ &\beta_{R-1} = \beta_{R-2} = ... = \beta_1 = \beta_0 = +1. \\ &R \in \left\{1, 2, 3, ..., \nu - 1\right\} \end{aligned} \tag{3.2}$$

Next, we have one example:

**Example 3.1.** The prime number Q=5686306475353569551690334109408678048393607420608184 33 is a factor of $F_{12}=2^{4096}+1$ . From the equation (2.6) we have $\nu+1=178$, and then from equation 2.1 we have

$$Q=2^{178}+2^{177}-2^{176}+2^{175}+2^{174}+2^{173}+2^{172}-2^{171}+2^{170}+2^{169}+2^{168}+2^{167}+2^{166}$$
$$+2^{165}-2^{164}+2^{163}-2^{162}-2^{161}-2^{160}-2^{159}+2^{158}+2^{157}+2^{156}-2^{155}-2^{154}-2^{153}-2^{152}$$
$$-2^{151}+2^{150}-2^{149}+2^{148}-2^{147}-2^{146}+2^{145}-2^{144}+2^{143}-2^{142}-2^{141}-2^{140}+2^{139}+2^{138}$$
$$-2^{137}-2^{136}+2^{135}-2^{134}-2^{133}+2^{132}-2^{131}+2^{130}-2^{129}+2^{128}-2^{127}+2^{126}-2^{125}-2^{124}$$
$$-2^{123}-2^{122}-2^{121}+2^{120}-2^{119}+2^{118}-2^{117}+2^{116}-2^{115}+2^{114}-2^{113}-2^{112}-2^{111}-2^{110}$$
$$-2^{109}-2^{108}+2^{107}-2^{106}+2^{105}-2^{104}+2^{103}-2^{102}+2^{101}-2^{100}+2^{99}+2^{98}-2^{97}+2^{96}-2^{95}$$
$$-2^{94}+2^{93}-2^{92}+2^{91}+2^{90}-2^{89}+2^{88}-2^{87}+2^{86}+2^{85}+2^{84}-2^{83}+2^{82}-2^{81}+2^{80}+2^{79}$$
$$-2^{78}-2^{77}-2^{76}-2^{75}+2^{74}+2^{73}-2^{72}-2^{71}-2^{70}+2^{69}+2^{68}+2^{67}+2^{66}+2^{65}+2^{64}-2^{63}$$
$$-2^{62}+2^{61}-2^{60}-2^{59}-2^{58}-2^{57}-2^{56}+2^{55}-2^{54}-2^{53}-2^{52}-2^{51}-2^{50}-2^{49}+2^{48}+2^{47}$$
$$-2^{46}+2^{45}+2^{44}+2^{43}+2^{42}-2^{41}-2^{40}+2^{39}-2^{38}-2^{37}-2^{36}+2^{35}-2^{34}-2^{33}+2^{32}+2^{31}$$
$$-2^{30}+2^{29}+2^{28}+2^{27}+2^{26}+2^{25}+2^{24}-2^{23}+2^{22}+2^{21}+2^{20}-2^{19}-2^{18}-2^{17}-2^{16}+2^{15}$$
$$+2^{14}-2^{13}-2^{12}-2^{11}-2^{10}-2^{9}-2^{8}-2^{7}-2^{6}-2^{5}-2^{4}-2^{3}-2^{2}-2^{1}-1$$

So the factor 5686306475353569551690334109408678048393607420608184 33 of $F_{12}$ has symmetry $L$ (5686306475353569551690334109408678048393607420608184 33)=14.

As a consequence of definition 3.1, for odd numbers of the form $Q$ is $\beta_0=-1$ and for odd numbers of the form $D$ is $\beta_0=+1$. Also, from equation (2.1) and definitions 3.1 it can be easily proved that the *Q* odds are written in the form

$$\begin{aligned} Q&=2^{L+1}\cdot K+1 \\ K&=odd \end{aligned} \tag{3.3}$$

and the $D$ odds in the form

$$\begin{aligned} D&=2^{R+1}\cdot K-1 \\ K&=odd \end{aligned}. \tag{3.4}$$

*Proof*. We prove equation (3.3) and (3.4) is proven similarly. From definition 3.1 we get

$$Q=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+...+\beta_{L+1}2^{L+1}+2^{L}-2^{L-1}-2^{L-2}-...-2^{2}-2^{1}-1$$

$$\nu+1=\left\lceil\frac{\ln Q}{\ln 2}\right\rceil$$

$$Q=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+...+\beta_{L+1}2^{L+1}+2^{L}-\left(2^{L-1}+2^{L-2}+...+2^{2}+2^{1}+1\right)$$

$$Q=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+...+\beta_{L+1}2^{L+1}+2^{L}-\left(2^{L}-1\right)$$

$$Q=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+...+\beta_{L+1}2^{L+1}+1$$

$$Q = 2^{L+1}\left(2^{\nu-L} + 2^{\nu-L-1} + \beta_{\nu-1}2^{\nu-L-2} + \beta_{\nu-2}2^{\nu-L-3}... + \beta_{L+1}\right)+1$$

$$Q = 2^{L+1}K + 1$$

$$K = 2^{\nu-L} + 2^{\nu-L-1} + \beta_{\nu-1}2^{\nu-L-2} + \beta_{\nu-2}2^{\nu-L-3}... + \beta_{L+1}. \square$$

It follows from Lucas theorem for the Fermat numbers (see, [1,2]) that the factors of the Fermat numbers are of the form (3.3).Equations (3.3), (3.4) provides the simplest way for the determination of the symmetry of a number. We give two examples.

**Example 3.2.** For odd number 18303 we have

$$\begin{matrix} 18303-1 = 2^1 \times 9151 \\ 18303+1 = 2^7 \times 143 \end{matrix}.$$

Therefore, $R(18303) = 7-1 = 6$. Indeed, from equation (2.6) we get $\nu = 13$and from equation (2.1) we obtain

$$18303 = 2^{14} + 2^{13} - 2^{12} - 2^{11} - 2^{10} + 2^9 + 2^8 + 2^7 - 2^6 + 2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 1.$$

**Example 3.3.** For the number C1133 which is composite factor of $F_{12}$ with 1133 digits, we have

$$C1133 - 1 = 2^{14} \cdot K.$$

Therefore, $L(C1133) = 14-1 = 13$.

It is easy to prove the following corollary:

**Corollary 3.1.**

1. $Q_1Q_2 = Q$.

2. $D_1D_2 = Q$.

3. $Q_1D_1 = D$.

4. $L(Q_1) < L(Q_2) \Rightarrow L(Q_1Q_2) = L(Q_1)$.

5. $L(Q) < R(D) \Rightarrow R(QD) = L(Q)$.

6. $R(D) < L(Q) \Rightarrow R(QD) = R(D)$.

7. $R(D_1) < R(D_2) \Rightarrow L(D_1D_2) = R(D_1)$.

8. $Symmetry(\Pi_1) = Symmetry(\Pi_2) \Rightarrow Symmetry(\Pi_1\Pi_2) > Symmetry(\Pi_1) = Symmetry(\Pi_2)$.

We give two examples:

**Example 3.4.** $L$ (641)=6< $L$ (114689)=13 => $L$ (641×114689)=6.

**Example 3.5.** $R$ (607)= 4< $R$ (16633)=6 => $L$ (607×16633)=4.

We now prove the following corollary:

**Corollary 3.2.** *Every composite number* $C$ *of the form*

$$C = 2^{\nu+1} \pm 1$$

*has at least two factors the symmetries of which have equal values.*

*Proof.* Corollary 3.2 is a direct consequence of corollary 3.1. □

From definitions 2.1 and 3.1 it emerges that for every conjugate pair $\left(\Pi, \Pi^*\right)$, one is of form $Q$ and the other of form $D$.

