Tutorial: (1) The Finite Lightspeed Electromagnetic Extension of the Galilean Boost (2) Thomas Precession: Surprising Rotation from Acceleration not Parallel to Velocity

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Abstract The Galilean-boost space-time transformation implies that accelerations are unchanged by changes of the relative constant velocity of the measuring instruments. Newton's Second Law then suggests that forces are also unchanged, but the torque a constant-velocity charge exerts on a compass needle at its point of closest approach changes with that velocity. This conflict motivates us to solve for the electromagnetic fields of a point charge moving at constant velocity, and to then inspect the results for the correct extension of the Galilean boost, a program we carry out by first deriving the electromagnetic wave equations, which we solve by Fourier methods, including indirectly via obtaining their causal Green's function. The constant-velocity point charge's electromagnetic fields display the space part of the Lorentz boost, which for infinite lightspeed becomes the space part of the Galilean boost. The time part of the Lorentz boost follows from its space part and the reciprocity of measuring instruments which have relative constant velocity. The frequently-presented consequences of the Lorentz boost are then developed in detail, as is the less familiar surprising Thomas-precession rotation produced by acceleration not parallel to velocity.

Galilean-boosted fields of a static charge: zero magnetic field and excess electric-field symmetry

Pondering the velocity of a person walking on the deck of a moving boat versus that of a person walking on a nearby stationary wharf, Galileo arrived at the hypothesis of *the additivity of constant velocities.* That hypothesis was broadened into the more comprehensive *Galilean-boost space-time transformation* of coordinates $(\mathbf{r}, t)$ *measured by stationary instruments* to the corresponding coordinates $(\mathbf{r}', t')$ *measured by instruments traveling at constant velocity* $\mathbf{v}$,

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t \ \text{ and } \ t' = t. \tag{1}$$

Two successive such Galilean-boost transformations *with constant velocities* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *respectively* yields,

$$\mathbf{r}'' = \mathbf{r}' - \mathbf{v}_2 t' = \mathbf{r} - \mathbf{v}_1 t - \mathbf{v}_2 t = \mathbf{r} - (\mathbf{v}_1 + \mathbf{v}_2)t, \text{ taking } \ t'' = t' = t \text{ into account,} \tag{2}$$

in accord with Galileo's hypothesis of *the additivity of the constant velocities* $\mathbf{v}_1$ and $\mathbf{v}_2$.

Additional important consequences of the Galilean-boost space-time transformation of Eq. (1) are that time intervals and *simultaneous-endpoint* space intervals *are the same* when measured by instruments traveling at any constant velocity $\mathbf{v}$ as they are when measured by stationary instruments,

$$\Delta t' = (t'_2 - t'_1) = (t_2 - t_1) = \Delta t, \text{ and also,}$$
$$\Delta \mathbf{r}' = (\mathbf{r}'_2 - \mathbf{r}'_1) = (\mathbf{r}_2 - \mathbf{r}_1) - \mathbf{v}(t_2 - t_1) = (\mathbf{r}_2 - \mathbf{r}_1) = \Delta \mathbf{r} \ \textit{for}\ t_2 = t_1, \text{ i.e. } \textit{for}\ \Delta t = 0, \tag{3}$$

and that the boost velocity $\mathbf{v}$ *adds to velocities,* while accelerations *are the same,*

$$d\mathbf{r}/dt = d(\mathbf{r}' + \mathbf{v}t)/dt = d\mathbf{r}'/dt + \mathbf{v} = d\mathbf{r}'/dt' + \mathbf{v}, \text{ and also,}$$
$$d(d\mathbf{r}/dt)/dt = d(d\mathbf{r}'/dt' + \mathbf{v})/dt = d(d\mathbf{r}'/dt')/dt = d(d\mathbf{r}'/dt')/dt', \tag{4}$$

which in conjunction with Newton's Second Law suggests that *forces are also the same.* Indeed, the constant-velocity-$\mathbf{v} \neq \mathbf{0}$ Galilean boost of the fields $\mathbf{E} = q\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{B} = \mathbf{0}$ of a stationary charge-q point entity is,

$$\mathbf{E} = q(\mathbf{r} - \mathbf{v}t)/|\mathbf{r} - \mathbf{v}t|^3 \text{ and } \mathbf{B} = \mathbf{0}, \tag{5}$$

in *disagreement* with the peak torque a constant-velocity-$\mathbf{v} \neq \mathbf{0}$ point charge exerts on a magnetic-dipole compass needle. The Eq. (5) constant-velocity-$\mathbf{v} \neq \mathbf{0}$ Galilean-boosted point-charge field results *are invalid* except *in the limit* that the constant c in the electromagnetic field equations *is made infinite;* c among other things is the vacuum speed of light. The *vanishing magnetic field* result in Eq. (5) for a constant-velocity-$\mathbf{v} \neq \mathbf{0}$, charge-$q$ point entity occurs *in spite of that entity's nonvanishing current density* $\mathbf{j}$, namely,

$$\mathbf{j} = q\mathbf{v}\delta^{(3)}(\mathbf{r} - \mathbf{v}t), \tag{6}$$

whose insertion into the electromagnetic-field Biot-Savart/Maxwell Law, i.e.,

$$\nabla \times \mathbf{B} = (1/c)(4\pi\mathbf{j} + \partial\mathbf{E}/\partial t) = 4\pi q(\mathbf{v}/c)\delta^{(3)}(\mathbf{r} - \mathbf{v}t) + (1/c)(\partial\mathbf{E}/\partial t), \tag{7}$$

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obviously *must yield a nonvanishing magnetic field.* But *in the* $c \to \infty$ *limit* the Eq. (7) Biot-Savart/Maxwell Law *no longer implies that current densities produce magnetic fields.* The Galilean-boost transformation becomes "valid" *only* in *this physically-inapplicable* $c \to \infty$ *limit.* However, because c *is over twenty thousand times greater* than speeds which produce *permanent departure* from the earth's gravity, *earthbound biological creatures* have over the eons developed *the overwhelmingly strong intuition* that the Eqs. (2) through (4) implications of *the unphysical* $c \to \infty$ Galilean boost *are actually true.*

The magnetic field implied by the Eq. (7) Biot-Savart/Maxwell Law is *orthogonal* to the constant velocity $\mathbf{v}$ of the point charge which produces it, just as the magnetic field produced by a current flowing through a straight wire *is orthogonal to that wire.* Therefore the magnetic field produced by a constant-velocity point charge *has cylindrical symmetry only,* in marked contrast to *the spherical symmetry about* $\mathbf{r} = \mathbf{v}t$ exhibited by *the Galilean-boosted electric field of* Eq. (5).

But notwithstanding that the electic field of a static or a sufficiently slowly-moving point charge has or tends toward spherical symmetry, the electric field *of any moving point charge* in fact *gets modified* by its accompanying time-varying *only-cylindrically-symmetric magnetic field* because of *the influence of time-varying magnetic fields on electric fields* set out in Faraday's Law, i.e.,

$$\nabla \times \mathbf{E} = -(1/c)(\partial \mathbf{B}/\partial t). \tag{8}$$

Therefore *it is to be expected* that the spherical symmetry of the electic field of a static point charge *gets degraded,* to at least some degree, *to cylindrical symmetry* when that charge *has nonzero constant velocity* $\mathbf{v}$. However *in the physically-inapplicable* $c \to \infty$ limit the Eq. (8) Faraday's Law shows that the time-varying, only-cylindrically-symmetric magnetic field *can no longer modify the electric field,* which accords with the spherical symmetry about $\mathbf{r} = \mathbf{v}t$ of the *unphysical* Eq. (5) Galilean-boosted point-charge electric field.

Having seen that the electromagnetic-field Biot-Savart/Maxwell Law and Faraday's Law *are incompatible with the Galilean boost,* as exemplified by the *unphysical* Eq. (5) Galilean-boosted electic and magnetic fields for constant-velocity-$\mathbf{v} \neq \mathbf{0}$, we are now *obliged* to solve for *the correct electric and magnetic fields produced by a constant-velocity-*$\mathbf{v} \neq \mathbf{0}$*, charge-*$q$ *point entity* using the four electromagnetic-field Laws, which, besides the Eq. (7) Biot-Savart/Maxwell Law and the Eq. (8) Faraday's Law, include Coulomb's Law,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{9}$$

and Gauss' Law,

$$\nabla \cdot \mathbf{B} = 0, \tag{10}$$

with the charge density ρ and current density $\mathbf{j}$ of that constant-velocity-$\mathbf{v}$, charge-q point entity, which are,

$$\rho = q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t) \quad \text{and} \quad \mathbf{j} = q\,\mathbf{v}\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t) = \mathbf{v}\rho, \tag{11}$$

followed *by inspection of the results to find the correct finite-*c *replacement* for the $c \to \infty$ Galilean boost.

