Tutorial: (1) The Finite Lightspeed Electromagnetic Extension of the Galilean Boost (2) Thomas Precession: Surprising Rotation from Acceleration not Parallel to Velocity

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Abstract The Galilean-boost space-time transformation implies that accelerations are unchanged by changes of the relative constant velocity of the measuring instruments. Newton's Second Law then suggests that forces are also unchanged, but the torque a constant-velocity charge exerts on a compass needle at its point of closest approach changes with that velocity. This conflict motivates us to solve for the electromagnetic fields of a point charge moving at constant velocity, and to then inspect the results for the correct extension of the Galilean boost, a program we carry out by first deriving the electromagnetic wave equations, which we solve by Fourier methods, including indirectly via obtaining their causal Green's function. The constant-velocity point charge's electromagnetic fields display the space part of the Lorentz boost, which for infinite lightspeed becomes the space part of the Galilean boost. The time part of the Lorentz boost follows from its space part and the reciprocity of measuring instruments which have relative constant velocity. The frequently-presented consequences of the Lorentz boost are then developed in detail, as is the less familiar surprising Thomas-precession rotation produced by acceleration not parallel to velocity.

Galilean-boosted fields of a static charge: zero magnetic field and excess electric-field symmetry

Pondering the velocity of a person walking on the deck of a moving boat versus that of a person walking on a nearby stationary wharf, Galileo arrived at the hypothesis of *the additivity of constant velocities.* That hypothesis was broadened into the more comprehensive *Galilean-boost space-time transformation* of coordinates $(\mathbf{r}, t)$ *measured by stationary instruments* to the corresponding coordinates $(\mathbf{r}', t')$ *measured by instruments traveling at constant velocity* $\mathbf{v}$,

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t \;\text{ and }\; t' = t. \tag{1}$$

Two successive such Galilean-boost transformations *with constant velocities* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *respectively* yields,

$$\mathbf{r}'' = \mathbf{r}' - \mathbf{v}_2 t' = \mathbf{r} - \mathbf{v}_1 t - \mathbf{v}_2 t = \mathbf{r} - (\mathbf{v}_1 + \mathbf{v}_2)t, \text{ taking } \; t'' = t' = t \text{ into account}, \tag{2}$$

in accord with Galileo's hypothesis of *the additivity of the constant velocities* $\mathbf{v}_1$ and $\mathbf{v}_2$.

Additional important consequences of the Galilean-boost space-time transformation of Eq. (1) are that time intervals and *simultaneous-endpoint* space intervals *are the same* when measured by instruments traveling at any constant velocity $\mathbf{v}$ as they are when measured by stationary instruments,

$$\Delta t' = (t'_2 - t'_1) = (t_2 - t_1) = \Delta t, \text{ and also,}$$
$$\Delta\mathbf{r}' = (\mathbf{r}'_2 - \mathbf{r}'_1) = (\mathbf{r}_2 - \mathbf{r}_1) - \mathbf{v}(t_2 - t_1) = (\mathbf{r}_2 - \mathbf{r}_1) = \Delta\mathbf{r} \textit{ for } t_2 = t_1, \text{ i.e. } \textit{for } \Delta t = 0, \tag{3}$$

and that the boost velocity $\mathbf{v}$ *adds to velocities,* while accelerations *are the same,*

$$d\mathbf{r}/dt = d(\mathbf{r}' + \mathbf{v}t)/dt = d\mathbf{r}'/dt + \mathbf{v} = d\mathbf{r}'/dt' + \mathbf{v}, \text{ and also,}$$
$$d(d\mathbf{r}/dt)/dt = d(d\mathbf{r}'/dt' + \mathbf{v})/dt = d(d\mathbf{r}'/dt')/dt = d(d\mathbf{r}'/dt')/dt', \tag{4}$$

which in conjunction with Newton's Second Law suggests that *forces are also the same.* Indeed, the constant-velocity-$\mathbf{v} \neq \mathbf{0}$ Galilean boost of the fields $\mathbf{E} = q\mathbf{r}/|\mathbf{r}|^3$ and $\mathbf{B} = \mathbf{0}$ of a stationary charge-q point entity is,

$$\mathbf{E} = q(\mathbf{r} - \mathbf{v}t)/|\mathbf{r} - \mathbf{v}t|^3 \text{ and } \mathbf{B} = \mathbf{0}, \tag{5}$$

in *disagreement* with the peak torque a constant-velocity-$\mathbf{v} \neq \mathbf{0}$ point charge exerts on a magnetic-dipole compass needle. The Eq. (5) constant-velocity-$\mathbf{v} \neq \mathbf{0}$ Galilean-boosted point-charge field results *are invalid* except *in the limit* that the constant c in the electromagnetic field equations *is made infinite;* c among other things is the vacuum speed of light. The *vanishing magnetic field* result in Eq. (5) for a constant-velocity-$\mathbf{v} \neq \mathbf{0}$, charge-$q$ point entity occurs *in spite of that entity's nonvanishing current density* $\mathbf{j}$, namely,

$$\mathbf{j} = q\mathbf{v}\delta^{(3)}(\mathbf{r} - \mathbf{v}t), \tag{6}$$

whose insertion into the electromagnetic-field Biot-Savart/Maxwell Law, i.e.,

$$\nabla \times \mathbf{B} = (1/c)(4\pi\mathbf{j} + \partial\mathbf{E}/\partial t) = 4\pi q(\mathbf{v}/c)\delta^{(3)}(\mathbf{r} - \mathbf{v}t) + (1/c)(\partial\mathbf{E}/\partial t), \tag{7}$$