# 4. Transpose of odd number

We now give the following definition:

**Definition 4.1.** 1. *We write the odd D in the form of equation (2.1),*

$$\begin{aligned} &D = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_1 2^1 + 1 \\ &\nu + 1 = \left\lfloor \frac{\ln D}{\ln 2} \right\rfloor \end{aligned} \quad . \tag{4.1}$$

*We define as transpose* $T\left(D\right)$ *of* $D$ *the odd that emerges by inverting the powers of 2 in equation (4.1) and multiplying the resulting number by* $2^{\nu+1}$*,*

$$T\left(D\right) = \left( \frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} + 1 \right) \cdot 2^{\nu+1} = 2^{\nu+1} + 3 + \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} . \tag{4.2}$$

2. *We write the odd* $Q$ *in the form of equation (2.1),*

$$\begin{aligned} &Q = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + \ldots + \beta_1 2^1 - 1 \\ &\nu + 1 = \left\lfloor \frac{\ln Q}{\ln 2} \right\rfloor \end{aligned} \quad . \tag{4.3}$$

*We define as transpose* $T\left(Q\right)$ *of* $Q$ *the opposite of the odd that emerges by inverting the powers of 2 in equation (4.3) and multiplying the resulting number by* $2^{\nu+1}$*,*

$$T\left(Q\right) = -\left( \frac{1}{2^{\nu+1}} + \frac{1}{2^{\nu}} + \frac{\beta_{\nu-1}}{2^{\nu-1}} + \frac{\beta_{\nu-2}}{2^{\nu-2}} + \ldots + \frac{\beta_1}{2^1} - 1 \right) \cdot 2^{\nu+1} = 2^{\nu+1} - 3 - \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} . \tag{4.4}$$

3. *We set*

$$T\left(1\right) = 1. \tag{4.5}$$

4. *From equations (4.2), (4.4), (4.5) we get the general equation*

$$\begin{aligned} &T\left(\Pi\right) = 2^{\nu+1} + \beta_0 \cdot \left( 3 + \sum_{k=1}^{\nu-1} \beta_k \cdot 2^{\nu+1-k} \right) \\ &\nu + 1 = \left\lfloor \frac{\ln \Pi}{\ln 2} \right\rfloor \end{aligned} \quad . \tag{4.6}$$

**Algorithm for the calculation of the transpose.** For the odd $\Pi$ we calculate $\nu+1=\left[\frac{\ln\Pi}{\ln 2}\right]$ from equation (2.6). Next, applying the algorithm of example 2.1 we write $\Pi$ in the form of equation (2.1), we calculate $\beta_i=\pm 1, i=1,2,3,...,\nu-1$ and the transpose $T(\Pi)$ of $\Pi$ from equation (4.6).

We now prove three theorems about the transpose of an odd number:

**Theorem 4.1.**

1. $T(\Pi)=1\Leftrightarrow\Pi=2^{\nu}-3,\nu\geq 2,\nu\in\mathbb{N}$. (4.7)

2.

$$\left.\begin{array}{l} T(D)=D \\ \nu+1=\left[\frac{\ln D}{\ln 2}\right] \end{array}\right\}\Leftrightarrow\begin{cases}\beta_1=1\\ \beta_{\nu-k}=\beta_{k+1}\\ k=1,2,3,...,\frac{\nu-2}{2},\nu=even\\ k=1,2,3,...,\frac{\nu-1}{2},\nu=odd\end{cases} \tag{4.8}$$

$$\left.\begin{array}{l} T(Q)=Q \\ \nu+1=\left[\frac{\ln Q}{\ln 2}\right] \end{array}\right\}\Leftrightarrow\begin{cases}\beta_1=-1\\ \beta_{\nu-k}=-\beta_{k+1}\\ k=1,2,3,...,\frac{\nu-2}{2},\nu=even\cdot\\ k=1,2,3,...,\frac{\nu-1}{2},\nu=odd\end{cases} \tag{4.9}$$

*Proof.* 1. For $\Pi=2^{\nu}-3$ we get

$$\Pi=2^{\nu}-3$$
$$\Pi=\left(2^{\nu}-1\right)-2=\left(2^{\nu-1}+2^{\nu-2}+2^{\nu-2}+...+2^1+1\right)-2$$
$$\Pi=2^{\nu-1}+2^{\nu-2}+2^{\nu-2}+...+2^1-1=Q$$

and from equation (4.4) we get $T(\Pi)=1$.

Now, let $T(\Pi)=1$. The odd $\Pi$ is either of the form $D$ or of the form $Q$. We prove the theorem for $\Pi=Q$, and the proof is similar for $\Pi=D$.

For $\Pi=Q$ we get

$$\Pi=Q=2^{n+1}+2^n+\beta_{n-1}2^{n-1}+\beta_{n-2}2^{n-2}+...+\beta_2 2^2+\beta_1 2^1-1$$
$$n+1=\left[\frac{\ln\Pi}{\ln 2}\right] \tag{4.10}$$

and from equation (4.4) we get

$$T(\Pi) = -1 - 2^1 - \beta_{n-1}2^2 - ... - \beta_2 2^n + 2^{n+1}$$

so we get

$$T(\Pi) = 1$$

$$-1 - 2^1 - \beta_{n-1}2^2 - ... - \beta_2 2^n + 2^{n+1} = 1$$

$$-1 - 2^1 - \beta_{n-1}2^2 - ... - \beta_2 2^n + 2^{n+1} + 2^{n+2} = 2^{n+2} + 1$$

$$-1 - 2^1 - \beta_{n-1}2^2 - ... - \beta_2 2^n + 2^{n+1} + 2^{n+2} = 2^{n+2} + 2^n - 2^{n-1} - 2^{n-2} - ... - 2^2 - 2^1 - 1$$

and taking into account that every odd $\Pi$ is written in a unique way in the form of equation (2.1) we get

$$\beta_2 = \beta_3 = \beta_3 = ... = \beta_{n-1} = +1$$

and from equation (4.10) we get

$$\Pi = 2^{n+1} + 2^n + 2^{n-1} ... + 2^1 - 1$$

$$\Pi = 2\left(2^n + 2^{n-1} + 2^{n-2} + ... + 2^1 + 1\right) - 1$$

$$\Pi = 2\left(2^{n+1} - 1\right) - 1$$

$$\Pi = 2^{n+2} - 3$$

and setting $n+2 = \nu$ we obtain $\Pi = 2^{\nu} - 3$.

2. We prove the equivalence (4.8), and (4.9) is similarly proven. We write the odd $D$ in the form of equation (2.1),

$$D = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + ... + \beta_1 2^1 + 1$$

$$\nu + 1 = \left[\frac{\ln D}{\ln 2}\right] \quad . \qquad (4.11)$$

From equations (4.11), (4.2) we get

$$T(D) = 1 + 2 + \beta_{\nu-1}2^2 + \beta_{\nu-2}2^3 + ... + \beta_2 2^{\nu} + 2^{\nu+1} . \qquad (4.12)$$

We next get

$$T(D) = D \Leftrightarrow$$

$$1 + 2 + \beta_{\nu-1}2^2 + \beta_{\nu-2}2^3 + ... + \beta_2 2^{\nu} + 2^{\nu+1} = 2^{\nu+1} + 2^{\nu} + \beta_{\nu-1}2^{\nu-1} + \beta_{\nu-2}2^{\nu-2} + ... + \beta_2 2^2 + \beta_1 2^1 + 1$$

and taking into account that the odd $D$ is written in a unique way in the form of equation (2.1) we get equivalence (4.8). □

**Theorem 4.2.**

1. $T(D) - T\left(D^*\right) = 6$. (4.13)

2. $T(Q) - T\left(Q^*\right) = -6$. (4.14)

*Proof.* We prove equation (4.13) and (4.14) is proven similarly. From equation (4.11) we get

$$D^* = 2^{\nu+1} + 2^{\nu} - \beta_{\nu-1}2^{\nu-1} - \beta_{\nu-2}2^{\nu-2} - ... - \beta_1 2^1 - 1 = Q. \quad (4.15)$$

From equation (4.4) we get

$$T\left(D^*\right) = -1 - 2 + \beta_{\nu-1}2^2 + \beta_{\nu-2}2^3 + ... + \beta_1 2^{\nu} + 2^{\nu+1}. \quad (4.16)$$

From equation (4.12), (4.16) we obtain $T(D) - T\left(D^*\right) = 6$. □

**Theorem 4.3.** *For every odd* $\Pi$, $\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right]$, $\Pi \in \Omega_\nu = \left[2^{\nu+1}, 2^{\nu+2}\right]$ *the following inequality holds*

$$T(\Pi) < 2^{\nu+2}. \quad (4.17)$$

*Proof.* We prove inequality (4.17) for the $D$ odds and the proof is similar for the $Q$ odds. From equation (4.12) and taking into account that $\beta_i = \pm 1, i = 0,1,2,...,\nu - 1$ we obtain

$$T(D) = 1 + 2 + \beta_{\nu-1}2^2 + \beta_{\nu-2}2^3 + ... + \beta_2 2^{\nu} + 2^{\nu+1} \le 1 + 2 + 2^2 + ... + 2^{\nu} + 2^{\nu+1} = 2^{\nu+2} - 1$$
$$T(D) \le 2^{\nu+2} - 1 < 2^{\nu+2}$$

. □

From inequality (4.17) it follows that if an odd $\Pi$ belongs to the interval $\Omega_\nu = \left[2^{\nu+1}, 2^{\nu+2}\right]$, its transpose $T(\Pi)$ can be found in intervals $\Omega_n, n \le \nu$ but cannot be found in intervals $\Omega_N, N > \nu$.