Fields of a constant-velocity point charge and the space-time boost transformation they imply

The Biot-Savart/Maxwell Law of Eq. (7) and the Faraday's Law of Eq. (8) couple the $\mathbf{B}$ and $\mathbf{E}$ fields to each other. We will use vector calculus to manipulate these two Laws to produce one equation which pertains exclusively to $\mathbf{B}$ together with a second equation which pertains exclusively to $\mathbf{E}$. Combining Gauss' Law of Eq. (10) with the equation for $\mathbf{B}$ and Coulomb's Law of Eq. (9) with the equation for $\mathbf{E}$ will then produce *identical-form wave equations for* $\mathbf{B}$ *and* $\mathbf{E}$ *that differ only in their sources.*

The *two sources* for the identical-form wave equations for $\mathbf{E}$ and $\mathbf{B}$ turn out to be linear combinations of *partial first derivatives* of the charge density ρ and the current density $\mathbf{j}$, but Eq. (11) *provides those densities themselves* for a constant-velocity charged point entity. To circumvent that disconnect, it is standard practice to define a scalar potential ϕ which satisfies the *same* wave-equation form, *but with* $4\pi\rho$ *as its source,* and to as well define a vector potential $\mathbf{A}$ which also satisfies the *same* wave equation form, *but with* $(4\pi/c)\mathbf{j}$ *as its source.* If the wave equations for these *potentials* ϕ and $\mathbf{A}$ *can be solved,* then the electromagnetic *fields* $\mathbf{E}$ and $\mathbf{B}$ will simply be linear combinations of *partial first derivatives of these potentials.*

Using Fourier-transform methods we will solve for the scalar potential ϕ whose source is 4π times the ρ of Eq. (11). The corresponding $\mathbf{j}$ of Eq. (11) is $\mathbf{v}\rho$, where $\mathbf{v}$ is constant, so the vector potential $\mathbf{A}$, whose source is $(4\pi/c)\mathbf{j} = (\mathbf{v}/c)(4\pi\rho)$, follows immediately from the scalar potential ϕ as,

$$\mathbf{A} = (\mathbf{v}/c)\phi. \tag{12}$$

Partial first differentiation of $\mathbf{A}$ and ϕ then yields $\mathbf{B}$ and $\mathbf{E}$. These results are to be inspected for the correct finite-c replacement of the unphysical $c \to \infty$ Galilean-boost space-time transformation given by Eq. (1).

We now turn to the calculational details of the electromagnetic field equation steps which we have so far in this section been outlining in words. Taking the curl of both sides of the Eq. (7) Biot-Savart/Maxwell Law, which is $(\nabla \times \mathbf{B}) = (1/c)(4\pi\mathbf{j} + (\partial\mathbf{E}/\partial t))$, yields,

$$\nabla \times (\nabla \times \mathbf{B}) = (4\pi/c)(\nabla \times \mathbf{j}) + (1/c)(\partial(\nabla \times \mathbf{E})/\partial t), \tag{13a}$$

so since it is an *identity* that $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B}$, and the Eq. (8) Faraday's Law tells us that $\nabla \times \mathbf{E} = -(1/c)(\partial\mathbf{B}/\partial t)$, Eq. (13a) becomes,

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B} = (4\pi/c)(\nabla \times \mathbf{j}) - (1/c)^2(\partial^2\mathbf{B}/\partial t^2). \tag{13b}$$

The Eq. (10) Gauss' Law tells us that $\nabla \cdot \mathbf{B} = 0$, which eliminates the first term of Eq. (13b), which then can readily be rearranged into the wave-equation form,

$$(1/c)^2(\partial^2\mathbf{B}/\partial t^2) - \nabla^2\mathbf{B} = (4\pi/c)(\nabla \times \mathbf{j}). \tag{13c}$$

Also, taking the curl of both sides of the Eq. (8) Faraday's Law, which is $(\nabla\times\mathbf{E}) = -(1/c)(\partial\mathbf{B}/\partial t)$, produces,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -(1/c)(\partial(\nabla \times \mathbf{B})/\partial t). \tag{14a}$$

We now substitute the right side of the Eq. (7) Biot-Savart/Maxwell Law, which is $\nabla \times \mathbf{B} = (1/c)(4\pi\mathbf{j} + (\partial\mathbf{E}/\partial t))$, for the occurrence of $\nabla \times \mathbf{B}$ within the right side of the second equality of Eq. (14a) to obtain,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -4\pi(1/c)^2(\partial\mathbf{j}/\partial t) - (1/c)^2(\partial^2\mathbf{E}/\partial t^2). \tag{14b}$$

The Eq. (9) Coulomb's Law tells us that $\nabla \cdot \mathbf{E} = 4\pi\rho$. The entity $\nabla \cdot \mathbf{E}$ *also* occurs within the first term on the left side of Eq. (14b); we now substitute $4\pi\rho$ for $\nabla \cdot \mathbf{E}$ in that term, after which the terms of Eq. (14b) are readily rearranged into the following wave-equation form,

$$(1/c)^2(\partial^2\mathbf{E}/\partial t^2) - \nabla^2\mathbf{E} = -4\pi\big(\nabla\rho + (1/c)^2(\partial\mathbf{j}/\partial t)\big). \tag{14c}$$

We already mentioned that the source terms on the right-hand sides of the Eq. (13c) and (14c) magnetic field and electric field wave equations are linear combinations of *partial first derivatives* of the charge density ρ and the current density $\mathbf{j}$, *not* linear combinations of ρ and $\mathbf{j}$ *themselves.* We also mentioned that that disconnect is circumvented by *instead* seeking the solutions of the closely-related electromagnetic scalar potential ϕ and vector potential $\mathbf{A}$ wave equations, which are,

$$(1/c)^2(\partial^2\phi/\partial t^2) - \nabla^2\phi = 4\pi\rho \quad\text{and}\quad (1/c)^2(\partial^2\mathbf{A}/\partial t^2) - \nabla^2\mathbf{A} = (4\pi/c)\mathbf{j}, \tag{15a}$$

since it is readily verified from Eqs. (13c), (14c) and (15a) that,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad\text{and}\quad \mathbf{E} = -\nabla\phi - (1/c)(\partial\mathbf{A}/\partial t). \tag{15b}$$

Our next task is to work out the scalar potential ϕ for a charge-q point entity which travels at constant velocity $\mathbf{v}$ that arises via the Eq. (15a) scalar-potential wave equation $(1/c)^2(\partial^2\phi/\partial t^2) - \nabla^2\phi = 4\pi\rho$ from the Eq. (11) charge density $\rho = q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t)$. We note once again from Eqs. (12) and (11) that in this particular case of a charged point entity that travels at constant velocity $\mathbf{v}$, $\mathbf{j} = \mathbf{v}\rho$ where $\mathbf{v}$ is constant, which implies that $\mathbf{A} = (\mathbf{v}/c)\phi$, obviating the need to separately solve for $\mathbf{A}$.

One approach to working out ϕ is via propagation of its source ρ with the causal Green's function for this electromagnetic form of wave equation. We will later pursue that approach since that Green's function's nonzero locus is an important space-time invariant under constant-velocity changes. Here, however, we approach ϕ *in a specialized way* based on the form of its source $\rho = q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t)$. The two approaches give *the same result for* ϕ via *initial algebraic forms of* ϕ *which have a markedly dissimilar appearance.*

Since Eq. (15a) tells us that to obtain the ϕ produced by $\rho = q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t)$, we must solve,

$$(1/c)^2(\partial^2\phi/\partial t^2) - \nabla^2\phi = 4\pi q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t) = (q/(2\pi^2))\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}d^3\mathbf{k}, \tag{16a}$$

we make the *specialized* Fourier *ansatz* that,

$$\phi = \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}f(\mathbf{k})d^3\mathbf{k}, \tag{16b}$$

which, when inserted into Eq. (16a) yields,

$$f(\mathbf{k}) = (q/(2\pi^2))\big(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2\big)^{-1}, \tag{16c}$$

which inserted in turn into Eq. (16b) presents the solution ϕ of Eq. (16a) as the integral,

$$\phi = (q/(2\pi^2)) \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}\big(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2\big)^{-1} d^3\mathbf{k}. \tag{16d}$$

We assume that the charged point entity's constant speed $|\mathbf{v}|$ is less than that of light c, *so we can seek a linear change of the vector integration variable* $\mathbf{k}$ *to* $\mathbf{K}$ *such that* $|\mathbf{K}|^2 = |\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2$. To work out $\mathbf{K}$, it is useful to introduce the charged point entity's fixed travel-direction unit vector,

$$\hat{\mathbf{u}} \equiv (\mathbf{v}/|\mathbf{v}|), \text{ from which it follows that } |\hat{\mathbf{u}}|^2 = 1. \tag{17a}$$

We now decompose $\mathbf{k}$ into two parts which are respectively parallel to and perpendicular to both $\hat{\mathbf{u}}$ and $\mathbf{v}$,

$$\mathbf{k} = \mathbf{k}_\parallel + \mathbf{k}_\perp, \text{ where } \mathbf{k}_\parallel \equiv \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{k}) \text{ and } \mathbf{k}_\perp \equiv \mathbf{k} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{k}), \text{ which imply that,}$$

$$|\mathbf{v}|^2|\mathbf{k}_\parallel|^2 = ((\mathbf{k}\cdot\hat{\mathbf{u}})|\mathbf{v}|)^2 = (\mathbf{k}\cdot\mathbf{v})^2, \quad (\mathbf{v}\cdot\mathbf{k}_\perp) = 0, \quad (\mathbf{k}_\parallel\cdot\mathbf{k}_\perp) = 0 \text{ and } |\mathbf{k}|^2 = |\mathbf{k}_\perp|^2 + |\mathbf{k}_\parallel|^2. \tag{17b}$$

The Eq. (17b) orthogonal decomposition of $\mathbf{k}$ enables us to arrive at the *linear* relation of $\mathbf{K}$ to $\mathbf{k}$,

$$|\mathbf{K}|^2 = |\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2 = |\mathbf{k}|^2 - |\mathbf{v}/c|^2|\mathbf{k}_\parallel|^2 = |\mathbf{k}_\perp|^2 + (1 - |\mathbf{v}/c|^2)|\mathbf{k}_\parallel|^2 = |\mathbf{k}_\perp + \sqrt{1-|\mathbf{v}/c|^2}\,\mathbf{k}_\parallel|^2, \tag{17c}$$

where the last equality reflects *the orthogonality of* $\mathbf{k}_\perp$ *to* $\mathbf{k}_\parallel$. The *linear* relation of $\mathbf{K}$ to $\mathbf{k}$ thus is,

$$\mathbf{K} = \mathbf{k}_\perp + \sqrt{(1-|\mathbf{v}/c|^2)}\,\mathbf{k}_\parallel = \mathbf{k} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{k}) + \sqrt{1-|\mathbf{v}/c|^2}\,\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{k}), \tag{17d}$$

where the second expression reflects the explicit formulas of Eq. (17b) for $\mathbf{k}_\perp$ and $\mathbf{k}_\parallel$ in terms of $\hat{\mathbf{u}}$ and $\mathbf{k}$.