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obviously *must yield a nonvanishing magnetic field.* But *in the $c \to \infty$ limit* the Eq. (7) Biot-Savart/Maxwell Law *no longer implies that current densities produce magnetic fields.* The Galilean-boost transformation becomes "valid" *only* in *this physically-inapplicable $c \to \infty$ limit.* However, because *c is over twenty thousand times greater* than speeds which produce *permanent departure* from the earth's gravity, *earthbound biological creatures* have over the eons developed *the overwhelmingly strong intuition* that the Eqs. (2) through (4) implications of *the unphysical $c \to \infty$* Galilean-boost *are actually true.*

The magnetic field implied by the Eq. (7) Biot-Savart/Maxwell Law is *orthogonal* to the constant velocity $\mathbf{v}$ of the point charge which produces it, just as the magnetic field produced by a current flowing through a straight wire *is orthogonal to that wire.* Therefore the magnetic field produced by a constant-velocity point charge *has cylindrical symmetry only,* in marked contrast to *the spherical symmetry about* $\mathbf{r} = \mathbf{v}t$ exhibited by *the Galilean-boosted electric field of* Eq. (5).

But notwithstanding that the electic field of a static or a sufficiently slowly-moving point charge has or tends toward spherical symmetry, the electric field *of any moving point charge* in fact *gets modified* by its accompanying time-varying *only-cylindrically-symmetric magnetic field* because of *the influence of time-varying magnetic fields on electric fields* set out in Faraday's Law, i.e.,

$$\nabla \times \mathbf{E} = -(1/c)(\partial \mathbf{B}/\partial t). \tag{8}$$

Therefore *it is to be expected* that the spherical symmetry of the electic field of a static point charge *gets degraded,* to at least some degree, *to cylindrical symmetry* when that charge *has nonzero constant velocity* $\mathbf{v}$. However *in the physically-inapplicable $c \to \infty$* limit the Eq. (8) Faraday's Law shows that the time-varying, only-cylindrically-symmetric magnetic field *can no longer modify the electric field,* which accords with the spherical symmetry about $\mathbf{r} = \mathbf{v}t$ of the *unphysical* Eq. (5) Galilean-boosted point-charge electric field.

Having seen that the electromagnetic-field Biot-Savart/Maxwell Law and Faraday's Law *are incompatible with the Galilean boost,* as exemplified by the *unphysical* Eq. (5) Galilean-boosted electic and magnetic fields for constant-velocity-$\mathbf{v} \neq \mathbf{0}$, we are now *obliged* to solve for *the correct electric and magnetic fields produced by a constant-velocity-$\mathbf{v} \neq \mathbf{0}$, charge-$q$ point entity* using the four electromagnetic-field Laws, which, besides the Eq. (7) Biot-Savart/Maxwell Law and the Eq. (8) Faraday's Law, include Coulomb's Law,

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{9}$$

and Gauss' Law,

$$\nabla \cdot \mathbf{B} = 0, \tag{10}$$

with the charge density ρ and current density $\mathbf{j}$ of that constant-velocity-$\mathbf{v}$, charge-q point entity, which are,

$$\rho = q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t) \quad \text{and} \quad \mathbf{j} = q\,\mathbf{v}\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t) = \mathbf{v}\rho, \tag{11}$$

followed *by inspection of the results to find the correct finite-c replacement* for the $c \to \infty$ Galilean boost.

Fields of a constant-velocity point charge and the space-time boost transformation they imply

The Biot-Savart/Maxwell Law of Eq. (7) and the Faraday's Law of Eq. (8) couple the $\mathbf{B}$ and $\mathbf{E}$ fields to each other. We will use vector calculus to manipulate these two Laws to produce one equation which pertains exclusively to $\mathbf{B}$ together with a second equation which pertains exclusively to $\mathbf{E}$. Combining Gauss' Law of Eq. (10) with the equation for $\mathbf{B}$ and Coulomb's Law of Eq. (9) with the equation for $\mathbf{E}$ will then produce *identical-form wave equations for* $\mathbf{B}$ *and* $\mathbf{E}$ *that differ only in their sources.*

The *two sources* for the identical-form wave equations for $\mathbf{E}$ and $\mathbf{B}$ turn out to be linear combinations of *partial first derivatives* of the charge density ρ and the current density $\mathbf{j}$, but Eq. (11) *provides those densities themselves* for a constant-velocity charged point entity. To circumvent that disconnect, it is standard practice to define a scalar potential ϕ which satisfies the *same* wave-equation form, *but with $4\pi\rho$ as its source,* and to as well define a vector potential $\mathbf{A}$ which also satisfies the *same* wave equation form, *but with* $(4\pi/c)\mathbf{j}$ *as its source.* If the wave equations for these *potentials* ϕ and $\mathbf{A}$ *can be solved,* then the electromagnetic *fields* $\mathbf{E}$ and $\mathbf{B}$ will simply be linear combinations of *partial first derivatives of these potentials.*

Using Fourier-transform methods we will solve for the scalar potential ϕ whose source is 4π times the ρ of Eq. (11). The corresponding $\mathbf{j}$ of Eq. (11) is $\mathbf{v}\rho$, where $\mathbf{v}$ is constant, so the vector potential $\mathbf{A}$, whose source is $(4\pi/c)\mathbf{j} = (\mathbf{v}/c)(4\pi\rho)$, follows immediately from the scalar potential ϕ as,

$$\mathbf{A} = (\mathbf{v}/c)\phi. \tag{12}$$

Partial first differentiation of $\mathbf{A}$ and ϕ then yields $\mathbf{B}$ and $\mathbf{E}$. These results are to be inspected for the correct finite-c replacement of the unphysical $c \to \infty$ Galilean-boost space-time transformation given by Eq. (1).