# 5. The odd number octet

We now give the following definitions:

**Definition 5.1.** *We define as the octet of odd number* $\Pi$ *the non ordered octet*

$$\left(\Pi, T(\Pi), \left(T(\Pi)\right)^*, T\left(\left(T(\Pi)\right)^*\right), \Pi^*, T\left(\Pi^*\right), \left(T\left(\Pi^*\right)\right)^*, T\left(\left(T\left(\Pi^*\right)\right)^*\right)\right). \quad (5.1)$$

**Definition 5.2.** 1. *We define as the* $8^*$ *transformation of the (5.1) octet, the octet that contains the conjugates of the (5.1) octet.*

2. *We define as the* $T_8$ *transformation of the (5.1) octet, the octet that contains the transposes of the (5.1) octet.*

**Definition 5.3.** *We define as symmetric every octet that remains unchanged under the* $T_8$ *transformation.*

**Properties of the symmetric octet.** As a consequence of definition 5.3 and the theorems of the previous section, every symmetric octet has the following properties:

1. It consists of pairs of conjugate odds (common property of all octets).
2. It is unchanged under the $8^*$ transformation (common property of all octets).
3. It is unchanged under the $T_8$ transformation (definition): for every odd $\Pi$ of the symmetric octet, it is $T\left(T(\Pi)\right) = \Pi$.

4. It consists of pairs of odds that differ by 6 (consequence of theorem 4.2).

5. It consists of 8 or 4 or 2 different odds.

6. The numbers of the symmetric octet belong to the same interval $\Omega_\nu = \left[2^{\nu+1}, 2^{\nu+2}\right]$,

$\nu + 1 = \left[\frac{\ln \Pi_k}{\ln 2}\right]$, where $\Pi_k, k = 1, 2, 3, ..., 8$ the octet numbers.

7. The smallest odd $m$ of a symmetric octet is always of the form $Q$, $m = Q$ and the largest $M$ is its conjugate, $M = Q^*$.

8. An odd $\Pi$ can belong to exactly one symmetric octet (while every odd which belongs to a symmetric octet, also belongs to infinite non symmetric octets, as we shall see later).

In order to determine the octet of an odd $\Pi$ we use the algorithm for the calculation of the transpose, and equation (2.17) for the calculation of the conjugate. We now give an example which also shows the ways in which we can write a symmetric octet.

**Example 5.1.** From equation (5.1) we get the symmetric octet in which $\Pi = 889$ belongs,

(889, 529, 1007, 895, 647, 535, 1001, 641).

In order to discern the pairs of transposes and of conjugates, we write the octet in the form

$$\begin{array}{ccccccc} 889 & \overset{T}{\longleftrightarrow} & 529 & \overset{*}{\longleftrightarrow} & 1007 & \overset{T}{\longleftrightarrow} & 895 \\ {\scriptstyle *}\updownarrow & & & & & & \updownarrow{\scriptstyle *} \\ 647 & \overset{T}{\longleftrightarrow} & 535 & \overset{*}{\longleftrightarrow} & 1001 & \overset{T}{\longleftrightarrow} & 641 \end{array} .$$

Because of equation (2.16) $\left(\Pi^*\right)^* = \Pi$ two conjugates are always connected, in all symmetries, by the symbol $\overset{*}{\longleftrightarrow}$, $\Pi \overset{*}{\longleftrightarrow} \Pi^*$. With $\Pi_1 \overset{T}{\longleftrightarrow} \Pi_2$ we denote that $T\left(\Pi_1\right) = \Pi_2$ and $T\left(\Pi_2\right) = \Pi_1$. If $T\left(\Pi_1\right) = \Pi_2$ and $T\left(\Pi_2\right) \neq \Pi_1$, we write $\Pi_1 \overset{T}{\longrightarrow} \Pi_2$. In our example, $\Pi_1 \overset{T}{\longleftrightarrow} \Pi_2$ for all of the octet numbers, therefore, it is unchanged under the $T_8$ transformation and consequently it is symmetric (definition). The $8^*$ and $T_8$ transformations only change the relative position of the numbers within the symmetric octet. The octet symmetries are easily seen when we place the numbers on the corners of a regular octagon,

$$
\begin{array}{ccccc}
 & & 889 & \overset{T}{\longleftrightarrow} & 529 & & \\
 & \overset{*}{\swarrow\!\nearrow} & & & & \overset{*}{\nwarrow\!\searrow} & \\
647 & & & & & & 1007 \\
T\updownarrow & & & & & & \updownarrow T \\
535 & & & & & & 895 \\
 & \overset{*}{\nwarrow\!\searrow} & & & & \overset{*}{\swarrow\!\nearrow} & \\
 & & 1001 & \overset{T}{\longleftrightarrow} & 641 & &
\end{array}
\quad . \qquad (5.2)
$$

A symmetric octet can be composed of 8 different numbers, like the one of the previous example, or of 4 different numbers or of 2 different numbers (with the exception of the degenerate octets $(1,1,1,1,1,1,1,1)$ of $1$ and $(3,3,3,3,3,3,3,3)$ of $3$). From the definitions of the conjugate and the transpose, the following equations are easily proven

$$
\begin{array}{l}
2^{\nu}+1 \overset{*}{\longleftrightarrow} 2^{\nu+1}-1 \\
2^{\nu}+7 \overset{*}{\longleftrightarrow} 2^{\nu+1}-7 \\
2^{\nu}+1 \overset{T}{\longleftrightarrow} 2^{\nu+1}-7 \\
2^{\nu}+7 \overset{T}{\longleftrightarrow} 2^{\nu}+7 \\
2^{\nu+1}-1 \overset{T}{\longleftrightarrow} 2^{\nu+1}-1
\end{array}
\quad . \qquad (5.3)
$$

Considering equations (5.3) we get the symmetric octets

$$
\begin{array}{l}
\left(2^{\nu+1}+1, 2^{\nu+2}-7, 2^{\nu+1}+7, 2^{\nu+1}+7, 2^{\nu+2}-1, 2^{\nu+2}-1, 2^{\nu+1}+1, 2^{\nu+2}-7\right) \\
\nu \geq 3, \nu \in \mathbb{N}
\end{array}
\quad . \qquad (5.4)
$$

The (5.4) symmetric octets consist of 4 different numbers. Fermat numbers for $\nu+1=2^{S}, S \in \mathbb{N}$ and Mersenne numbers for $\nu+2=p=prime$ belong to the (5.4) symmetric octets. The symmetric octet $(9,9,15,15,15,15,9,9)$ of conjugates $(\Pi,\Pi^{*})=(9,15)$ consists of 2 numbers.

**Definition 5.4.** *We define as non-symmetric or asymmetric every octet that contains a pair of conjugates* $(\Pi_1,\Pi_1^{*})$ *for which*

$$
\begin{array}{l}
\Pi_1 \overset{T}{\longrightarrow} \Pi_2 \wedge \Pi_2 \overset{T}{\longleftrightarrow} \Pi_3 \neq \Pi_1 \\
\Pi_1^{*} \overset{T}{\longrightarrow} \Pi_4 \wedge \Pi_4 \overset{T}{\longleftrightarrow} \Pi_5 \neq \Pi_1^{*}
\end{array}
. \qquad (5.5)
$$

**Asymmetric octets as generators of symmetric octets.** An asymmetric octet contains pairs of conjugates that belong in different intervals $\Omega_{\nu}$. At the position of these pairs, "bifurcations" emerge outside the (5.1) octet. At each of these bifurcations, an asymmetric octet "produces" a symmetric one.