To change the vector integration variable in Eq. (16d) from $\mathbf{k}$ to $\mathbf{K}$ we *also* need to solve Eq. (17d) for $\mathbf{k}$ in terms of $\mathbf{K}$, which requires only a few steps. Contracting both sides of Eq. (17d) with $\hat{\mathbf{u}}$ yields,

$$(\hat{\mathbf{u}}\cdot\mathbf{K}) = \sqrt{1-|\mathbf{v}/c|^2}\,(\hat{\mathbf{u}}\cdot\mathbf{k}) \quad\Rightarrow\quad (\hat{\mathbf{u}}\cdot\mathbf{k}) = \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}(\hat{\mathbf{u}}\cdot\mathbf{K}). \tag{17e}$$

Insertion of the Eq. (17e) result for $(\hat{\mathbf{u}}\cdot\mathbf{k})$ into the right side of Eq. (17d) in turn yields,

$$\mathbf{K} = \mathbf{k} - \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) \quad\Rightarrow$$

$$\mathbf{k} = \mathbf{K} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) + \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) = \mathbf{K}_\perp + \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}\mathbf{K}_\parallel. \tag{17f}$$

To effect the desired change of the vector integration variable in Eq. (16d), the right side of Eq. (17f) is substituted for all occurrences of $\mathbf{k}$ in Eq. (16d). A simple instance of such an occurrence in Eq. (16d) is,

$$d^3\mathbf{k} = d^2\mathbf{K}_\perp\left[\left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1} d\mathbf{K}_\parallel\right] = \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1} d^3\mathbf{K} = \gamma\, d^3\mathbf{K}, \tag{18a}$$

where we henceforth use the standard abbreviation γ for $\left(\sqrt{1-|\mathbf{v}/c|^2}\,\right)^{-1}$. The occurrence in Eq. (16d) of $(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2)^{-1}$ becomes just $|\mathbf{K}|^{-2}$, according to Eq. (17c). Finally, $\mathbf{k}$ occurs in Eq. (16d) within $e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}$. Inserting the Eq. (17f) change of vector integration variable, namely $\mathbf{k} = \mathbf{K} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) + \gamma\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K})$, into $\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)$ and noting that $\mathbf{v}$ is perpendicular to $(\mathbf{K} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}))$ produces, upon changing the orders of factors to send all occurrences of $\mathbf{K}$ to the left side of the expression, that,

$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t) = \mathbf{K}\cdot(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))). \tag{18b}$$

Inserting the three consequences of the change of integration variable from $\mathbf{k}$ to $\mathbf{K}$ into Eq. (16d) yields,

$$\phi = (q\gamma/(2\pi^2)) \int e^{i\mathbf{K}\cdot(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t))))}|\mathbf{K}|^{-2}d^3\mathbf{K}. \tag{18c}$$

Unlike the Eq. (16d) *integral for* ϕ, the Eq. (18c) integral for ϕ *is just as easy to evaluate when* $\mathbf{v} \neq \mathbf{0}$ *as it is to evaluate when* $\mathbf{v} = \mathbf{0}$, the case for which *both* the Eq. (16d) integral *and* the Eq.(18c) integral *clearly yield* $\phi = q/|\mathbf{r}|$, *the static Coulomb potential.* So from its $\mathbf{v} = \mathbf{0}$ case, Eq. (18c) obviously yields,

$$\phi = q\gamma/|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|. \tag{19a}$$

It is pointed out in Eq. (12) and also below Eq. (15b) that $\mathbf{A} = (\mathbf{v}/c)\phi$, so,

$$\mathbf{A} = q\gamma(\mathbf{v}/c)/|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t)))|. \tag{19b}$$

The $\mathbf{B}$ and $\mathbf{E}$ fields are obtained from $\mathbf{A}$ and ϕ by partial first differentiation using Eq. (15b),

$$\mathbf{B} = \nabla\times\mathbf{A} = q\gamma((\mathbf{v}/c)\times\mathbf{r})/|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t)))|^3. \tag{19c}$$

$$\mathbf{E} = -\nabla\phi - (1/c)(\partial\mathbf{A}/\partial t) = q\gamma(\mathbf{r} - \mathbf{v}t)/|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t)))|^3. \tag{19d}$$

In the *physically inapplicable* $c \to \infty$ *limit,* $\gamma \to 1$ and Eqs. (19c) and (19d) *revert to the* Eq. (5) *Galilean-boost results,* i.e., to $\mathbf{B} = \mathbf{0}$, *the nonexistence of magnetic fields in a Galilean universe,* and to $\mathbf{E} = q(\mathbf{r}-\mathbf{v}t)/|\mathbf{r}-\mathbf{v}t|^3$, *which is spherically symmetric around* $\mathbf{r} = \mathbf{v}t$, as expected in a Galilean universe.

The *actual* cylindrically-symmetric Eq. (19c) magnetic field *has vanishing component in the point-charge entity's direction of motion,* and so departs drastically from spherical symmetry. Electric fields of a point charge at rest are spherically symmetric, but Eq. (19d) shows that they technically shed spherical symmetry as soon as the point charge has nonzero velocity; in the extreme case that the point charge is moving at a speed sufficiently near that of light that $\gamma \gg 1$, Eq. (19d) shows that its electric field strongly "pancakes" in the charge's direction of motion; a behavior somewhat akin to a charge's magnetic field having zero component in the charge's direction of motion. Electric and magnetic fields become somewhat more similar as their sources approach the speed of light, albeit the Eq. (19c) magnetic field *is always everywhere orthogonal to the* Eq. (19d) *electric field.*

The denominators of the potentials and fields of Eqs. (19a) through (19d) *all display the vector* $(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t))))$, *which reverts to* $\mathbf{r}$ *in the static* $|\mathbf{v}| = 0$ *case,* as it of course must. In the unphysical $c \to \infty$ limit, which causes $\gamma \to 1$, it becomes $(\mathbf{r}-\mathbf{v}t)$, *the right side of the space part of the Galilean boost.* It also has *another,* more subtle, feature *in common* with the right side of the space part of the Galilean boost, namely that its part which is perpendicular to $\mathbf{v}$ (and therefore perpendicular to $\hat{\mathbf{u}} = (\mathbf{v}/|\mathbf{v}|)$) *is exactly the same as the part of merely* $\mathbf{r}$ *itself which is perpendicular to* $\mathbf{v}$.

The *part of an arbitrary vector* $\boldsymbol{\Xi}$ *which is parallel to* $\mathbf{v}$ is $\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\boldsymbol{\Xi})$, *where the unit vector* $\hat{\mathbf{u}} \equiv (\mathbf{v}/|\mathbf{v}|)$, so *the part of this arbitrary vector* $\boldsymbol{\Xi}$ *which is perpendicular* $\mathbf{v}$ *is* $(\boldsymbol{\Xi} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\boldsymbol{\Xi}))$. Thus the part of $\mathbf{r}$ *which is perpendicular to* $\mathbf{v}$ is $(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}))$. The part of $(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t))))$ *which is parallel to* $\mathbf{v}$ is $\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t)))$, so *the part of* $(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t))))$ *which is perpendicular to* $\mathbf{v}$ *is also* $(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}))$. Similarly, the part of $(\mathbf{r} - \mathbf{v}t)$, the right side of the space part of the Galilean boost, which is *parallel to* $\mathbf{v}$ is $(\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) - \mathbf{v}t)$, so the part of $(\mathbf{r} - \mathbf{v}t)$ *which is perpendicular to* $\mathbf{v}$ *is as well* $(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}))$.

Therefore the Galilean boost, whose space part is $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, *leaves invariant the part of* $\mathbf{r}$ *which is perpendicular to* $\mathbf{v}$, namely $(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}))$. Likewise, if we were to take $\mathbf{r}' = (\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t))))$ *to be the space part of the finite-c extension of the physically inapplicable* $c \to \infty$ *Galilean boost,* then *that finite-c Galilean-boost extension would as well leave invariant the part of* $\mathbf{r}$ *which is perpendicular to* $\mathbf{v}$. *Other features the* Eqs. (19a)–(19d) *electromagnetic-field motivated* $\mathbf{r}' = (\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t))))$ *has in its favor to be the space part of the correct finite-c Galilean-boost extension* is that *it correctly reduces to* $\mathbf{r}' = \mathbf{r}$ *when* $|\mathbf{v}| = 0$ and that *it reduces to* $\mathbf{r}' = \mathbf{r} - \mathbf{v}t$, the space part of the Galilean boost, in the limit $c \to \infty$.