We now turn to the calculational details of the electromagnetic field equation steps which we have so far in this section been outlining in words. Taking the curl of both sides of the Eq. (7) Biot-Savart/Maxwell Law, which is $(\nabla \times \mathbf{B}) = (1/c)(4\pi\mathbf{j} + (\partial\mathbf{E}/\partial t))$, yields,

$$\nabla \times (\nabla \times \mathbf{B}) = (4\pi/c)(\nabla \times \mathbf{j}) + (1/c)(\partial(\nabla \times \mathbf{E})/\partial t), \tag{13a}$$

so since it is an *identity* that $\nabla \times (\nabla \times \mathbf{B}) = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B}$, and the Eq. (8) Faraday's Law tells us that $\nabla \times \mathbf{E} = -(1/c)(\partial\mathbf{B}/\partial t)$, Eq. (13a) becomes,

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2\mathbf{B} = (4\pi/c)(\nabla \times \mathbf{j}) - (1/c)^2(\partial^2\mathbf{B}/\partial t^2). \tag{13b}$$

The Eq. (10) Gauss' Law tells us that $\nabla \cdot \mathbf{B} = 0$, which eliminates the first term of Eq. (13b), which then can readily be rearranged into the wave-equation form,

$$(1/c)^2(\partial^2\mathbf{B}/\partial t^2) - \nabla^2\mathbf{B} = (4\pi/c)(\nabla \times \mathbf{j}). \tag{13c}$$

Also, taking the curl of both sides of the Eq. (8) Faraday's Law, which is $(\nabla\times\mathbf{E}) = -(1/c)(\partial\mathbf{B}/\partial t)$, produces,

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -(1/c)(\partial(\nabla \times \mathbf{B})/\partial t). \tag{14a}$$

We now substitute the right side of the Eq. (7) Biot-Savart/Maxwell Law, which is $\nabla \times \mathbf{B} = (1/c)(4\pi\mathbf{j} + (\partial\mathbf{E}/\partial t))$, for the occurrence of $\nabla \times \mathbf{B}$ within the right side of the second equality of Eq. (14a) to obtain,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E} = -4\pi(1/c)^2(\partial\mathbf{j}/\partial t) - (1/c)^2(\partial^2\mathbf{E}/\partial t^2). \tag{14b}$$

The Eq. (9) Coulomb's Law tells us that $\nabla \cdot \mathbf{E} = 4\pi\rho$. The entity $\nabla \cdot \mathbf{E}$ *also* occurs within the first term on the left side of Eq. (14b); we now substitute $4\pi\rho$ for $\nabla \cdot \mathbf{E}$ in that term, after which the terms of Eq. (14b) are readily rearranged into the following wave-equation form,

$$(1/c)^2(\partial^2\mathbf{E}/\partial t^2) - \nabla^2\mathbf{E} = -4\pi\big(\nabla\rho + (1/c)^2(\partial\mathbf{j}/\partial t)\big). \tag{14c}$$

We already mentioned that the source terms on the right-hand sides of the Eq. (13c) and (14c) magnetic field and electric field wave equations are linear combinations of *partial first derivatives* of the charge density ρ and the current density $\mathbf{j}$, *not* linear combinations of ρ and $\mathbf{j}$ *themselves.* We also mentioned that that disconnect is circumvented by *instead* seeking the solutions of the closely-related electromagnetic scalar potential ϕ and vector potential $\mathbf{A}$ wave equations, which are,

$$(1/c)^2(\partial^2\phi/\partial t^2) - \nabla^2\phi = 4\pi\rho \quad\text{and}\quad (1/c)^2(\partial^2\mathbf{A}/\partial t^2) - \nabla^2\mathbf{A} = (4\pi/c)\mathbf{j}, \tag{15a}$$

since it is readily verified from Eqs. (13c), (14c) and (15a) that,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad\text{and}\quad \mathbf{E} = -\nabla\phi - (1/c)(\partial\mathbf{A}/\partial t). \tag{15b}$$

Our next task is to work out the scalar potential ϕ for a charge-q point entity which travels at constant velocity $\mathbf{v}$ that arises via the Eq. (15a) scalar-potential wave equation $(1/c)^2(\partial^2\phi/\partial t^2) - \nabla^2\phi = 4\pi\rho$ from the Eq. (11) charge density $\rho = q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t)$. We note once again from Eqs. (12) and (11) that in this particular case of a charged point entity that travels at constant velocity $\mathbf{v}$, $\mathbf{j} = \mathbf{v}\rho$ where $\mathbf{v}$ is constant, which implies that $\mathbf{A} = (\mathbf{v}/c)\phi$, obviating the need to separately solve for $\mathbf{A}$.