We now present one example of an asymmetric octet in which one can see the way in which we can write it so that the asymmetry is evident and so are the symmetric octet that it produces.

**Example 5.2.** The conjugate pair $(91,101)$ gives the asymmetric octet

$$\begin{array}{ccccccc} & & 47 & & & & \\ & & \updownarrow T & & & & \\ 91 & \xrightarrow{T} & 55 & \xleftrightarrow{*} & 41 & \xleftrightarrow{T} & 41 \\ {}^{*}\updownarrow & & & & & & \updownarrow^{*} \\ 101 & \xrightarrow{T} & 49 & \xleftrightarrow{*} & 47 & \xleftrightarrow{T} & 55 \\ & & \updownarrow T & & & & \\ & & 49 & & & & \end{array}.$$

The asymmetric octet has produces the symmetric octet

$$\begin{array}{ccccccc} 47 & \xleftrightarrow{T} & 55 & \xleftrightarrow{*} & 41 & \xleftrightarrow{T} & 41 \\ {}^{*}\updownarrow & & & & & & \updownarrow^{*} \\ 49 & \xleftrightarrow{T} & 49 & \xleftrightarrow{*} & 47 & \xleftrightarrow{T} & 55 \end{array}$$

which emerges by replacing the pair of conjugates $(91,101)$ by the pair of conjugates $(47,49)$.

We now prove the following theorem:

**Theorem 5.1.** (Fundamental octet theorem)

A. *The octets of the numbers*

$$\begin{aligned} &\Pi = D = 2^2 \cdot K - 1 \\ &\Pi^* = Q = 2^2 \cdot \left(3 \cdot 2^{\nu-1} - K\right) + 1 \\ &\nu + 1 = \left[\frac{\ln \Pi}{\ln 2}\right] \geq 3, K = odd \end{aligned} \tag{5.6}$$

*are asymmetric.*

B.

1. *The asymmetric octets are given by the odds*

$$\begin{aligned} &D_1 = 11 + 8m, m \in \mathbb{N} \\ &Q_1 = 13 + 8m, m \in \mathbb{N} \end{aligned} \tag{5.7}$$

*except from the asymmetric octet* $(5,1,1,1,7,7,5,1)$.

2. *The symmetric octets are given by the odds*

$$\begin{aligned} &D_2 = 7 + 8m, m \in \mathbb{N}^* \\ &Q_2 = 9 + 8m, m \in \mathbb{N} \end{aligned}. \tag{5.8}$$

*Proof.* A. From 2 of theorem 4.1 it follows that for the odds $\Pi$, for which $\beta_0\beta_1=-1$, it holds that $T(\Pi)\neq\Pi$,

$$\beta_0\beta_1=-1\Rightarrow T(\Pi)\neq\Pi. \tag{5.9}$$

$D=2^3+2^2-2+1=11$ and $Q=2^3+2^2+2-1=13$ are the smallest odds for which it can be $\beta_0\beta_1=-1$. Therefore, the theorem holds for $\nu+1\geq 3$.

We prove the theorem for $\Pi=D$ and it is similarly proven for $\Pi=Q$. From equation (4.11) for $\beta_0\beta_1=-1$ we get

$$\begin{aligned}\Pi=D&=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+\ldots+\beta_2 2^2-2+1\\&=2^2\cdot\left(2^{\nu-1}+2^{\nu-2}+\beta_{\nu-1}2^{\nu-3}+\beta_{\nu-2}2^{\nu-3}+\ldots+\beta_2\right)-2+1\\&=2^2\cdot\left(2^{\nu-1}+2^{\nu-2}+\beta_{\nu-1}2^{\nu-3}+\beta_{\nu-2}2^{\nu-3}+\ldots+\beta_2\right)-1\end{aligned}$$

and setting

$$K=2^{\nu-1}+2^{\nu-2}+\beta_{\nu-1}2^{\nu-3}+\beta_{\nu-2}2^{\nu-3}+\ldots+\beta_2 \tag{5.10}$$

we get $D=2^2\cdot K-1$. Considering equation (5.10) we calculate the conjugate of $D$ which is $\Pi^*=Q=2^2\cdot\left(3\cdot 2^{\nu-1}-K\right)+1$.

B. Equations (5.6) give the consecutive pairs $(D,Q)=(11,13),(19,21),(27,29),\ldots$, that is, equations (5.7). Taking into account the fourth property of the symmetric octets we conclude that the intermediate pairs of odds give the symmetric octets. These pairs are given by equations (5.8). Additionally, equations (5.7) give all odds that produce asymmetric octets (with the exception of the asymmetric octet $(5,1,1,1,7,7,5,1)$) and the equations give all the odds that produce symmetric octets (otherwise the fourth property of the symmetric octets would not hold, which cannot be true due to theorem 4.2)

The octet

$$(5,1,1,1,7,7,5,1) \tag{5.11}$$

of the conjugate pair $(\Pi,\Pi^*)=(5,7)$ is the only asymmetric octet that does not belong to the (5.7) octets, since for this conjugate pair it is

$$\left[\frac{\ln 5}{\ln 2}\right]=\left[\frac{\ln 7}{\ln 2}\right]=\nu+1=2<3.$$

The asymmetric octet (5.11) is given by the terms of the $\varepsilon_n$ sequence of odds,

$$\begin{matrix}\varepsilon_n=3+2^{n+1}\\ n=1,2,3,\ldots\end{matrix}. \tag{5.12}$$

The (5.11) octet emerges from the terms of the $\varepsilon_n$ sequence because of equation

$$T\left(\gamma_n\right)=T\left(2^{n+1}+3\right)=\varepsilon_1=7 \quad . \qquad (5.13)$$
$$n=1,2,3,...$$

The terms of the $\varepsilon_n$ sequence are of the form $D$. From equation (4.13) we get

$$T\left(2^{n+1}+3\right)-T\left(\left(2^{n+1}+3\right)^*\right)=6$$

$$T\left(2^{n+1}+3\right)-T\left(2^{n+2}-3\right)=6$$

and with equation (4.7) we obtain

$$T\left(2^{n+1}+3\right)-1=6$$

$$T\left(2^{n+1}+3\right)=7$$

which is equation (5.13). □

For the odds $Q$, $D$ that belong to a symmetric octet we make the following conjecture:

**Conjecture 5.1.** A. 1. *For every odd* $Q$ *it holds that*

$$T\left(2^n\cdot\left(Q+3\right)-3\right)=T\left(Q\right) \qquad (5.14)$$
$$\forall n\in\mathbb{N}$$

$$T\left(2^n\cdot\left(Q+3\right)+3\right)=\left(T\left(Q\right)\right)^* \quad . \qquad (5.14a)$$
$$\forall n\in\mathbb{N}$$

2. *For every odd* $D$ *it holds that*

$$T\left(2^n\cdot\left(D-3\right)+3\right)=T\left(D\right) \qquad (5.15)$$
$$\forall n\in\mathbb{N}$$

$$T\left(2^n\cdot\left(D-3\right)-3\right)=\left(T\left(D\right)\right)^* \quad . \qquad (5.15a)$$
$$\forall n\in\mathbb{N}$$

B. 1. *The sequences*

$$\gamma_n\left(Q\right)=2^n\cdot\left(Q-3\right)-3$$
$$\Gamma_n\left(Q\right)=2^n\cdot\left(Q-3\right)+3 \qquad (5.16)$$
$$n\in\mathbb{N}^*$$

*derive the same symmetric octet for each* $n\in\mathbb{N}^*$.

2. *The sequences*

$$\Delta_n(D) = 2^n \cdot (D+3) + 3$$
$$\delta_n(Q) = 2^n \cdot (D+3) - 3 \qquad (5.17)$$
$$n \in \mathbb{N}^*$$

*derive the same symmetric octet for each* $n \in \mathbb{N}^*$.

From conjecture 5.1 and the definition of the symmetric octet, the following corollary emerges directly:

**Corollary 5.1.** 1. *For every odd* $Q$ *that belongs to a symmetric octet the sequence*

$$a_n(Q) = 2^n \cdot (Q+3) - 3$$
$$A_n(Q) = 2^n \cdot (Q+3) + 3 \qquad (5.18)$$
$$n \in \mathbb{N}^*$$

*gives (infinite) asymmetric octets that produce the symmetric octet to which* $Q$ *belongs*.