Of course the space part *alone* of a proposed constant-velocity boost transformation *isn't enough,* we need *its time part as well.* The time part of a constant-velocity boost transformation in fact *follows* from its space part *and the principle of relativistic reciprocity,* namely that *both* of a pair of instrument sets which have relative constant velocity $\mathbf{v}$ *are subject to the same transformation rules,* which implies that a constant-velocity-$\mathbf{v}$ boost transformation from unprimed to primed space-time coordinates *remains valid when the sign of* $\mathbf{v}$ *is reversed and the unprimed and primed space-time coordinates are swapped.*

As an illustration, the time part of the Galilean boost *follows* from its space part,

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t, \tag{19e}$$

and the relativistic reciprocal of that space part, which of course is,

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t'. \tag{19f}$$

To show this, we first *contract both* Eqs. (19e) *and* (19f) *with* $\mathbf{v}$ to obtain the pair of *scalar equations,*

$$(\mathbf{v}\cdot\mathbf{r}') = (\mathbf{v}\cdot\mathbf{r}) - |\mathbf{v}|^2 t \quad\text{and}\quad (\mathbf{v}\cdot\mathbf{r}) = (\mathbf{v}\cdot\mathbf{r}') + |\mathbf{v}|^2 t'. \tag{19g}$$

The time part of the boost gives t' in terms of t and $(\mathbf{v}\cdot\mathbf{r})$, so we *eliminate* $(\mathbf{v}\cdot\mathbf{r}')$ by substituting the first of the Eq. (19g) pair of equations into the second one to obtain,

$$(\mathbf{v}\cdot\mathbf{r}) = (\mathbf{v}\cdot\mathbf{r}) - |\mathbf{v}|^2 t + |\mathbf{v}|^2 t', \tag{19h}$$

which yields that,

$$|\mathbf{v}|^2(t' - t) = 0, \tag{19i}$$

so *the time part* of any nontrivial Galilean boost, namely one having $\mathbf{v} \neq \mathbf{0}$, is,

$$t' = t, \tag{19j}$$

which is precisely the time part that Eq. (1) *gives for the Galilean boost.*

We now take the space part of the correct finite-c Galilean-boost extention to be the electromagnetic,

$$\mathbf{r}' = \mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t))), \tag{20a}$$

and obtain its time part *from this space part and its relativistic reciprocal,* which of course is,

$$\mathbf{r} = \mathbf{r}' - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}') + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}' + \mathbf{v}t'))). \tag{20b}$$

We next *contract both* Eq. (20a) *and* Eq. (20b) *with* $\mathbf{v}$ to obtain the pair of *scalar equations,*

$$(\mathbf{v}\cdot\mathbf{r}') = \gamma((\mathbf{v}\cdot\mathbf{r}) - |\mathbf{v}|^2 t) \quad \text{and} \quad (\mathbf{v}\cdot\mathbf{r}) = \gamma((\mathbf{v}\cdot\mathbf{r}') + |\mathbf{v}|^2 t'). \tag{20c}$$

The time part of the boost gives t' in terms of t and $(\mathbf{v}\cdot\mathbf{r})$, so we *eliminate* $(\mathbf{v}\cdot\mathbf{r}')$ by substituting the first of the Eq. (20c) pair of equations into the second one to obtain,

$$(\mathbf{v}\cdot\mathbf{r}) = \gamma(\gamma((\mathbf{v}\cdot\mathbf{r}) - |\mathbf{v}|^2 t) + |\mathbf{v}|^2 t'), \tag{20d}$$

which, with help from the identity $\gamma^2 - 1 = (|\mathbf{v}|^2/c^2)\gamma^2$, is readily solved for t' in terms of t and $(\mathbf{v}\cdot\mathbf{r})$,

$$t' = \gamma(t - ((\mathbf{v}\cdot\mathbf{r})/c^2)). \tag{20e}$$

Eqs. (20a) and (20e) together comprise the *the electromagnetic extension to finite c of the physically-inapplicable $c \to \infty$* Eq. (1) *Galilean boost.* It is universally known as *the Lorentz boost;* we obtained it via inspection of the electromagnetic fields produced by a point charge moving at constant velocity $\mathbf{v}$, which were first worked out by Oliver Heaviside. It was *because* George Fitzgerald had read Heaviside's paper *and had seen the electromagnetic fields of* Eqs. (19c) *and* (19d) that he was able to provide the physics world with its first solid clue as to proper interpretation of the unexpected null result of the Michelson-Morley experiment: although the physically-inapplicable $c \to \infty$ Galilean boost predicts, as we have seen in Eq. (3), that the boost velocity $\mathbf{v}$ *has no effect on space and time interval measurements,* that *isn't true of the Lorentz boost given by* Eqs. (20a) *and* (20e).

Consequences of the finite lightspeed Lorentz-boost extension of the unphysical Galilean boost

The Eq. (20e) time part of the finite-c Lorentz boost predicts that *unlike* the Eq. (3) result for the $c \to \infty$ Galilean boost, *the boost velocity* $\mathbf{v}$ *shortens the time interval measured by a stationary clock.* For time intervals $\Delta t' \equiv (t'_2 - t'_1)$ and $\Delta t \equiv (t_2 - t_1)$, and space interval $\Delta\mathbf{r} \equiv (\mathbf{r}_2 - \mathbf{r}_1)$, Eq. (20e) yields,

$$\Delta t' = \gamma(\Delta t - ((\mathbf{v}\cdot\Delta\mathbf{r})/c^2)) = \gamma\Delta t \text{ for a clock } \textit{at a fixed location,} \text{ which obviously requires } \Delta\mathbf{r} = \mathbf{0}. \tag{21a}$$

Therefore,

$$\Delta t = (1/\gamma)(\Delta t') = \sqrt{1 - |\mathbf{v}/c|^2}\,(\Delta t'), \tag{21b}$$

so a time interval $\Delta t'$ recorded by a clock moving at constant velocity $\mathbf{v}$ *is recorded by a stationary clock as shortened by the factor* $\sqrt{1 - |\mathbf{v}/c|^2}$, i.e., a stationary observer who consults his own stationary clock notes that the moving clock ticks at a slower rate than his stationary clock. That may seem counterintuitive, but the intuition which *earthbound biological creatures* have developed over the eons *is inapplicable to that tick-rate disparity.* Such *earthbound creatures* could not possibly, over the eons, have developed an intuition for speeds as high as 43,000 km per hour, because that speed *permanently removes entities which attain it from the earth's gravitational influence.* Yet a clock moving at that speed has its tick rate *slowed by less* *than one part in a billion relative to the tick rate of a stationary clock,* which *isn't* within the competence *of the inherent sense of time of biological creatures* to detect.

The Eq. (20a) space part of the finite-c Lorentz boost predicts, somewhat analogously to its Eq. (20e) time part, that *unlike* the Eq. (3) result for the $c \to \infty$ Galilean boost, *the boost velocity* $\mathbf{v}$ *shortens the component of a space interval which is oriented in its direction, as measured by stationary instruments.* For space intervals $\Delta\mathbf{r}' \equiv (\mathbf{r}_2' - \mathbf{r}_1')$ and $\Delta\mathbf{r} \equiv (\mathbf{r}_2 - \mathbf{r}_1)$, and time interval $\Delta t \equiv (t_2 - t_1)$, Eq. (20a) yields,

$$\Delta\mathbf{r}' = \Delta\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r}) + \gamma\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\Delta\mathbf{r} - \mathbf{v}\Delta t)). \tag{21c}$$

As usual for a space-interval measurement, that measurement must be simultaneous at the two ends of the space interval, so $t_2 = t_1$, which imples $\Delta t = 0$ in Eq. (21c), and therefore,

$$\Delta\mathbf{r}' = \Delta\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r}) + \gamma\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r}). \tag{21d}$$

Using $\hat{\mathbf{u}}$, the unit-vector direction of the constant velocity $\mathbf{v}$, we now decompose both $\Delta\mathbf{r}'$ and $\Delta\mathbf{r}$ into two orthogonal vectors that are respectively perpendicular and parallel to $\mathbf{v}$,

$$\Delta\mathbf{r}' = \Delta\mathbf{r}'_\perp + \Delta\mathbf{r}'_\parallel, \quad \text{where} \quad \Delta\mathbf{r}'_\perp \equiv \Delta\mathbf{r}' - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r}') \quad \text{and} \quad \Delta\mathbf{r}'_\parallel \equiv \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r}'), \tag{21e}$$

and likewise for $\Delta\mathbf{r}$, i.e., $\Delta\mathbf{r} = \Delta\mathbf{r}_\perp + \Delta\mathbf{r}_\parallel$, where $\Delta\mathbf{r}_\perp \equiv \Delta\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r})$ and $\Delta\mathbf{r}_\parallel \equiv \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r})$. When these perpendicular/parallel decompositions of $\Delta\mathbf{r}'$ and $\Delta\mathbf{r}$ *are combined with* Eq. (21d), we obtain,

$$\Delta\mathbf{r}' = \Delta\mathbf{r}'_\perp + \Delta\mathbf{r}'_\parallel = (\Delta\mathbf{r}' - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r}')) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r}') = (\Delta\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r})) + \gamma\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\Delta\mathbf{r}) = \Delta\mathbf{r}_\perp + \gamma\Delta\mathbf{r}_\parallel, \tag{21f}$$

which yields that,

$$\Delta\mathbf{r}'_\perp = \Delta\mathbf{r}_\perp \;\text{ and }\; \Delta\mathbf{r}'_\parallel = \gamma\Delta\mathbf{r}_\parallel, \tag{21g}$$

so $\Delta\mathbf{r}_\perp$ is unaffected by the boost, but for $\Delta\mathbf{r}_\parallel$ we have that,

$$\Delta\mathbf{r}_\parallel = (1/\gamma)(\Delta\mathbf{r}'_\parallel) = \sqrt{1 - |\mathbf{v}/c|^2}\,(\Delta\mathbf{r}'_\parallel). \tag{21h}$$

Thus a space interval moving at constant velocity $\mathbf{v}$ appears *to stationary instruments to be shortened in the direction of* $\mathbf{v}$ *by the factor* $\sqrt{1 - |\mathbf{v}/c|^2}$ *relative to the length measured by instruments that are traveling with that space interval.* This is the "Fitzgerald contraction" of moving objects in their direction of motion, which helped physicists begin to understand the results of the Michelson-Morley experiment.

That an object moving at constant velocity contracts in length in its direction of motion also seems counterintuitive. Once again the answer to concern about contradicting the intuition developed over the eons by *earthbound biological creatures* is that they could not possibly have developed an intuition for a speed as high as 43,000 km per hour because that speed *permanently removes entities which attain it from the earth's gravitational influence,* but the Fitzgerald contraction produced by that speed is less than one part in a billion, which *isn't* within the competence *of biological organs* to detect.