One approach to working out ϕ is via propagation of its source ρ with the causal Green's function for this electromagnetic form of wave equation. We will later pursue that approach since that Green's function's nonzero locus is an important space-time invariant under constant-velocity changes. Here, however, we approach ϕ *in a specialized way* based on the form of its source $\rho = q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t)$. The two approaches give *the same result for* ϕ via *initial algebraic forms of* ϕ *which have a markedly dissimilar appearance.*

Since Eq. (15a) tells us that to obtain the ϕ produced by $\rho = q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t)$, we must solve,

$$(1/c)^2(\partial^2\phi/\partial t^2) - \nabla^2\phi = 4\pi q\,\delta^{(3)}(\mathbf{r} - \mathbf{v}t) = (q/(2\pi^2))\int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}d^3\mathbf{k}, \tag{16a}$$

we make the *specialized* Fourier *ansatz* that,

$$\phi = \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}f(\mathbf{k})d^3\mathbf{k}, \tag{16b}$$

which, when inserted into Eq. (16a) yields,

$$f(\mathbf{k}) = (q/(2\pi^2))\big(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2\big)^{-1}, \tag{16c}$$

which inserted in turn into Eq. (16b) presents the solution ϕ of Eq. (16a) as the integral,

$$\phi = (q/(2\pi^2)) \int e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}\big(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2\big)^{-1} d^3\mathbf{k}. \tag{16d}$$

We assume that the charged point entity's constant speed $|\mathbf{v}|$ is less than that of light c, *so we can seek a linear change of the vector integration variable* $\mathbf{k}$ *to* $\mathbf{K}$ *such that* $|\mathbf{K}|^2 = |\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2$. To work out $\mathbf{K}$, it is useful to introduce the charged point entity's fixed travel-direction unit vector,

$$\hat{\mathbf{u}} \equiv (\mathbf{v}/|\mathbf{v}|), \text{ from which it follows that } |\hat{\mathbf{u}}|^2 = 1. \tag{17a}$$

We now decompose $\mathbf{k}$ into two parts which are respectively parallel to and perpendicular to both $\hat{\mathbf{u}}$ and $\mathbf{v}$,

$$\mathbf{k} = \mathbf{k}_{\|} + \mathbf{k}_{\perp}, \text{ where } \mathbf{k}_{\|} \equiv \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{k}) \text{ and } \mathbf{k}_{\perp} \equiv \mathbf{k} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{k}), \text{ which imply that,}$$

$$|\mathbf{v}|^2|\mathbf{k}_{\|}|^2 = ((\mathbf{k}\cdot\hat{\mathbf{u}})|\mathbf{v}|)^2 = (\mathbf{k}\cdot\mathbf{v})^2, \quad (\mathbf{v}\cdot\mathbf{k}_{\perp}) = 0, \quad (\mathbf{k}_{\|}\cdot\mathbf{k}_{\perp}) = 0 \text{ and } |\mathbf{k}|^2 = |\mathbf{k}_{\perp}|^2 + |\mathbf{k}_{\|}|^2. \tag{17b}$$

The Eq. (17b) orthogonal decomposition of $\mathbf{k}$ enables us to arrive at the *linear* relation of $\mathbf{K}$ to $\mathbf{k}$,

$$|\mathbf{K}|^2 = |\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2 = |\mathbf{k}|^2 - |\mathbf{v}/c|^2|\mathbf{k}_{\|}|^2 = |\mathbf{k}_{\perp}|^2 + (1-|\mathbf{v}/c|^2)|\mathbf{k}_{\|}|^2 = |\mathbf{k}_{\perp} + \sqrt{1-|\mathbf{v}/c|^2}\,\mathbf{k}_{\|}|^2, \tag{17c}$$

where the last equality reflects *the orthogonality of* $\mathbf{k}_{\perp}$ *to* $\mathbf{k}_{\|}$. The *linear* relation of $\mathbf{K}$ to $\mathbf{k}$ thus is,

$$\mathbf{K} = \mathbf{k}_{\perp} + \sqrt{(1-|\mathbf{v}/c|^2)}\,\mathbf{k}_{\|} = \mathbf{k} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{k}) + \sqrt{1-|\mathbf{v}/c|^2}\,\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{k}), \tag{17d}$$

where the second expression reflects the explicit formulas of Eq. (17b) for $\mathbf{k}_{\perp}$ and $\mathbf{k}_{\|}$ in terms of $\hat{\mathbf{u}}$ and $\mathbf{k}$.

To change the vector integration variable in Eq. (16d) from $\mathbf{k}$ to $\mathbf{K}$ we *also* need to solve Eq. (17d) for $\mathbf{k}$ in terms of $\mathbf{K}$, which requires only a few steps. Contracting both sides of Eq. (17d) with $\hat{\mathbf{u}}$ yields,

$$(\hat{\mathbf{u}}\cdot\mathbf{K}) = \sqrt{1-|\mathbf{v}/c|^2}\,(\hat{\mathbf{u}}\cdot\mathbf{k}) \quad\Rightarrow\quad (\hat{\mathbf{u}}\cdot\mathbf{k}) = \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}(\hat{\mathbf{u}}\cdot\mathbf{K}). \tag{17e}$$

Insertion of the Eq. (17e) result for $(\hat{\mathbf{u}}\cdot\mathbf{k})$ into the right side of Eq. (17d) in turn yields,

$$\mathbf{K} = \mathbf{k} - \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) \quad\Rightarrow$$

$$\mathbf{k} = \mathbf{K} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) + \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) = \mathbf{K}_{\perp} + \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}\mathbf{K}_{\|}. \tag{17f}$$