2. *For every odd* $D$ *that belongs to a symmetric octet the sequence*

$$B_n(D) = 2^n \cdot (D-3) + 3$$
$$\beta_n(D) = 2^n \cdot (D-3) - 3 \qquad (5.19)$$
$$n \in \mathbb{N}^*$$

*gives (infinite) asymmetric octets that produce the symmetric octet to which* $D$ *belongs*.

The $a_n(Q)$ sequence has exactly one term in every interval

$$\Omega_N = \left[2^{N+1}, 2^{N+2}\right], N \in \mathbb{N}, N+1 > \left[\frac{\ln Q}{\ln 2}\right].$$

The $B_n(D)$ sequence has exactly one term in every interval

$$\Omega_N = \left[2^{N+1}, 2^{N+2}\right], N \in \mathbb{N}, N+1 > \left[\frac{\ln D}{\ln 2}\right].$$

We give an example:

**Example 5.3.** We pick a number from the symmetric octet (5.2), for example $647 = D$ and a random $n = 20$. From equation (5.19) we get $B_{20}(647) = 2^{20} \cdot (647-3) + 3 = 675282947$. The odd $675282947$ gives the asymmetric octet

$$
\begin{array}{ccccccc}
 & & 647 & & & & \\
 & & \updownarrow T & & & & \\
675282947 & \xrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \\
\overset{*}{\updownarrow} & & & & & & \updownarrow^{*} \\
935329789 & \xrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895 \\
 & & \updownarrow T & & & & \\
 & & 889 & & & &
\end{array}
$$

which produces the symmetric octet

$$
\begin{array}{ccccccc}
647 & \xleftrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \\
\overset{*}{\updownarrow} & & & & & & \updownarrow^{*} \\
889 & \xleftrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895
\end{array}
$$

which is (5.2). $B_{20}\left(647\right)=675282947\in\Omega_{28}$ belongs to the interval $\Omega_{28}=\left[2^{29},2^{30}\right]$.

$B_{21}\left(647\right)=2^{21}\cdot\left(647-3\right)+3=1350565891\in\Omega_{29}$ also produces the (5.2) symmetric octet and belongs to the next interval $\Omega_{29}=\left[2^{30},2^{31}\right]$,

$$
\begin{array}{ccccccc}
 & & 647 & & & & \\
 & & \updownarrow T & & & & \\
1350565891 & \xrightarrow{T} & 535 & \xleftrightarrow{*} & 1001 & \xleftrightarrow{T} & 641 \\
\overset{*}{\updownarrow} & & & & & & \updownarrow^{*} \\
1870659581 & \xrightarrow{T} & 529 & \xleftrightarrow{*} & 1007 & \xleftrightarrow{T} & 895 \\
 & & \updownarrow T & & & & \\
 & & 889 & & & &
\end{array}
.
$$

We now prove the following corollary:

**Corollary 5.2.** 1. *For every odd* $Q$ *belonging to a symmetric octet, it holds that*

$$T\left(2^{n}\cdot\left(T\left(Q\right)+3\right)-3\right)=T\left(T\left(Q\right)\right)=Q \quad \forall n\in\mathbb{N}. \tag{5.20}$$

2. *For every odd* $D$ *belonging to a symmetric octet, it holds that*

$$T\left(2^{n}\cdot\left(T\left(D\right)-3\right)+3\right)=T\left(T\left(D\right)\right)=D \quad \forall n\in\mathbb{N}. \tag{5.21}$$

*Proof.* We prove equation (5.20) and the proof of (5.21) is similar. From definition 5.3 of the symmetric octet it follows that if $Q$ belongs to a symmetric octet, then $T\left(Q\right)$ belongs to the same symmetric octet. Therefore, equation (5.14) also holds for $T\left(Q\right)$,

$$T\left(2^n\cdot\left(T\left(Q\right)+3\right)-3\right)=T\left(T\left(Q\right)\right)$$

$$\forall n\in\mathbb{N}^*$$

and taking into account that for the symmetric octet it is (definition) $T\left(T\left(Q\right)\right)=Q$ we get equation (5.20). □

Also, applying definition 5.1 for *even numbers* $2^n\cdot\Pi, n\in\mathbb{N}^*, \Pi = odd$ we obtain the following corollary:

**Corollary 5.3.** *For every odd* $\Pi$ *it holds that*

$$\begin{matrix} T\left(2^n\cdot\Pi\right)=T\left(\Pi\right) \\ \forall n\in\mathbb{N},\Pi=odd \end{matrix}. \tag{5.22}$$

*Proof.* We prove equation (5.22) for the $D$ odds and the proof is similar for the $Q$ odds.

$$D=2^{\nu+1}+2^{\nu}+\beta_{\nu-1}2^{\nu-1}+\beta_{\nu-2}2^{\nu-2}+...+\beta_1 2^1+1$$

$$\nu+1=\left[\frac{\ln D}{\ln 2}\right]$$

$$2^nD=2^{n+\nu+1}+2^{n+\nu}+\beta_{\nu-1}2^{n+\nu-1}+\beta_{\nu-2}2^{n+\nu-2}+...+\beta_1 2^{n+1}+2^n$$

$$T\left(2^nD\right)=\left(\frac{1}{2^{n+\nu+1}}+\frac{1}{2^{n+\nu}}+\frac{\beta_{\nu-1}}{2^{n+\nu-1}}+\frac{\beta_{\nu-2}}{2^{n+\nu-2}}+...+\frac{\beta_1}{2^{n+1}}+\frac{1}{2^n}\right)\cdot 2^{n+\nu+1}$$

$$T\left(2^nD\right)=\left(\frac{1}{2^{\nu+1}}+\frac{1}{2^{\nu}}+\frac{\beta_{\nu-1}}{2^{\nu-1}}+\frac{\beta_{\nu-2}}{2^{\nu-2}}+...+\frac{\beta_1}{2^{1}}+\frac{1}{2^{0}}\right)\cdot 2^{\nu+1}=T\left(D\right).\square$$

We give an example:

**Example 5.4.** $\Pi=Q=2021=2^{10}+2^9+2^8+2^7+2^6+2^5+2^4-2^3-2^2+2^1-1$

$$2^n\Pi=2^{10+n}+2^{9+n}+2^{8+n}+2^{7+n}+2^{6+n}+2^{5+n}+2^{4+n}-2^{3+n}-2^{2+n}+2^{1+n}-2^n$$

$$n\in\mathbb{N}$$

$$T\left(2^n\cdot\Pi\right)=-\left(\frac{1}{2^{10+n}}+\frac{1}{2^{9+n}}+\frac{1}{2^{8+n}}+\frac{1}{2^{7+n}}+\frac{1}{2^{6+n}}+\frac{1}{2^{5+n}}+\frac{1}{2^{4+n}}-\frac{1}{2^{3+n}}-\frac{1}{2^{2+n}}+\frac{1}{2^{1+n}}-\frac{1}{2^n}\right)\cdot 2^{10+n}$$

$$T\left(2^n\cdot\Pi\right)=-\left(\frac{1}{2^{10}}+\frac{1}{2^{9}}+\frac{1}{2^{8}}+\frac{1}{2^{7}}+\frac{1}{2^{6}}+\frac{1}{2^{5}}+\frac{1}{2^{4}}-\frac{1}{2^{3}}-\frac{1}{2^{2}}+\frac{1}{2^{1}}-1\right)\cdot 2^{10}=T\left(\Pi\right).$$

We now prove the following corollary:

**Corollary 5.4.** *For every odd* $A$ *that doesn't belong to a symmetric octet, it holds that*

$$A\in\Omega_\nu\Rightarrow T\left(A\right)\in\Omega_\mu,\mu<\nu. \tag{5.23}$$

*Proof.* Corollary 5.4 is a direct consequence of theorem 4.3 and definition 5.4. □

We complete section 5 with the following definition:

**Definition 5.5.** 1. *We define as asymmetric numbers, the numbers that don't belong to a symmetric octet (equations (5.6), (5.7))*

2. *We define as symmetric numbers, the numbers that belong to a symmetric octet (equations (5.8).*

# 6. The odd number quadruple

From the definition of the symmetric octet (5.1) and theorem 4.2 we have that every symmetric octet consists of two ordered symmetric quadruple of the form

$$\left(Q, Q+6, Q^*-6, Q^*\right)=\left(q_1, d_1, q_2, d_2\right), \tag{6.1}$$

so

$$\left(Q_1, Q_1+6, Q_1^*-6, Q_1^*, Q_2, Q_2+6, Q_2^*-6, Q_2^*\right). \tag{6.2}$$

If $Q_1 \neq Q_2$ then the symmetric octet consists of eight different numbers, while if $Q_1 = Q_2$ then it consists of four different numbers.