That the finite-c Lorentz-boost extension of the $c \to \infty$ physically-inapplicable Galilean boost, *unlike* the latter, *doesn't* preserve space and time intervals, raises the question of whether there exists *any* space-time locus that the Lorentz boost *does* preserve. The fact that the speed of light c *is a constant of the electromagnetic field equations* suggests that the space-time locus of a light wavefront—whose evolution in time is of course governed by the constant speed c of light—*is* preserved by the finite-c Lorentz boost. In the theory of light wavefront evolution (Huygens), the underlying fundamental "building block" wavefront (from which *any other* wavefront results *only* through reinforcement) is the expanding spherical-shell wavefront whose radius grows uniformly with time at the rate c and has the space-time locus,

$$|\mathbf{r} - \mathbf{r}_0|^2 = c^2(t - t_0)^2 \text{ for } t \geq t_0. \tag{22a}$$

This lightspeed-expanding spherical-shell wavefront space-time locus turns out to be *faithfully embedded in the functional form* of the causal Green's function $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$ for electromagnetic wave equations,

$$(1/c)^2(\partial^2 G(\mathbf{r}, t; \mathbf{r}_0, t_0)/\partial t^2) - \nabla^2_{\mathbf{r}} G(\mathbf{r}, t; \mathbf{r}_0, t_0) = 4\pi\,\delta^{(3)}(\mathbf{r} - \mathbf{r}_0)\,\delta(c(t - t_0)). \tag{22b}$$

In the next section we solve Eq. (22b) for the causal Green's function $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$, and then use the result to propagate the effect of the charge density $\rho = q\delta^{(3)}(\mathbf{r} - \mathbf{v}t)$ of a point entity of charge q which moves at constant velocity $\mathbf{v}$ to alternatively obtain the Eq. (19a) scalar potential ϕ.

But first we verify *a key property* of the Lorentz boost of Eqs. (20a) and (20e), namely that it preserves the space-time quadratic form $|\mathbf{r}|^2 - (ct)^2$ for all constant $\mathbf{v}$ which satisfy $|\mathbf{v}| < c$. That the Lorentz boost preserves *only* $|\mathbf{r}|^2 - (ct)^2$ shows that *it needn't preserve the lengths of space and time intervals,* but *does imply* that it in particular preserves the lightspeed-expanding spherical-shell wavefront space-time locus $|\mathbf{r}|^2 = (ct)^2$. This last property was Einstein's basic postulate *for deriving the finite-c Lorentz boost,* but *one must in addition* postulate specific properties of the boost *which a given boost velocity* $\mathbf{v}$ *entails,* and one must furthermore *stipulate* that the $c \to \infty$ limit of the finite-c boost is the Galilean boost.

The Eq. (20a) $\mathbf{r}'$ together with the Eq. (20e) t' imply that $|\mathbf{r}'|^2 - (ct')^2 = |\mathbf{r}|^2 - (ct)^2$ because,

$$|\mathbf{r}'|^2 - (ct')^2 = \big|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t)))\big|^2 - \big(\gamma(ct - ((\mathbf{v}/c)\cdot\mathbf{r}))\big)^2 =$$

$$|\mathbf{r}|^2 - (\hat{\mathbf{u}}\cdot\mathbf{r})^2 + \gamma^2\Big[\big((\hat{\mathbf{u}}\cdot\mathbf{r}) - |\mathbf{v}|t\big)^2 - \big(ct - (|\mathbf{v}|/c)(\hat{\mathbf{u}}\cdot\mathbf{r})\big)^2\Big] =$$

$$|\mathbf{r}|^2 - (\hat{\mathbf{u}}\cdot\mathbf{r})^2 + \gamma^2\Big[\big(1 - (|\mathbf{v}|/c)^2\big)(\hat{\mathbf{u}}\cdot\mathbf{r})^2 - \big(1 - (|\mathbf{v}|/c)^2\big)(ct)^2\Big] = |\mathbf{r}|^2 - (ct)^2. \tag{22c}$$

Solution for and application of the electromagnetic wave-equation causal Green's function

Since $\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) = (2\pi)^{-3}\int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)}$, we make for the $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$ of Eq. (22b) the Fourier *ansatz,*

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \int d^3\mathbf{k}\; h(\mathbf{k}, t; t_0)\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)}, \tag{23a}$$

which when inserted into Eq. (22b) yields that $h(\mathbf{k}, t; t_0)$ satisfies the time differential equation,

$$(1/c)^2(\partial^2 h(\mathbf{k}, t; t_0)/\partial t^2) + |\mathbf{k}|^2 h(\mathbf{k}, t; t_0) = (2\pi^2 c)^{-1}\delta(t - t_0). \tag{23b}$$

In order to be *causal,* $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$, and therefore also $h(\mathbf{k}, t; t_0)$, must vanish when $t < t_0$. Using the Heaviside step function $\theta(t - t_0)$, which vanishes when $t < t_0$ and is equal to unity when $t \geq t_0$, we construct the following form for $h(\mathbf{k}, t; t_0)$ that is a causally proper solution of Eq. (23b) in both of the time regions $t < t_0$ and $t > t_0$,

$$h(\mathbf{k}, t; t_0) = K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0) + \eta), \tag{23c}$$

where the constants K and η in Eq. (23c) are to be chosen to produce the singular function $(2\pi^2 c)^{-1}\delta(t - t_0)$ on the right side of Eq. (23b). Since the second time derivative of $\theta(t - t_0)$ is $\delta'(t - t_0)$, *yet no such derivative of a delta function occurs on the right side of* Eq. (23b), $h(\mathbf{k}, t; t_0)$ must be *continuous at* $t = t_0$, so $\eta = 0$. With η set to zero, we now prepare to insert the right side of Eq. (23c) into Eq. (23b) to determine K,

$$(1/c)(\partial(K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0)))/\partial t) =$$

$$(K/c)[\delta(t - t_0)\sin(|\mathbf{k}|c(t - t_0)) + |\mathbf{k}|c\theta(t - t_0)\cos(|\mathbf{k}|c(t - t_0))] =$$

$$K|\mathbf{k}|\theta(t - t_0)\cos(|\mathbf{k}|c(t - t_0)). \tag{23d}$$

Using the Eq. (23d) result to differentiate $K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0))$ *again* with respect to time yields,

$$(1/c)^2(\partial^2(K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0)))/\partial t^2) = (1/c)(\partial(K|\mathbf{k}|\theta(t - t_0)\cos(|\mathbf{k}|c(t - t_0)))/\partial t) =$$

$$(K|\mathbf{k}|/c)[\delta(t - t_0)\cos(|\mathbf{k}|c(t - t_0)) - |\mathbf{k}|c\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0))] =$$

$$(K|\mathbf{k}|/c)\delta(t - t_0) - |\mathbf{k}|^2 K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0)). \tag{23e}$$

Comparison of Eq. (23e) with Eq. (23b) shows that $K = (2\pi^2|\mathbf{k}|)^{-1}$, which inserted into Eq. (23c) (with $\eta = 0$) yields $h(\mathbf{k}, t; t_0) = (2\pi^2|\mathbf{k}|)^{-1}\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0))$, which is then inserted into Eq. (23a) to obtain,

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \theta(t - t_0)(2\pi^2)^{-1}\int d^3\mathbf{k}\;(\sin(|\mathbf{k}|c(t - t_0))/|\mathbf{k}|)\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)} =$$

$$(\theta(t - t_0)/\pi)\int_0^\infty kdk\;\sin(kc(t - t_0))\int_{-1}^{1} d\alpha\; e^{ik|\mathbf{r}-\mathbf{r}_0|\alpha} =$$

$$\big(2\theta(t - t_0)/(\pi|\mathbf{r} - \mathbf{r}_0|)\big)\int_0^\infty dk\;\sin(kc(t - t_0))\sin(k|\mathbf{r} - \mathbf{r}_0|) =$$

$$\big(\theta(t - t_0)/(2\pi|\mathbf{r} - \mathbf{r}_0|)\big)\int_{-\infty}^{\infty} dk\big[\cos\big(k(|\mathbf{r} - \mathbf{r}_0| - c(t - t_0))\big) - \cos\big(k(|\mathbf{r} - \mathbf{r}_0| + c(t - t_0))\big)\big] =$$

$$\big(\theta(t - t_0)/|\mathbf{r} - \mathbf{r}_0|\big)\delta(|\mathbf{r} - \mathbf{r}_0| - c(t - t_0)). \tag{23f}$$

The causal Heaviside step function $\theta(t-t_0)$ *eliminates a second delta function of argument* $|\mathbf{r}-\mathbf{r}_0|+c(t-t_0)$ *from the* Eq. (23f) *result.* A slightly more compact, neater form of that result for $G(\mathbf{r},t;\mathbf{r}_0,t_0)$ is,

$$G(\mathbf{r},t;\mathbf{r}_0,t_0) = 2\theta(t-t_0)\,\delta(|\mathbf{r}-\mathbf{r}_0|^2 - c^2(t-t_0)^2), \tag{23g}$$

which is nonzero *only on precisely the* Eq. (22a) lightspeed-expanding spherical-shell wavefront space-time locus $|\mathbf{r}-\mathbf{r}_0|^2 = c^2(t-t_0)^2$ for $t \geq t_0$.