To effect the desired change of the vector integration variable in Eq. (16d), the right side of Eq. (17f) is substituted for all occurrences of $\mathbf{k}$ in Eq. (16d). A simple instance of such an occurrence in Eq. (16d) is,

$$d^3\mathbf{k} = d^2\mathbf{K}_{\perp}\left[\left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}d\mathbf{K}_{\|}\right] = \left(\sqrt{1-|\mathbf{v}/c|^2}\right)^{-1}d^3\mathbf{K} = \gamma\, d^3\mathbf{K}, \tag{18a}$$

where we henceforth use the standard abbreviation γ for $\left(\sqrt{1-|\mathbf{v}/c|^2}\;\right)^{-1}$. The occurrence in Eq. (16d) of $(|\mathbf{k}|^2 - (\mathbf{k}\cdot(\mathbf{v}/c))^2)^{-1}$ becomes just $|\mathbf{K}|^{-2}$, according to Eq. (17c). Finally, $\mathbf{k}$ occurs in Eq. (16d) within $e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)}$. Inserting the Eq. (17f) change of vector integration variable, namely $\mathbf{k} = \mathbf{K} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}) + \gamma\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K})$, into $\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t)$ and noting that $\mathbf{v}$ is perpendicular to $(\mathbf{K} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{K}))$ produces, upon changing the orders of factors to send all occurrences of $\mathbf{K}$ to the left side of the expression, that,

$$\mathbf{k}\cdot(\mathbf{r}-\mathbf{v}t) = \mathbf{K}\cdot(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))). \tag{18b}$$

Inserting the three consequences of the change of integration variable from $\mathbf{k}$ to $\mathbf{K}$ into Eq. (16d) yields,

$$\phi = (q\gamma/(2\pi^2)) \int e^{i\mathbf{K}\cdot(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t))))}|\mathbf{K}|^{-2}d^3\mathbf{K}. \tag{18c}$$

Unlike the Eq. (16d) *integral for* ϕ, the Eq. (18c) integral for ϕ *is just as easy to evaluate when* $\mathbf{v} \neq \mathbf{0}$ *as it is to evaluate when* $\mathbf{v} = \mathbf{0}$, the case for which *both* the Eq. (16d) integral *and* the Eq.(18c) integral *clearly yield* $\phi = q/|\mathbf{r}|$, *the static Coulomb potential.* So from its $\mathbf{v} = \mathbf{0}$ case, Eq. (18c) obviously yields,

$$\phi = q\gamma/|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|. \tag{19a}$$

It is pointed out in Eq. (12) and also below Eq. (15b) that $\mathbf{A} = (\mathbf{v}/c)\phi$, so,

$$\mathbf{A} = q\gamma(\mathbf{v}/c)/|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|. \tag{19b}$$

The $\mathbf{B}$ and $\mathbf{E}$ fields are obtained from $\mathbf{A}$ and ϕ by partial first differentiation using Eq. (15b),

$$\mathbf{B} = \nabla\times\mathbf{A} = q\gamma((\mathbf{v}/c)\times\mathbf{r})/|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|^3. \tag{19c}$$

$$\mathbf{E} = -\nabla\phi - (1/c)(\partial\mathbf{A}/\partial t) = q\gamma(\mathbf{r}-\mathbf{v}t)/|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|^3. \tag{19d}$$

In the *physically inapplicable* $c\to\infty$ *limit,* $\gamma\to 1$ and Eqs. (19c) and (19d) *revert to the* Eq. (5) *Galilean-boost results,* i.e., to $\mathbf{B} = \mathbf{0}$, *the nonexistence of magnetic fields in a Galilean universe,* and to $\mathbf{E} = q(\mathbf{r}-\mathbf{v}t)/|\mathbf{r}-\mathbf{v}t|^3$, *which is spherically symmetric around* $\mathbf{r} = \mathbf{v}t$, as expected in a Galilean universe.

The *actual* cylindrically-symmetric Eq. (19c) magnetic field *has vanishing component in the point-charge entity's direction of motion,* and so departs drastically from spherical symmetry. Electric fields of a point charge at rest are spherically symmetric, but Eq. (19d) shows that they technically shed spherical symmetry as soon as the point charge has nonzero velocity; in the extreme case that the point charge is moving at a speed sufficiently near that of light that $\gamma \gg 1$, Eq. (19d) shows that its electric field strongly "pancakes" in the charge's direction of motion; a behavior somewhat akin to a charge's magnetic field having zero component in the charge's direction of motion. Electric and magnetic fields become somewhat more similar as their sources approach the speed of light, albeit the Eq. (19c) magnetic field *is always everywhere orthogonal to the* Eq. (19d) *electric field.*

The denominators of the potentials and fields of Eqs. (19a) through (19d) *all display the vector* $(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t))))$, *which reverts to* $\mathbf{r}$ *in the static* $|\mathbf{v}| = 0$ *case,* as it of course must. In the unphysical $c\to\infty$ limit, which causes $\gamma\to 1$, it becomes $(\mathbf{r}-\mathbf{v}t)$, *the right side of the space part of the Galilean boost.* It also has *another,* more subtle, feature *in common* with the right side of the space part of the Galilean boost, namely that its part which is perpendicular to $\mathbf{v}$ (and therefore perpendicular to $\hat{\mathbf{u}} = (\mathbf{v}/|\mathbf{v}|)$) *is exactly the same as the part of merely* $\mathbf{r}$ *itself which is perpendicular to* $\mathbf{v}$.

The *part of an arbitrary vector* $\boldsymbol{\Xi}$ *which is parallel to* $\mathbf{v}$ is $\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\boldsymbol{\Xi})$, *where the unit vector* $\hat{\mathbf{u}} \equiv (\mathbf{v}/|\mathbf{v}|)$, so *the part of this arbitrary vector* $\boldsymbol{\Xi}$ *which is perpendicular* $\mathbf{v}$ *is* $(\boldsymbol{\Xi} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\boldsymbol{\Xi}))$. Thus the part of $\mathbf{r}$ *which is perpendicular to* $\mathbf{v}$ is $(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}))$. The part of $(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t))))$ *which is parallel to* $\mathbf{v}$ is $\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))$, so *the part of* $(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t))))$ *which is perpendicular to* $\mathbf{v}$ *is also* $(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}))$. Similarly, the part of $(\mathbf{r}-\mathbf{v}t)$, the right side of the space part of the Galilean boost, which is *parallel to* $\mathbf{v}$ is $(\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) - \mathbf{v}t)$, so the part of $(\mathbf{r}-\mathbf{v}t)$ *which is perpendicular to* $\mathbf{v}$ *is as well* $(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}))$.