From the theorem 4.2 and the definition 5.1 we have that every odd number $\Pi$ belonging to a symmetric octet it belongs to the quadruple

$$\left(\Pi, T\left(\left(T\left(\Pi\right)\right)^*\right), T\left(\left(T\left(\Pi^*\right)\right)^*\right), \Pi^*\right). \tag{6.3}$$

If $\Pi = Q$ doesn't belong to a symmetric octet, then the quadruple in which it belongs is given by equation (6.1) in the form

$$\left(Q, Q+6, Q^*-6, Q^*\right)=\left(q_1, d_1, q_2, d_2\right). \tag{6.4}$$

So, the equation (6.4) gives the quadruple in which every odd number $\Pi$ belongs. Moreover, the quadruple (6.4) is easily calculated from equations (2.6) and (2.16) which give the conjugate $\Pi^*$ of the odd $\Pi$.

We now prove the following theorem:

**Theorem 6.1.** 1. *If the quadruple*

$$\left(q_1, d_1, q_2, d_2\right)$$

*belongs to the interval* $\Omega_v$ *then the quadruple*

$$\left(2^n\cdot\left(q_1+3\right)-3, 2^n\cdot\left(d_1-3\right)+3, 2^n\cdot\left(q_2+3\right)-3, 2^n\cdot\left(d_1-3\right)+3\right) \tag{6.5}$$

*belongs to the interval* $\Omega_{v+n}$.

2. *If the quadruple*

$$\left(q_1, d_1, q_2, d_2\right)$$

*belongs to the interval* $\Omega_\nu$ *then the quadruple*

$$\left(2^n\cdot\left(d_1-3\right)-3,2^n\cdot\left(q_1+3\right)+3,2^n\cdot\left(d_1-3\right)-3,2^n\cdot\left(q_2+3\right)+3\right) \tag{6.6}$$

*belongs to the interval* $\Omega_{\nu+n}$.

*Proof.* $\Pi$ That's the same quadruple which belongs to the interval $\Omega_{\nu+n}$,

$$2^n\cdot\left(q_1+3\right)-3+2^n\cdot\left(d_1-3\right)+3=2^n\cdot\left(d_1-3\right)+3+2^n\cdot\left(q_2+3\right)-3$$
$$=2^n\left(q_1+d_1\right)=2^n\left(q_2+d_2\right)=2^n\times3\times2^{\nu+1}=3\times2^{(\nu+n)+1}$$

and satisfies the equation (6.4). □

If in equation (6.4) we have $Q^*-Q=6$ then the quadruple consists of two different numbers. In addition, if we use the equation $Q^*+Q=3\times2^{\nu+1},\nu\in\mathbb{N}$ we take the quadruple

$$\left(q_1=3\times2^{\nu+1}-3,d_1=3\times2^{\nu+1}+3,q_2=3\times2^{\nu+1}-3,d_2=3\times2^{\nu+1}+3\right)$$
$$\nu\in\mathbb{N}$$

which consists of the pair

$$\left(Q=3\times2^{\nu+1}-3,Q^*=3\times2^{\nu+1}+3\right) \quad . \tag{6.7}$$
$$\nu\in\mathbb{N}$$

# 7. Strictly decreasing sequences of symmetric quadruples

We now prove the following theorem:

**Theorem 7.1.** *For every symmetric number* $\Pi$ *the number*

$$E\left(\Pi,\omega\right)=\Pi+4-8\omega \tag{7.1}$$
$$\omega\in\mathbb{N}$$

*is asymmetric*.

*Proof.* We prove the theorem for the $\Pi=Q=8m+9$ odds and the proof is similar for the $\Pi=D=8m+7$ odds. For the numbers of form $\Pi=8m+9$ (see equations (5.8)) we have

$E\left(9+8m,\omega\right)=9+8m+4-8\omega=8\left(m-\theta\right)+13$. □

By successive use of the asymmetric numbers (7.1) we get strictly decreasing (see corollary 5.4) sequences of symmetric quadruples. We give an example:

**Example 7.1.** For the symmetric number (see equation (5.4)) $\Pi=2^{32}+1$ and for $\omega=14562049$ in equation (7.1) we get the asymmetric number

$E\left(2^{32}+1,14562049\right)=2^{32}+1+4-8\cdot14562049=4178470909$

which produces the symmetric quadruple

$\left(6700417,6700423,5882489,5882495\right)$.

# 8. An algorithm for factorization of the symmetric odd numbers

Starting from a composite symmetric odd $\Pi = \alpha\beta \in \left[2^{\nu+1}, 2^{\nu+2}\right]$ we can determine the asymmetric odd (7.1) which derives a symmetric quadruple to which a factor of $\Pi$ belongs. For the execution of the algorithm, it is required the estimation of the intervals $\Omega_{\nu_1} = \left[2^{\nu_1+1}, 2^{\nu_1+2}\right]$ and $\Omega_{\nu_2} = \left[2^{\nu_2+1}, 2^{\nu_2+2}\right]$ to which the factors $\alpha$ and $\beta$ of the odd $\Pi = \alpha\beta$ belongs. From equations

$$\ln\Pi = \ln(\alpha\beta) = \ln\alpha + \ln\beta$$

and (2.1),

$$\nu + 1 = \left[\frac{\ln\Pi}{\ln 2}\right],\ \nu_1 + 1 = \left[\frac{\ln\alpha}{\ln 2}\right],\ \nu_2 + 1 = \left[\frac{\ln\alpha}{\ln 2}\right],$$

we get the possible values of $\nu_1$ and $\nu_2$. As a consequence of equation (2.1), the number of pairs $(\nu_1, \nu_2)$ increases logarithmically with respect to the increase of $\Pi$. If we cannot estimate the intervals to which the factors of $\Pi$ belongs then we execute the algorithm for all intervals of the form

$$\Omega_k = \left[2^k, 2^{k+1}\right], k = 1, 2, 3, ..., \nu - 1.$$

We present the four steps of the algorithm:

**A.** For numbers of the form $\Pi = Q = 8m + 9$.

**Step 1.** We estimate the intervals $\Omega_{\nu_1} = \left[2^{\nu_1}, 2^{\nu_1+1}\right]$, $\Omega_{\nu_2} = \left[2^{\nu_2}, 2^{\nu_2+1}\right]$ to which the factors of $\Pi$ belong.

**Step 2.** From equation (7.1) and replacing $Q+3$ with $K$ in equation (5.14) we get

$$\Pi + 4 - 8\omega = 2^n K - 3$$
$$K = odd$$
$$\Pi + 7 - 8\omega = 2^n K. \qquad (8.1)$$

**Step 3.** In equation (8.1) we gives values $\omega = 0, 1, 2, ....$ As we give these values to $\omega$, the natural $n$ in equation (8.1) changes. This change is periodic and we can determine the period for every value of the natural $n$. We choose the values of $\omega$ for which we have $n = \nu_2$ and $n = \nu_1$.

**Step 4.** For $n = \nu_1$, $n = \nu_2$ we calculate the values of $K$ in equation (8.1). We get the odd numbers

$$\Pi_1 = K + 2\theta,\ \theta \in \mathbb{Z}. \qquad (8.2)$$

For small values of $\theta$ a factor of $\Pi$ belongs to the octet of $\Pi_1$. There exists a value of $\omega$ for which $K = \alpha \vee K = \beta$, since every odd symmetric numbers belongs to its octet.

**B.** For numbers of the form $\Pi = D = 8m + 7$.

**Step 1.** We estimate the intervals $\Omega_{\nu_1} = \left[2^{\nu_1}, 2^{\nu_1+1}\right]$, $\Omega_{\nu_2} = \left[2^{\nu_2}, 2^{\nu_2+1}\right]$ to which the factors of $\Pi$ belong.