We now alternatively obtain the Eq. (19a) scalar potential ϕ for a charge-q, constant-velocity-$\mathbf{v}$ point entity by propagating its charge density $\rho(\mathbf{r},t) = q\delta^{(3)}(\mathbf{r}-\mathbf{v}t)$ with the Eq. (23g) causal Green's function,

$$\begin{gathered}\phi(\mathbf{r},t) = \int G(\mathbf{r},t;\mathbf{r}_0,t_0)\,\rho(\mathbf{r}_0,t_0)\,d^3\mathbf{r}_0\,d(ct_0) = \\ 2q\int\theta(t-t_0)\,\delta(|\mathbf{r}-\mathbf{r}_0|^2 - c^2(t-t_0)^2)\,\delta^{(3)}(\mathbf{r}_0-\mathbf{v}t_0)\,d^3\mathbf{r}_0\,d(ct_0) = \\ 2qc\int_{-\infty}^{t}\delta(|\mathbf{r}-\mathbf{v}t_0|^2 - c^2(t-t_0)^2)\,dt_0 = \\ 2qc\int_{-\infty}^{t}\delta\big(|\mathbf{r}|^2-(ct)^2+2(ct_0)[(ct)-((\mathbf{v}/c)\cdot\mathbf{r})]-(ct_0)^2(1-|\mathbf{v}/c|^2)\big)dt_0 = \\ 2qc\int_{-\infty}^{t}\delta\big((ct_0)^2(1-|\mathbf{v}/c|^2)-2(ct_0)[(ct)-((\mathbf{v}/c)\cdot\mathbf{r})]+(ct)^2-|\mathbf{r}|^2\big)dt_0.\end{gathered} \tag{24a}$$

where in the last step the sign of argument of the delta function was reversed, which is permitted since the delta function is an even function. The next step will be to change the variable of integration from from t_0 to $y = -(ct_0/\gamma)$, where, as usual, $\gamma = \big(1/\sqrt{1-|\mathbf{v}/c|^2}\,\big)$. Therefore $(ct_0) = -\gamma y$ and $dt_0 = -(\gamma/c)dy$,

$$\phi(\mathbf{r},t) = 2q\gamma\int_{-(ct/\gamma)}^{\infty}\delta\big(y^2+2y\gamma[(ct)-((\mathbf{v}/c)\cdot\mathbf{r})]+(ct)^2-|\mathbf{r}|^2\big)dy. \tag{24b}$$

The final change of the variable of integration is from y to $l = y+\gamma[(ct)-((\mathbf{v}/c)\cdot\mathbf{r})]$, so $dy = dl$. The lower limit of integration gets shifted from $-(ct/\gamma)$ to $-\gamma[((\mathbf{v}/c)\cdot\mathbf{r})-(1-(1/\gamma^2))(ct)] = -\gamma(\mathbf{v}/c)\cdot(\mathbf{r}-\mathbf{v}t)$,

$$\phi(\mathbf{r},t) = 2q\gamma\int_{-\gamma(\mathbf{v}/c)\cdot(\mathbf{r}-\mathbf{v}t)}^{\infty}\delta\big(l^2-[|\mathbf{r}|^2+\gamma^2((ct)-((\mathbf{v}/c)\cdot\mathbf{r}))^2-(ct)^2]\big)dl \tag{24c}$$

We now express the entity $[|\mathbf{r}|^2+\gamma^2((ct)-((\mathbf{v}/c)\cdot\mathbf{r}))^2-(ct)^2]$ within the Eq. (24c) delta function in terms of the constant-velocity direction unit vector $\hat{\mathbf{u}} \equiv (\mathbf{v}/|\mathbf{v}|)$, *and we as well apply the identity* $\gamma^2-1 = \gamma^2|\mathbf{v}/c|^2$ *twice in order to obtain the particular algebraic form of this entity that we want,*

$$\begin{gathered}[|\mathbf{r}|^2+\gamma^2((ct)-((\mathbf{v}/c)\cdot\mathbf{r}))^2-(ct)^2] = [|\mathbf{r}|^2+\gamma^2((ct)-|\mathbf{v}/c|(\hat{\mathbf{u}}\cdot\mathbf{r}))^2-(ct)^2] = \\ |\mathbf{r}|^2+(\gamma^2-1)(ct)^2-2\gamma^2|\mathbf{v}|t(\hat{\mathbf{u}}\cdot\mathbf{r})+\gamma^2|\mathbf{v}/c|^2(\hat{\mathbf{u}}\cdot\mathbf{r})^2 = \\ |\mathbf{r}|^2+\gamma^2|\mathbf{v}/c|^2c^2t^2-2\gamma^2|\mathbf{v}|t(\hat{\mathbf{u}}\cdot\mathbf{r})+(\gamma^2-1)(\hat{\mathbf{u}}\cdot\mathbf{r})^2 = \\ |\mathbf{r}|^2-(\hat{\mathbf{u}}\cdot\mathbf{r})^2+\gamma^2[(\hat{\mathbf{u}}\cdot\mathbf{r})^2-2(\hat{\mathbf{u}}\cdot\mathbf{r})|\mathbf{v}|t+|\mathbf{v}|^2t^2] = \\ |\mathbf{r}|^2-(\hat{\mathbf{u}}\cdot\mathbf{r})^2+\gamma^2((\hat{\mathbf{u}}\cdot\mathbf{r})-|\mathbf{v}|t)^2 = |\mathbf{r}|^2-(\hat{\mathbf{u}}\cdot\mathbf{r})^2+(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))^2 = \\ |\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|^2.\end{gathered} \tag{24d}$$

We insert the result of Eq. (24d) into Eq. (24c) to obtain,

$$\begin{gathered}\phi(\mathbf{r},t) = 2q\gamma\int_{-|\mathbf{v}/c|(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))}^{\infty}\delta\big(l^2-|\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|^2\big)dl = \\ q\gamma/|\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|,\end{gathered} \tag{24e}$$

the Eq. (19a) result, obtained there using a different approach. As one sees from Eq. (24d), however, *getting the algebraic forms of the identical results* of the two approaches *to align* is awkward and tedious.

Lorentz boosts as "hyperbolic rotations of space-time planes" in four dimensions

Eq. (22c) shows that Lorentz boosts preserve $|\mathbf{r}|^2-(ct)^2 = (x_1)^2+(x_2)^2+(x_3)^2-(x_4)^2$, where we designate ct as x_4. That is *reminiscent* of the preservation of $|\mathbf{r}|^2 = (x_1)^2+(x_2)^2+(x_3)^2$ by ordinary space rotations, *which change only a plane* within the (x_1, x_2, x_3) space. To show that Lorentz boosts *change only a plane* within the (x_1, x_2, x_3, x_4) space, we re-express Eqs. (20a) and (20e) in terms of $x_4 \equiv ct$ and $\beta \equiv |\mathbf{v}/c|$,

$$\mathbf{r}' = \mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\gamma((\hat{\mathbf{u}}\cdot\mathbf{r})-\beta x_4)) \;\text{ and }\; x_4' = \gamma(x_4-\beta(\hat{\mathbf{u}}\cdot\mathbf{r})), \tag{25a}$$

which *changes only the* $(\hat{\mathbf{u}}\cdot\mathbf{r})$*—*$x_4$ *plane* within the four-dimensional $(\mathbf{r}, x_4)$ space, and leaves *unchanged* the four-dimensional vector $(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}), 0)$, which is *orthogonal* to that plane.

Furthermore, since, as is readily shown from Eq. (25a) and the definitions of β and γ in terms of $|\mathbf{v}/c|$,

$$(\hat{\mathbf{u}}\cdot\mathbf{r}') = \gamma((\hat{\mathbf{u}}\cdot\mathbf{r}) - \beta x_4),\quad x'_4 = \gamma(x_4 - \beta(\hat{\mathbf{u}}\cdot\mathbf{r}))\ \text{ and }\ \gamma^2 = 1/(1-\beta^2), \tag{25b}$$

it is straightforward to work out that,

$$(\hat{\mathbf{u}}\cdot\mathbf{r}')^2 - (x'_4)^2 = \gamma^2\big[((\hat{\mathbf{u}}\cdot\mathbf{r}) - \beta x_4)^2 - (x_4 - \beta(\hat{\mathbf{u}}\cdot\mathbf{r}))^2\big] = \gamma^2(1-\beta^2)\big[(\hat{\mathbf{u}}\cdot\mathbf{r})^2 - (x_4)^2\big] = (\hat{\mathbf{u}}\cdot\mathbf{r})^2 - (x_4)^2, \tag{25c}$$

so an Eq. (25a) Lorentz boost *also leaves the* $(\hat{\mathbf{u}}\cdot\mathbf{r})$*—*$x_4$ *planar quadratic form* $(\hat{\mathbf{u}}\cdot\mathbf{r})^2 - (x_4)^2$ *invariant.*

Using ordinary space rotations, we now build *as close an analogy to an* Eq. (25a) *Lorentz boost as we can achieve.* Since an Eq. (25a) Lorentz boost changes *only planes which contain the* x_4-axis, our analogy is restricted to rotations of *only planes which contain the* $(\hat{\mathbf{j}}\cdot\mathbf{r})$-axis, *where* $\hat{\mathbf{j}}$ *is a selected fixed unit vector,* i.e., $|\hat{\mathbf{j}}|^2 = 1$. Our analogy to an Eq. (25a) Lorentz boost also requires all *nonzero vectors* $\mathbf{s}$ *which are orthogonal to* $\hat{\mathbf{j}}$, i.e., $\mathbf{s} \neq \mathbf{0}$ and $\mathbf{s}\cdot\hat{\mathbf{j}} = 0$, and *unit vectors* $\hat{\mathbf{n}}$ *which are orthogonal to* $\hat{\mathbf{j}}$, i.e., $|\hat{\mathbf{n}}|^2 = 1$ and $\hat{\mathbf{n}}\cdot\hat{\mathbf{j}} = 0$. Our analogy to an Eq. (25a) Lorentz boost is specifically *the following rotation in the* $(\hat{\mathbf{n}}\cdot\mathbf{s})$—$(\hat{\mathbf{j}}\cdot\mathbf{r})$ *plane,*

$$\mathbf{s}' = \mathbf{s} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{s}) + \hat{\mathbf{n}}(\cos\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}) + \sin\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r})) = \mathbf{s} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{s}) + \hat{\mathbf{n}}(\cos\theta\,((\hat{\mathbf{n}}\cdot\mathbf{s}) + \tan\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r})))\ \text{and,}$$

$$(\hat{\mathbf{j}}\cdot\mathbf{r}') = \cos\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r}) - \sin\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}) = \cos\theta\,((\hat{\mathbf{j}}\cdot\mathbf{r}) - \tan\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s})). \tag{26a}$$