Therefore the Galilean boost, whose space part is $\mathbf{r}' = \mathbf{r}-\mathbf{v}t$, *leaves invariant the part of* $\mathbf{r}$ *which is perpendicular to* $\mathbf{v}$, namely $(\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}))$. Likewise, if we were to take $\mathbf{r}' = (\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t))))$ *to be the space part of the finite-c extension of the physically inapplicable* $c\to\infty$ *Galilean boost,* then *that finite-c Galilean-boost extension would as well leave invariant the part of* $\mathbf{r}$ *which is perpendicular to* $\mathbf{v}$. *Other features the* Eqs. (19a)–(19d) *electromagnetic-field motivated* $\mathbf{r}' = (\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t))))$ *has in its favor to be the space part of the correct finite-c Galilean-boost extension* is that *it correctly reduces to* $\mathbf{r}' = \mathbf{r}$ *when* $|\mathbf{v}| = 0$ and that *it reduces to* $\mathbf{r}' = \mathbf{r}-\mathbf{v}t$, the space part of the Galilean boost, in the limit $c\to\infty$.

Of course the space part *alone* of a putative finite-c boost transformation *isn't enough,* we need *the time part as well.* The time part of a constant-velocity boost transformation is typically determined by the space part in conjunction with *the principle of relativistic reciprocity,* namely that two sets of instruments which have constant relative velocity $\mathbf{v}$ *are subject to exactly the same transformation rules.* Thus a boost transformation from unprimed to primed space-time coordinates *is required to also be valid when the unprimed and primed space-time coordinates are swapped,* with the understandable *additional stipulation that the sign of* $\mathbf{v}$ *is reversed as well.*

For example, the time part of the Galilean boost is obtained from its space part,

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t, \tag{19e}$$

and the relativistic reciprocal of this space part, which of course is,

$$\mathbf{r} = \mathbf{r}' + \mathbf{v}t', \tag{19f}$$

To show this, we first *contract both* Eqs. (19e) *and* (19f) *with* $\mathbf{v}$ to obtain the pair of *scalar equations,*

$$(\mathbf{v}\cdot\mathbf{r}') = (\mathbf{v}\cdot\mathbf{r}) - |\mathbf{v}|^2 t \quad \text{and} \quad (\mathbf{v}\cdot\mathbf{r}) = (\mathbf{v}\cdot\mathbf{r}') + |\mathbf{v}|^2 t'. \tag{19g}$$

The time part of the boost gives t' in terms of t and $(\mathbf{v}\cdot\mathbf{r})$, so we *eliminate* $(\mathbf{v}\cdot\mathbf{r}')$ by substituting the first of the Eq. (19g) pair of equations into the second one to obtain,

$$(\mathbf{v}\cdot\mathbf{r}) = (\mathbf{v}\cdot\mathbf{r}) - |\mathbf{v}|^2 t + |\mathbf{v}|^2 t', \tag{19h}$$

which yields that,

$$|\mathbf{v}|^2(t' - t) = 0, \tag{19i}$$

so *the time part* of any nontrivial Galilean boost, namely one having $\mathbf{v} \neq \mathbf{0}$, is,

$$t' = t, \tag{19j}$$

which is precisely the time part which Eq. (1) *gives for the Galilean boost.*

We now take the space part of the correct finite-c Galilean-boost extention to be the electromagnetic,

$$\mathbf{r}' = \mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t))), \tag{20a}$$

and obtain its time part *from this space part and its relativistic reciprocal,* which of course is,

$$\mathbf{r} = \mathbf{r}' - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}') + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}' + \mathbf{v}t'))). \tag{20b}$$

We next *contract both* Eq. (20a) *and* Eq. (20b) *with* $\mathbf{v}$ to obtain the pair of *scalar equations,*

$$(\mathbf{v}\cdot\mathbf{r}') = \gamma((\mathbf{v}\cdot\mathbf{r}) - |\mathbf{v}|^2 t) \quad \text{and} \quad (\mathbf{v}\cdot\mathbf{r}) = \gamma((\mathbf{v}\cdot\mathbf{r}') + |\mathbf{v}|^2 t'). \tag{20c}$$

The time part of the boost gives t' in terms of t and $(\mathbf{v}\cdot\mathbf{r})$, so we *eliminate* $(\mathbf{v}\cdot\mathbf{r}')$ by substituting the first of the Eq. (20c) pair of equations into the second one to obtain,

$$(\mathbf{v}\cdot\mathbf{r}) = \gamma(\gamma((\mathbf{v}\cdot\mathbf{r}) - |\mathbf{v}|^2 t) + |\mathbf{v}|^2 t'), \tag{20d}$$

which, with help from the identity $\gamma^2 - 1 = (|\mathbf{v}|^2/c^2)\gamma^2$, is readily solved for t' in terms of t and $(\mathbf{v}\cdot\mathbf{r})$,

$$t' = \gamma(t - ((\mathbf{v}\cdot\mathbf{r})/c^2)). \tag{20e}$$

Eqs. (20a) and (20e) together comprise the *the electromagnetic extension to finite c of the physically-inapplicable $c \to \infty$* Eq. (1) *Galilean boost.* It is universally known as *the Lorentz boost;* we obtained it via inspection of the electromagnetic fields produced by a point charge moving at constant velocity $\mathbf{v}$, which were first worked out by Oliver Heaviside. It was *because* George Fitzgerald had read Heaviside's paper *and had seen the electromagnetic fields of* Eqs. (19c) *and* (19d) that he was able to provide the physics world with its first solid clue as to proper interpretation of the unexpected null result of the Michelson-Morley experiment: although the physically-inapplicable $c \to \infty$ Galilean boost predicts, as we have seen in Eq. (3), that the boost velocity $\mathbf{v}$ *has no effect on space and time interval measurements,* that *isn't true of the Lorentz boost given by* Eqs. (20a) *and* (20e).