**Step 2.** From equation (7.1) and replacing $D-3$ with $K$ in equation (5.14) we get

$$\Pi + 4 - 8\omega = 2^n K + 3$$
$$K = odd$$
$$\Pi + 1 - 8\omega = 2^n K. \qquad (8.3)$$

**Step 3.** In equation (8.3) we give values $\omega = 0,1,2,....$ As we give these values to $\omega$ the natural $n$ in equation (8.3) changes. This change is periodic and we can determine the period for every value of the natural $n$. We choose the values of $\omega$ for which we have $n = \nu_1$ and $n = \nu_2$.

**Step 4.** For $n = \nu_1$, $n = \nu_2$ we calculate the values of $K$ in equation (8.3). We get the odd numbers

$$\begin{matrix} \Pi_1 = K + 2\theta \\ \theta \in \mathbb{Z} \end{matrix}. \qquad (8.4)$$

For small values of $\theta$ a factor of $\Pi$ belongs to the octet of $\Pi_1$. There exists a value of $\omega$ for which $K = \alpha \vee K = \beta$, since every odd symmetric numbers belongs to its octet.

We give two examples of application of the algorithm. In these examples we factorize small odd numbers in order to understand the algorithm.

**Example 8.1.** For the symmetric number $\Pi = Q = 2^{32} + 1 = 641 \times 6700417$, $641 \in \Omega_9$ and $6700417 \in \Omega_{22}$. Hence we have $\nu_1 = 9$ and $\nu_2 = 22$. For $n = \nu_2 = 22$, from equation (8.1) we get

$$\Pi + 7 - 8\omega = 2^{22} K$$
$$2^{32} + 1 + 7 - 8\omega = 2^{22} K.$$

Giving values $\omega = 0,1,2,...$ we determine the values of $\omega$ for which the power $2^{\nu_2} = 2^{22}$ is periodically appeared in the right side of the equation $\omega = 1 + (2k+1) \cdot 2^{19}, k \in \mathbb{N}$. To determine the equations of such type is something simple (see example 8.3). So we get the equation

$$\begin{matrix} 2^{32} + 1 + 7 - 8\left(1 + (2k+1) \cdot 2^{19}\right) = 2^{22} K \\ k = 0,1,2,... \end{matrix}. \qquad (8.5)$$

For $k = 0,1,2,...$ we get $K = 1023,1021,1019,....$ For $k = 8$ we get $K = 1007$ which gives the octet to which 641 belongs (see equation (5.2).

For $k = 191$ equation (8.5) gives

$$2^{32} + 1 + 7 - 8\left(1 + (2 \cdot 191 + 1) \cdot 2^{19}\right) = 2^{22} \cdot 641,$$
$$K = 641.$$

Every odd symmetric numbers belongs to its octet.

Giving values $\omega = 0,1,2,...$ we determine the values of $\omega$ for which the power $2^{\nu_1} = 2^9$ is periodically appeared in the right side of the equation $\omega = 65 + k \cdot 2^7, k \in \mathbb{N}$. So, we get the equation

$$\begin{matrix} 2^{32} + 1 + 7 - 8\left(65 + k \cdot 2^7\right) = 2^9 K \\ k = 0,1,2,... \end{matrix} . \qquad (8.6)$$

For $k = 0,1,2,...$ we get $K = 8388607, 8388605, 8388603, ....$ For $k = 113767$ we get $K = 8161073$ which gives the octet to which 6700417 belongs,

$$\begin{matrix} 8161073 & \overset{T}{\longleftrightarrow} & 6700417 & \overset{*}{\longleftrightarrow} & 5882495 & \overset{T}{\longleftrightarrow} & 8161079 \\ \overset{*}{\updownarrow} & & & & & & \updownarrow^{*} \\ 4421839 & \overset{T}{\longleftrightarrow} & 6700423 & \overset{*}{\longleftrightarrow} & 5882489 & \overset{T}{\longleftrightarrow} & 4421833 \end{matrix} .$$

For $k = 844095$ equation (8.6) gives

$2^{32} + 1 + 7 - 8\left(65 + 844095 \cdot 2^7\right) = 2^9 \cdot 6700417$,

$K = 6700417$.

**Example 8.2.** For the symmetric number $\Pi = D = 1030087 = 641 \times 1607$,

$641 \in \Omega_9$ and $1607 \in \Omega_{10}$. Hence we have $\nu_1 = 9$ and $\nu_2 = 10$. For $n = \nu_1 = 9$, from equation (8.4) we get

$\Pi + 1 - 8\omega = 2^9 K$

$1030087 + 1 - 8\omega = 2^9 K$.

Giving values $\omega = 0,1,2,...$ we determine the values of $\omega$ for which the power $2^{\nu_1} = 2^9$ is periodically appeared in the right side of the equation $\omega = 57 + (2k+1) \cdot 2^7$. So, we get the equation

$$\begin{matrix} 1030087 + 1 - 8\left(57 + \left(2k+1\right) \cdot 2^7\right) = 2^9 K \\ k = 0,1,2,... \end{matrix} . \qquad (8.7)$$

For $k = 0$ equation (8.7) gives

$1030087 + 1 - 8\left(57 + \left(2 \cdot 0 + 1\right) \cdot 2^7\right) = 2^9 \cdot 2011$,

$K = 2011$.

The odd $K = 2011$ gives the asymmetric octet

$$
\begin{array}{ccccccc}
 & & 1007 & & & & \\
 & & \updownarrow T & & & & \\
2011 & \xrightarrow{T} & 895 & \xleftrightarrow{*} & 641 & \xleftrightarrow{T} & 1001 \\
\overset{*}{\updownarrow} & & & & & & \updownarrow^{*} \\
1061 & \xrightarrow{T} & 889 & \xleftrightarrow{*} & 647 & \xleftrightarrow{T} & 535 \\
 & & \updownarrow T & & & & \\
 & & 529 & & & &
\end{array}
$$

which produces the symmetric octet

$$
\begin{array}{ccccccc}
1007 & \xleftrightarrow{T} & 895 & \xleftrightarrow{*} & 641 & \xleftrightarrow{T} & 1001 \\
\overset{*}{\updownarrow} & & & & & & \updownarrow^{*} \\
529 & \xleftrightarrow{T} & 889 & \xleftrightarrow{*} & 647 & \xleftrightarrow{T} & 535
\end{array}
$$

to which 641 belongs.

For $k=342$ equation (8.7) gives

$$1030087+1-8\left(57+\left(2\cdot 342+1\right)\cdot 2^{7}\right)=2^{9}\cdot 641,$$

$$K=641.$$

Giving values $\omega=0,1,2,...$ we determine the values of $\omega$ for which the power $2^{\nu_2}=2^{10}$ is periodically appeared in the right side of the equation $\omega=121+k\cdot 2^{8}$. So, we get the equation

$$1030087+1-8\left(121+k\cdot 2^{8}\right)=2^{10}K \quad . \qquad (8.8)$$
$$k=0,1,2,...$$

For $k=0$ equation (8.8) gives

$$1030087+1-8\left(121+0\cdot 2^{8}\right)=2^{10}\cdot 1005,$$

$$K=1005.$$

From equation (8.4) we get

$$\Pi_1=1005+2\theta$$
$$\theta\in\mathbb{Z}$$

and for $\theta=1$ we get $\Pi_1=1007$ which gives the octet of 641.

For $k=0,1,2,...$ we get $K=1005,1003,1001,...$. For $k=2$ we get $K=1001$ which gives the octet to which 641 belongs.