This rotation in the $(\hat{\mathbf{n}}\cdot\mathbf{s})$—$(\hat{\mathbf{j}}\cdot\mathbf{r})$ plane leaves the vector $\mathbf{s} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{s})$, which is *orthogonal* to that plane, *unchanged.* Furthermore, since Eq. (26a) implies that,

$$(\hat{\mathbf{n}}\cdot\mathbf{s}') = \cos\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}) + \sin\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r})\ \text{ and }\ (\hat{\mathbf{j}}\cdot\mathbf{r}') = \cos\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r}) - \sin\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}), \tag{26b}$$

it follows that,

$$(\hat{\mathbf{n}}\cdot\mathbf{s}')^2 + (\hat{\mathbf{j}}\cdot\mathbf{r}')^2 = \big(\cos\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}) + \sin\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r})\big)^2 + \big(\cos\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r}) - \sin\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s})\big)^2 = (\hat{\mathbf{n}}\cdot\mathbf{s})^2 + \big((\hat{\mathbf{j}}\cdot\mathbf{r})\big)^2, \tag{26c}$$

so this rotation in the $(\hat{\mathbf{n}}\cdot\mathbf{s})$—$(\hat{\mathbf{j}}\cdot\mathbf{r})$ plane *leaves the planar quadratic form* $(\hat{\mathbf{n}}\cdot\mathbf{s})^2 + ((\hat{\mathbf{j}}\cdot\mathbf{r}))^2$ *invariant.*

A *shortcoming* of our Eq. (26a) rotational analogy to an Eq. (25a) Lorentz boost *is the sign difference in their planar quadratic forms,* namely $(\hat{\mathbf{u}}\cdot\mathbf{r})^2 - (x_4)^2$ for an Eq. (25a) Lorentz boost, but $(\hat{\mathbf{n}}\cdot\mathbf{s})^2 + (\hat{\mathbf{j}}\cdot\mathbf{r})^2$ for our Eq. (26a) rotational analogy. So using our analogy to relate γ to $\cos\theta$ and β to $\tan\theta$ doesn't work. But inserting *an imaginary angle* $\theta = i\upsilon$ *into our analogy* is more encouraging: taking $\gamma = \cosh\upsilon$ and $\beta = \tanh\upsilon$ *is indeed viable since* $\gamma = 1/\sqrt{1-\beta^2}$. Thus an Eq. (25a) Lorentz boost *isn't an ordinary rotation, but a "hyperbolic" one,* which sometimes makes it useful to express Eq. (25a) in the form,

$$\mathbf{r}' = \mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\cosh\upsilon\,(\hat{\mathbf{u}}\cdot\mathbf{r}) - \sinh\upsilon\, x_4)\ \text{ and }\ x'_4 = \cosh\upsilon\, x_4 - \sinh\upsilon\,(\hat{\mathbf{u}}\cdot\mathbf{r}), \tag{27a}$$

where the unit vector $\hat{\mathbf{u}}$ and the rapidity υ are related to the boost velocity $\mathbf{v}$ as follows,

$$\hat{\mathbf{u}} = (\mathbf{v}/|\mathbf{v}|),\ \cosh\upsilon = 1/\sqrt{1-|\mathbf{v}/c|^2} = \gamma,\ \tanh\upsilon = |\mathbf{v}/c| = \beta,\ \sinh\upsilon = |\mathbf{v}/c|/\sqrt{1-|\mathbf{v}/c|^2} = \gamma\beta,$$

$$\exp(\upsilon) = \cosh\upsilon + \sinh\upsilon = \sqrt{(1+|\mathbf{v}/c|)/(1-|\mathbf{v}/c|)}\ \text{ and }\ \upsilon = (\ln(1+|\mathbf{v}/c|) - \ln(1-|\mathbf{v}/c|))/2, \tag{27b}$$

so when $|\mathbf{v}/c| \ll 1$, the rapidity υ is very close to $|\mathbf{v}/c| = \beta$, but as $|\mathbf{v}/c| \to 1$, the rapidity $\upsilon \to \infty$.

Both ordinary space rotations and Lorentz boosts *act only on planes,* but Lorentz boosts *furthermore act only on planes which include the* x_4*-axis.* If *the net effect* of *two successive Lorentz boosts* was to act on a plane *that didn't include the original* x_4*-axis* (Lorentz boosts, unlike Galilean boosts, *alter* the x_4 axis), those two successive Lorentz boosts *wouldn't be a Lorentz boost of the original coordinates.* In fact, *two successive Lorentz boosts in different directions never are a Lorentz boost of the original coordinates,* but successive Lorentz boosts *must preserve* $|\mathbf{r}|^2 - (x_4)^2$ *because all Lorentz boosts do,* so two Lorentz boosts in different directions *necessarily amount to a Lorentz boost followed by an ordinary space rotation—which is obliged to lie in the plane those two different directions determine.* An entity *whose acceleration* $\ddot{\mathbf{r}}(t)$ *is not parallel to its velocity* $\dot{\mathbf{r}}(t)$ goes from rest to velocity $\dot{\mathbf{r}}(t+\delta t) = \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\delta t$ *via two successive Lorentz boosts of velocities* $\dot{\mathbf{r}}(t)$ *and* $\ddot{\mathbf{r}}(t)\delta t$ *in different directions,* so that entity *undergoes an infinitesimal ordinary space rotation in the plane which* $\dot{\mathbf{r}}(t)$ *and* $\ddot{\mathbf{r}}(t)$ *determine.* The *angular rate* of those rotations times the unit normal to that plane is the *Thomas precession* vector angular velocity.

In Newtonian physics, a frictionless gyroscope always points in the same direction, for example, toward the North Star, regardless of acceleration of the vehicle carrying the gyroscope; that is the basis for inertial guidance systems. But Lorentz boosts *subject the gyroscope to Thomas precession,* slightly violating inertial guidance precepts. Another way to view this slight violation of inertial guidance precepts by Lorentz boosts is to note that from inside the vehicle the visible universe is Fitzgerald-contracted in the direction of the vehicle's velocity. Acceleration of the vehicle in a direction not parallel to its velocity continuously slightly rotates the direction of the Fitzgerald contraction of the visible universe, which continuously slightly shifts the apparent positions of a great many stars, to the slight detriment of inertial guidance.

We next work out the Thomas-precession vector angular velocity of an entity whose acceleration is not parallel to its velocity by finding the plane and infinitesimal angle of ordinary space rotation produced by two Lorentz boosts in different directions when the second boost has infinitesimal velocity.

Thomas-precession vector angular velocity from two Lorentz boosts in different directions

Before pondering the involved result of two Lorentz boosts in different directions, *it is extremely useful to be aware from* Eq. (27a) that for all *bona fide* Lorentz boosts, the x_4 partial derivative of $\mathbf{r}'(\mathbf{r}, x_4)$ and the spatial gradient $\nabla_{\mathbf{r}}$ of $x'_4(\mathbf{r}, x_4)$ *comprise redundant identical complete information about the boost velocity,*

$$-(\partial\,\mathbf{r}'(\mathbf{r}, x_4)/\partial x_4) = -\nabla_{\mathbf{r}}\, x'_4(\mathbf{r}, x_4) = \sinh\upsilon\,\hat{\mathbf{u}}. \tag{28a}$$

The *first* of the two Lorentz boosts in different directions has direction $\hat{\mathbf{u}}_1$ and rapidity υ_1 (see Eq. (27a)),

$$\mathbf{r}' = \mathbf{r} - \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1\cdot\mathbf{r}) + \hat{\mathbf{u}}_1(\cosh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \sinh\upsilon_1\,x_4) \;\text{ and }\; x'_4 = \cosh\upsilon_1\,x_4 - \sinh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}). \tag{28b}$$

The *second* boost has direction $\hat{\mathbf{u}}_2$, where $\hat{\mathbf{u}}_1\times\hat{\mathbf{u}}_2 \neq \mathbf{0}$, and infinitesimal rapidity $\delta\upsilon_2$ kept only to first order,

$$\mathbf{r}'' = \mathbf{r}' - \hat{\mathbf{u}}_2\,\delta\upsilon_2\,x'_4 \;\text{ and }\; x''_4 = x'_4 - \delta\upsilon_2\,(\hat{\mathbf{u}}_2\cdot\mathbf{r}'), \tag{28c}$$

Inserting Eq. (28b) into Eq. (28c) to eliminate $\mathbf{r}'$ and x'_4 from the latter produces the unwieldy result,

$$\mathbf{r}'' = \mathbf{r} - \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1\cdot\mathbf{r}) + \hat{\mathbf{u}}_1(\cosh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \sinh\upsilon_1\,x_4) - \hat{\mathbf{u}}_2\,\delta\upsilon_2(\cosh\upsilon_1\,x_4 - \sinh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r})) \;\text{ and}$$
$$x''_4 = \cosh\upsilon_1\,x_4 - \sinh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \delta\upsilon_2\,\hat{\mathbf{u}}_2\cdot(\mathbf{r} - \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1\cdot\mathbf{r}) + \hat{\mathbf{u}}_1(\cosh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \sinh\upsilon_1\,x_4)). \tag{28d}$$

which, after regrouping terms, reads,

$$\mathbf{r}'' = \mathbf{r} - \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1\cdot\mathbf{r}) + (\hat{\mathbf{u}}_1\cosh\upsilon_1 + \hat{\mathbf{u}}_2\,\delta\upsilon_2\sinh\upsilon_1)(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - (\hat{\mathbf{u}}_1\sinh\upsilon_1 + \hat{\mathbf{u}}_2\,\delta\upsilon_2\cosh\upsilon_1)x_4 \;\text{ and}$$
$$x''_4 = (\cosh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\sinh\upsilon_1)x_4 - (\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)(\cosh\upsilon_1 - 1))(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \delta\upsilon_2(\hat{\mathbf{u}}_2\cdot\mathbf{r}). \tag{28e}$$