Consequences of the finite lightspeed Lorentz-boost extension of the unphysical Galilean boost

The Eq. (20e) time part of the finite-c Lorentz boost predicts that *unlike* the Eq. (3) result for the $c \to \infty$ Galilean boost, *the boost velocity* $\mathbf{v}$ *shortens the time interval measured by a stationary clock.* For time intervals $\Delta t' \equiv (t'_2 - t'_1)$ and $\Delta t \equiv (t_2 - t_1)$, and space interval $\Delta\mathbf{r} \equiv (\mathbf{r}_2 - \mathbf{r}_1)$, Eq. (20e) yields,

$\Delta t' = \gamma(\Delta t - ((\mathbf{v}\cdot\Delta\mathbf{r})/c^2)) = \gamma\Delta t$ for a clock *at a fixed location,* which obviously requires $\Delta\mathbf{r} = \mathbf{0}$. (21a)

Therefore,

$$\Delta t = (1/\gamma)(\Delta t') = \sqrt{1 - |\mathbf{v}/c|^2}\,(\Delta t'), \tag{21b}$$

so a time interval $\Delta t'$ recorded by a clock moving at constant velocity $\mathbf{v}$ *is recorded by a stationary clock as shortened by the factor* $\sqrt{1 - |\mathbf{v}/c|^2}$, i.e., a stationary observer who consults his own stationary clock notes that the moving clock ticks at a slower rate than his stationary clock. That may seem counterintuitive, but the intuition which *earthbound biological creatures* have developed over the eons *is inapplicable to that tick-rate disparity.* Such *earthbound creatures* could not possibly, over the eons, have developed an intuition for speeds as high as 43,000 km per hour, because that speed *permanently removes entities which attain it* *from the earth's gravitational influence.* Yet a clock moving at that speed has its tick rate *slowed by less than one part in a billion relative to the tick rate of a stationary clock,* which *isn't* within the competence *of the inherent sense of time of biological creatures* to detect.

The Eq. (20a) space part of the finite-c Lorentz boost predicts, somewhat analogously to its Eq. (20e) time part, that *unlike* the Eq. (3) result for the $c \to \infty$ Galilean boost, *the boost velocity* $\mathbf{v}$ *shortens the component of a space interval which is oriented in its direction, as measured by stationary instruments.* For space intervals $\Delta\mathbf{r}' \equiv (\mathbf{r}'_2 - \mathbf{r}'_1)$ and $\Delta\mathbf{r} \equiv (\mathbf{r}_2 - \mathbf{r}_1)$, and time interval $\Delta t \equiv (t_2 - t_1)$, Eq. (20a) yields,

$$\Delta\mathbf{r}' = \Delta\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r}) + \gamma\hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot (\Delta\mathbf{r} - \mathbf{v}\Delta t)). \tag{21c}$$

As usual for a space-interval measurement, that measurement must be simultaneous at the two ends of the space interval, so $t_2 = t_1$, which imples $\Delta t = 0$ in Eq. (21c), and therefore,

$$\Delta\mathbf{r}' = \Delta\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r}) + \gamma\hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r}). \tag{21d}$$

Using $\hat{\mathbf{u}}$, the unit-vector direction of the constant velocity $\mathbf{v}$, we now decompose both $\Delta\mathbf{r}'$ and $\Delta\mathbf{r}$ into two orthogonal vectors that are respectively perpendicular and parallel to $\mathbf{v}$,

$$\Delta\mathbf{r}' = \Delta\mathbf{r}'_\perp + \Delta\mathbf{r}'_\parallel, \quad \text{where} \quad \Delta\mathbf{r}'_\perp \equiv \Delta\mathbf{r}' - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r}') \quad \text{and} \quad \Delta\mathbf{r}'_\parallel \equiv \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r}'), \tag{21e}$$

and likewise for $\Delta\mathbf{r}$, i.e., $\Delta\mathbf{r} = \Delta\mathbf{r}_\perp + \Delta\mathbf{r}_\parallel$, where $\Delta\mathbf{r}_\perp \equiv \Delta\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r})$ and $\Delta\mathbf{r}_\parallel \equiv \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r})$. When these perpendicular/parallel decompositions of $\Delta\mathbf{r}'$ and $\Delta\mathbf{r}$ *are combined with* Eq. (21d), we obtain,

$$\Delta\mathbf{r}' = \Delta\mathbf{r}'_\perp + \Delta\mathbf{r}'_\parallel = (\Delta\mathbf{r}' - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r}')) + \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r}') = (\Delta\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r})) + \gamma\hat{\mathbf{u}}(\hat{\mathbf{u}} \cdot \Delta\mathbf{r}) = \Delta\mathbf{r}_\perp + \gamma\Delta\mathbf{r}_\parallel, \tag{21f}$$

which yields that,

$$\Delta\mathbf{r}'_\perp = \Delta\mathbf{r}_\perp \ \text{ and } \ \Delta\mathbf{r}'_\parallel = \gamma\Delta\mathbf{r}_\parallel, \tag{21g}$$

so $\Delta\mathbf{r}_\perp$ is unaffected by the boost, but for $\Delta\mathbf{r}_\parallel$ we have that,

$$\Delta\mathbf{r}_\parallel = (1/\gamma)(\Delta\mathbf{r}'_\parallel) = \sqrt{1 - |\mathbf{v}/c|^2}\,(\Delta\mathbf{r}'_\parallel). \tag{21h}$$