**Example 8.3.** We calculate the $\omega$ of equation (8.1) for the $\Pi=Q=276569=193\times 1433$, $193\in\Omega_7$, $1433\in\Omega_8$, $\nu_1=7$, $\nu_2=8$. For $\nu_1=7$ we have

$$\Pi+7-8\omega=2^{7}K$$

$$276576-8\omega=2^7K. \tag{8.9}$$

We give values $\omega=0,1,2,3$ and we get

$$276576\text{-}8\cdot 0=2^5\cdot 8643$$
$$276576\text{-}8\cdot 1=2^3\cdot 34571$$
$$276576\text{-}8\cdot 2=2^4\cdot 17285$$
$$276576\text{-}8\cdot 3=2^3\cdot 34569$$

The $2^3$ power is **periodically** appeared for the odd values of $\omega$, so we choose the even values:

$$276576-8(2k)=2^7K \tag{8.10}$$
$$k=0,1,2,...$$

We give values $k=0,1,2,3$ and we get

$$276576\text{-}8\cdot 2\cdot 0=2^5\cdot 8643$$
$$276576\text{-}8\cdot 2\cdot 1=2^4\cdot 17285$$
$$276576\text{-}8\cdot 2\cdot 2=2^6\cdot 4321$$
$$276576\text{-}8\cdot 2\cdot 3=2^4\cdot 17283$$

The $2^4$ power is periodically appeared for the odd values of $k$, so we choose the even values:

$$276576-8(2\cdot 2\cdot k)=2^7K$$
$$k=0,1,2,...$$
$$276576-8(4\cdot k)=2^7K$$
$$k=0,1,2,...$$

We give values $k=0,1,2,3$ and we get

$$276576\text{-}8\cdot 4\cdot 0=2^5\cdot 8643$$
$$276576\text{-}8\cdot 4\cdot 1=2^6\cdot 4321$$
$$276576\text{-}8\cdot 4\cdot 2=2^5\cdot 8641$$
$$276576\text{-}8\cdot 4\cdot 3=2^6\cdot 135$$

The $2^5$ power is periodically appeared for the even values of $k$, so we choose the odd values:

$$276576-8(4\cdot(2k+1))=2^7K$$
$$k=0,1,2,...$$
$$276576-8(4+8k)=2^7K$$
$$k=0,1,2,...$$

We give values $k=0,1,2,3$ and we get

$$276576\text{-}8\cdot(4+8\cdot 0)=2^6\cdot 4321$$
$$276576\text{-}8\cdot(4+8\cdot 1)=2^{11}\cdot 135$$
$$276576\text{-}8\cdot(4+8\cdot 2)=2^6\cdot 4319$$
$$276576\text{-}8\cdot(4+8\cdot 3)=2^7\cdot 2159$$

where the $2^7$ power of 2 appears,

$$276576\text{-}8\cdot(4+8\cdot 3)=2^7\cdot 2159. \qquad (8.11)$$

We give additional values to $k$ and get

$$276576\text{-}8\cdot(4+8\cdot 7)=2^7\cdot 2157. \qquad (8.12)$$

From equations (8.11), (8.12) we get

$$276576-8\left(28+k\cdot 2^5\right)=2^7K$$
$$k=0,1,2,...$$

$$\omega=28+k\cdot 2^5$$
$$k=0,1,2,...$$

.

The sequence $\omega$ has a set of properties that have not been investigated in detail. One of them is that in some cases the periodicity of powers of 2 is lost. We give one example.

**Example 8.4.** For the symmetric number

$$\Pi=Q=41\cdot\left(9+2^{119}\right)=\ 27\ 249173\ 913590\ 775394\ 528044\ 735747\ 064177,$$

in the calculation of $\omega$, after 1+119=120 steps the periodicity of powers of 2 is lost:

$$\begin{aligned}
&27\ 249173\ 913590\ 775394\ 528044\ 735747\ 063808=2^{119}\times 41\\
&25\ 919945\ 917805\ 859521\ 624237\ 675466\ 719232=2^{119}\times 39\\
&24\ 590717\ 922020\ 943648\ 720430\ 615186\ 374656=2^{119}\times 37\\
&23\ 261489\ 926236\ 027775\ 816623\ 554906\ 030080=2^{119}\times 35
\end{aligned}. \qquad (8.13)$$

The odd numbers 41, 39, 37, 35 in equation (8.13) are consecutive.

The algorithm can be executed for all intervals

$$\Omega_k=\left[2^k,2^{k+1}\right],k=1,2,3,...,\nu-1,\ \nu+1=\left[\frac{\ln\Pi}{\ln 2}\right].$$

In this case, an open topic for algorithm improvement is the determination of the optimal value of $n$ in equations (8.1), (8.3).

# 9. A categorization of the composite odd numbers

From equations (5.7), (5.8) we get the following 10 categories for composite odd numbers:

$$Q=(8m_1+7)(8m_2+7)=8\Phi+49 \tag{9.1}$$

$$D=(8m_1+7)(8m_2+9)=8\Phi+63 \tag{9.2}$$

$$Q=(8m_1+7)(8m_2+11)=8\Phi+77 \tag{9.3}$$

$$D=(8m_1+7)(8m_2+13)=8\Phi+91 \tag{9.4}$$

$$Q=(8m_1+9)(8m_2+9)=8\Phi+81 \tag{9.5}$$

$$D=(8m_1+9)(8m_2+11)=8\Phi+99 \tag{9.6}$$

$$Q=(8m_1+9)(8m_2+13)=8\Phi+117 \tag{9.7}$$

$$Q=(8m_1+11)(8m_2+11)=8\Phi+121 \tag{9.8}$$

$$D=(8m_1+11)(8m_2+13)=8\Phi+143 \tag{9.9}$$

$$Q=(8m_1+13)(8m_2+13)=8\Phi+169 \tag{9.10}$$

$m_1, m_2, \Phi \in \mathbb{N}$.

As a consequence of equations

81-49=$8\cdot 4$

99-91=$8\cdot 1$

117-77=$8\cdot 5$

121-49=$8\cdot 9$

121-81=$8\cdot 5$

143-63=$8\cdot 10$

169-49=$8\cdot 15$

169-81=$8\cdot 11$

169-121=$8\cdot 6$

for every odd number for which we don't know its factors, we get at least two possible forms (9.1)-(9.10) as product of two factors. We give two examples.

**Example 9.1.** For the
RSA120=227010481295437363334259960947493668895875336466084780038173258247009
162675779735389791151574049166747880487470296548479

we have

RSA120-63=8Φ

RSA120-91=8Φ

RSA120-143=8Φ.

Therefore, it is either form (9.2),

$$D=(8m_1+7)(8m_2+9) \tag{9.11}$$

or from (9.4),

$$D=(8m_1+7)(8m_2+13) \tag{9.12}$$

or form (9.9),

$$D=(8m_1+11)(8m_2+13). \tag{9.13}$$

The factors of RSA120 are known and indeed are of the form (9.12).

**Example 9.2.** The factors of

RSA260=2211282552952966643528108525502623092761208950247001539441374831912882294140200198651272972656974659908590033003140005117074220456085927635795375718595429883895870922923849100670303412462054578456641366454068421436129301769402084639106587591479425143514445819 9

are not known.

However, we have

RSA260-63=8Φ

RSA260-143=8Φ.

Therefore, it is either form (9.2),

$$D=(8m_1+7)(8m_2+9) \tag{9.14}$$

or form (9.9),

$$D=(8m_1+11)(8m_2+13). \tag{9.15}$$

For every form (9.1)-(9.10) we can develop an algorithm similar to this of section 8.

There are five algorithms, one for each of the following subsets of equations (9.1) - (9.10):

1. (9.1), (9.5) (the A of the algorithm of section 8).

2. (9.2) (the B of the algorithm of section 8).

3. (9.3), (9.7),

$$\Pi+3-8\omega=2^nK. \tag{9.16}$$

4. (9.4), (9.6),

$$\Pi-3-8\omega=2^nK. \tag{9.17}$$

5. (9.8), (9.9), (9.10).

The algorithm finds symmetric factors of $\Pi$. Numbers (9.8), (9.9), (9.10) have no symmetric factors and therefore are not factorized by the algorithm. Factoring these numbers requires special study. Factors of the form $8m+11,\ 8m+13$ of the odd $\Pi$ can be found via conjecture 5.1. We will return to this topic in next article.

# 10. Conclusion

If we want to summarize this article in one sentence we would say that we study the symmetries of natural numbers which arise from theorem (2.1). These symmetries establish a new framework for the study of natural numbers which is entirely different from the context in which they have been studied so far.

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## References

[1] Apostol, Tom M. *Introduction to analytic number theory*. Springer Science & Business Media, 2013.

[2] Robinson, Raphael M. "A Report on Primes of the Form and On Factors of Fermat Numbers." *Proceedings of the American Mathematical Society* 9.5 (1958): 673-681.