Putting the Eq. (28e) space-time transformation *to the* Eq. (28a) *test for a bona fide Lorentz boost* yields,

$$-(\partial\,\mathbf{r}''(\mathbf{r}, x_4)/\partial x_4) = \sinh\upsilon_1\,\hat{\mathbf{u}}_1 + \delta\upsilon_2\,\cosh\upsilon_1\,\hat{\mathbf{u}}_2 \qquad \neq$$
$$-\nabla_{\mathbf{r}}\, x''_4(\mathbf{r}, x_4) = (\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)(\cosh\upsilon_1 - 1))\hat{\mathbf{u}}_1 + \delta\upsilon_2\,\hat{\mathbf{u}}_2, \tag{28f}$$

so *the two successive Lorentz boosts in different directions given by* Eq. (28e) *aren't a Lorentz boost,* and therefore *necessarily amount to a Lorentz boost followed by an ordinary space rotation.* Furthermore,

$$|\partial\,\mathbf{r}''(\mathbf{r}, x_4)/\partial x_4)|^2 = |\nabla_{\mathbf{r}}\, x''_4(\mathbf{r}, x_4)|^2 = \sinh^2\upsilon_1 + 2\delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\sinh\upsilon_1\cosh\upsilon_1 =$$
$$(\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\cosh\upsilon_1)^2 = (\sinh(\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)))^2, \tag{28g}$$

so $-(\partial\,\mathbf{r}''(\mathbf{r}, x_4)/\partial x_4)$ *is a rotation of* $-\nabla_{\mathbf{r}}\, x''_4(\mathbf{r}, x_4)$, which itself *definitely isn't rotated by the ordinary space rotation part of* Eq. (28e) *because* $x''_4(\mathbf{r}, x_4)$ *is the coordinate time component, a rotational invariant.*

The *two* Eq. (28a) *equalities for a bona fide boost, i.e.,* $-(\partial\,\mathbf{r}'(\mathbf{r}, x_4)/\partial x_4) = -\nabla_{\mathbf{r}}\, x'_4(\mathbf{r}, x_4) = \sinh\upsilon\,\hat{\mathbf{u}}$, *are modified as follows* for the Eq. (28e) Lorentz boost followed by an ordinary space rotation: the equality $-\nabla_{\mathbf{r}}\, x''_4(\mathbf{r}, x_4) = \sinh\upsilon\,\hat{\mathbf{u}}$ *still holds for the rotationally-invariant* $-\nabla_{\mathbf{r}}\, x''_4(\mathbf{r}, x_4)$; it determines the velocity of the Lorentz boost part of Eq. (28e). However, $-(\partial\,\mathbf{r}''(\mathbf{r}, x_4)/\partial x_4)$ *becomes a rotation of the rotationally-invariant* $-\nabla_{\mathbf{r}}\, x''_4(\mathbf{r}, x_4)$; it determines the ordinary space rotation part of Eq. (28e).

From $-\nabla_{\mathbf{r}}\, x''_4(\mathbf{r}, x_4) = \sinh\upsilon\,\hat{\mathbf{u}}$, together with the last equality of Eq. (28g) it follows that υ, the rapidity of the Lorentz boost part of Eq. (28e), is given by,

$$\upsilon = \upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2). \tag{28h}$$

From $-\nabla_{\mathbf{r}}\, x_4''(\mathbf{r}, x_4) = \sinh \upsilon\, \hat{\mathbf{u}}$ and Eq. (28h), the velocity $\mathbf{v}$ of the Lorentz boost part of Eq. (28e) is, $\mathbf{v} = c \tanh \upsilon\, \hat{\mathbf{u}} = c(-\nabla_{\mathbf{r}}\, x_4''(\mathbf{r}, x_4))/(\cosh \upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2) \sinh \upsilon_1)$. Inserting Eq. (28f) into this yields,

$$\mathbf{v} = c((\sinh \upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2)(\cosh \upsilon_1 - 1))\hat{\mathbf{u}}_1 + \delta\upsilon_2\, \hat{\mathbf{u}}_2)/(\cosh \upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2) \sinh \upsilon_1) =$$

$$c((\tanh \upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2)((\cosh \upsilon_1 - 1)/\cosh \upsilon_1))\hat{\mathbf{u}}_1 + (\delta\upsilon_2/\cosh \upsilon_1)\, \hat{\mathbf{u}}_2)(1 - \delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2) \tanh \upsilon_1) =$$

$$c((\tanh \upsilon_1 - \delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2)(\tanh^2 \upsilon_1/(\cosh \upsilon_1 + 1)))\hat{\mathbf{u}}_1 + (\delta\upsilon_2/\cosh \upsilon_1)\, \hat{\mathbf{u}}_2) =$$

$$\big[1 - \big(\mathbf{v}_1 \cdot \delta\mathbf{v}_2/\big(c^2(1 + \gamma_1)\big)\big)\big]\mathbf{v}_1 + \big[1/\gamma_1\big]\delta\mathbf{v}_2, \tag{28i}$$

where, of course, $\gamma_1 = 1/\sqrt{1 - |\mathbf{v}_1/c|^2} = \cosh \upsilon_1$.

The ordinary space rotation part of Eq. (28e) *is equal to the rotation between* $-\nabla_{\mathbf{r}}\, x_4''(\mathbf{r}, x_4)$ *and* $-(\partial\, \mathbf{r}''(\mathbf{r}, x_4)/\partial x_4)$, which differ only infinitesimally. Their cross product divided by the product of their equal norms gives the infinitesimal angle between them times the unit vector which is normal to the plane in which they lie. We denote that infinitesimal vector angle $\delta\boldsymbol{\Theta}$; *it as well provides the infinitesimal angle times the unit vector which is normal to the plane in which it lies of the ordinary space rotation part of* Eq. (28e). Using the Eq. (28f) and Eq. (28g) results for $-\nabla_{\mathbf{r}}\, x_4''(\mathbf{r}, x_4)$ and $-(\partial\, \mathbf{r}''(\mathbf{r}, x_4)/\partial x_4)$, we obtain for the infinitesimal vector angle $\delta\boldsymbol{\Theta}$,

$$\delta\boldsymbol{\Theta} = \big([-\nabla_{\mathbf{r}}\, x_4''(\mathbf{r}, x_4)] \times [-(\partial\, \mathbf{r}''(\mathbf{r}, x_4)/\partial x_4)]\big)\big/\big(|\nabla_{\mathbf{r}}\, x_4''(\mathbf{r}, x_4)|\, |\partial\, \mathbf{r}''(\mathbf{r}, x_4)/\partial x_4|\big) =$$

$$\big([(\sinh \upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2)(\cosh \upsilon_1 - 1))\hat{\mathbf{u}}_1 + \delta\upsilon_2\, \hat{\mathbf{u}}_2] \times [\sinh \upsilon_1\, \hat{\mathbf{u}}_1 + \delta\upsilon_2\, \cosh \upsilon_1\, \hat{\mathbf{u}}_2]\big)\big/$$

$$\big(\sinh^2 \upsilon_1 + 2\delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2) \sinh \upsilon_1 \cosh \upsilon_1\big) =$$

$$\delta\upsilon_2\, (\sinh \upsilon_1\, \cosh \upsilon_1 - \sinh \upsilon_1)(\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2)\big/\big(\sinh^2 \upsilon_1 + 2\delta\upsilon_2(\hat{\mathbf{u}}_1 \cdot \hat{\mathbf{u}}_2) \sinh \upsilon_1 \cosh \upsilon_1\big) =$$

$$\delta\upsilon_2((\cosh \upsilon_1 - 1)/\sinh \upsilon_1)(\hat{\mathbf{u}}_1 \times \hat{\mathbf{u}}_2). \tag{28j}$$

Eq. (28j) is written in terms of rapidities and velocity-direction unit vectors, but we need it expressed in terms of velocities. For the infinitesimal rapidity $\delta\upsilon_2$ that is straightforward because $\delta\upsilon_2 = |\delta\mathbf{v}_2/c|$, so $\delta\upsilon_2 \hat{\mathbf{u}}_2 = (\delta\mathbf{v}_2/c)$. For $\hat{\mathbf{u}}_1$ we proceed from its definition: $\hat{\mathbf{u}}_1 = (\mathbf{v}_1/|\mathbf{v}_1|) = (\mathbf{v}_1/(c \tanh \upsilon_1)) = (\cosh \upsilon_1/\sinh \upsilon_1)(\mathbf{v}_1/c)$. Putting the results $\delta\upsilon_2 \hat{\mathbf{u}}_2 = (\delta\mathbf{v}_2/c)$ and $\hat{\mathbf{u}}_1 = (\cosh \upsilon_1/\sinh \upsilon_1)(\mathbf{v}_1/c)$, into Eq. (28j) yields,

$$\delta\boldsymbol{\Theta} = (\cosh \upsilon_1(\cosh \upsilon_1 - 1)/\sinh^2 \upsilon_1)(\mathbf{v}_1 \times \delta\mathbf{v}_2)/c^2 =$$

$$(\cosh \upsilon_1/(\cosh \upsilon_1 + 1))(\mathbf{v}_1 \times \delta\mathbf{v}_2)/c^2 = \Big(1\Big/\Big(1 + \sqrt{1 - |\mathbf{v}_1/c|^2}\Big)\Big)(\mathbf{v}_1 \times \delta\mathbf{v}_2)/c^2. \tag{28k}$$

If an entity has trajectory $\mathbf{r}(t)$, then since $\dot{\mathbf{r}}(t + \delta t) = \dot{\mathbf{r}}(t) + \ddot{\mathbf{r}}(t)\delta t$, the boosts of $\mathbf{r}(t + \delta t)$ from instantaneous rest have the velocities $\mathbf{v}_1 = \dot{\mathbf{r}}(t)$ and $\delta\mathbf{v}_2 = \ddot{\mathbf{r}}(t)\delta t$. Since in addition, $d\boldsymbol{\Theta}(t)/dt = \delta\boldsymbol{\Theta}/\delta t$, Eq. (28k) implies,

$$d\boldsymbol{\Theta}(t)/dt = \Big(1\Big/\Big(1 + \sqrt{1 - |\dot{\mathbf{r}}(t)/c|^2}\Big)\Big)(\dot{\mathbf{r}}(t) \times \ddot{\mathbf{r}}(t))/c^2, \tag{28l}$$

the Thomas-precession vector angular velocity of an entity that has trajectory $\mathbf{r}(t)$.