Thus a space interval moving at constant velocity $\mathbf{v}$ appears *to stationary instruments to be shortened in the direction of* $\mathbf{v}$ *by the factor* $\sqrt{1 - |\mathbf{v}/c|^2}$ *relative to the length measured by instruments that are traveling with that space interval.* This is the "Fitzgerald contraction" of moving objects in their direction of motion, which helped physicists begin to understand the results of the Michelson-Morley experiment.

That an object moving at constant velocity contracts in length in its direction of motion also seems at odds with intuition, and makes a great many people long for *the physically-inapplicable* $c \to \infty$ *Galilean boost.* Again the answer to concerns about contradicting the intuition developed over the eons by *earthbound biological creatures* is that they could not possibly have developed an intuition for a speed as high as 43,000 km per hour because that speed *permanently removes entities which attain it from the earth's gravitational influence,* but its Fitzgerald contraction that contradicts intuition will still be less than one part in a billion, which *isn't* within the competence *of biological organs* to detect.

That the finite-c Lorentz-boost extension of the $c \to \infty$ physically-inapplicable Galilean boost, *unlike* the latter, *doesn't* preserve space and time intervals, raises the question of whether there exists *any* space-time locus that the Lorentz boost *does* preserve. The fact that the speed of light c *is a constant of the electromagnetic field equations* suggests that the space-time locus of a light wavefront—whose evolution in time is of course governed by the constant speed c of light—*is* preserved by the finite-c Lorentz boost. In the theory of light wavefront evolution (Huygens), the underlying fundamental "building block" wavefront (from which *any other* wavefront results *only* through reinforcement) is the expanding spherical-shell wavefront whose radius grows uniformly with time at the rate c and has the space-time locus,

$$|\mathbf{r} - \mathbf{r}_0|^2 = c^2(t - t_0)^2 \text{ for } t \geq t_0. \tag{22a}$$

This lightspeed-expanding spherical-shell wavefront space-time locus turns out to be *faithfully embedded in the functional form* of the causal Green's function $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$ for electromagnetic wave equations,

$$(1/c)^2(\partial^2 G(\mathbf{r}, t; \mathbf{r}_0, t_0)/\partial t^2) - \nabla^2_{\mathbf{r}} G(\mathbf{r}, t; \mathbf{r}_0, t_0) = 4\pi\, \delta^{(3)}(\mathbf{r} - \mathbf{r}_0)\, \delta(c(t - t_0)). \tag{22b}$$

In the next section we sketch the steps which solve Eq. (22b) for the causal Green's function $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$ for electromagnetic wave equations, and then also sketch an alternate calculation of the Eq. (19a) scalar potential ϕ for a point entity of charge q which moves at constant velocity $\mathbf{v}$ by propagating the effect of its source charge density $\rho = q\delta^{(3)}(\mathbf{r} - \mathbf{v}t)$ with the causal Green's function $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$ of Eq. (22b).

First, however, we verify *a key property* of the finite-c Lorentz boost transformation of Eqs. (20a) and (20e), namely that it preserves the space-time quadratic form $|\mathbf{r}|^2 - (ct)^2$ for all constant velocities $\mathbf{v}$ which satisfy $|\mathbf{v}| < c$; the fact that the Lorentz boost is guaranteed to preserve *only* $|\mathbf{r}|^2 - (ct)^2$ implies that *it doesn't necessarily preserve* the lengths of space and time intervals. Of course its preservation of $|\mathbf{r}|^2 - (ct)^2$ implies that it in particular preserves the lightspeed-expanding spherical-shell wavefront space-time locus $|\mathbf{r}|^2 = (ct)^2$. This last property was Einstein's basic postulate *for deriving the finite-c Lorentz boost,* but one *is obliged* to *add* to that Einstein postulate a mild *further postulate* that articulates the specific property of the boost which boost velocity $\mathbf{v}$ entails, and one *is further obliged* to *add* to the Einstein postulate the *requirement* that the Galilean transformation be the consequence of the $c \to \infty$ limit of the finite-c boost.

The Eq. (20a) $\mathbf{r}'$ together with the Eq. (20e) t' imply that $|\mathbf{r}'|^2 - (ct')^2 = |\mathbf{r}|^2 - (ct)^2$ because,

$$|\mathbf{r}'|^2 - (ct')^2 = \left|\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r} - \mathbf{v}t)))\right|^2 - \big(\gamma(ct - ((\mathbf{v}/c)\cdot\mathbf{r}))\big)^2 =$$

$$|\mathbf{r}|^2 - (\hat{\mathbf{u}}\cdot\mathbf{r})^2 + \gamma^2\Big[\big((\hat{\mathbf{u}}\cdot\mathbf{r}) - |\mathbf{v}|t\big)^2 - \big(ct - (|\mathbf{v}|/c)(\hat{\mathbf{u}}\cdot\mathbf{r})\big)^2\Big] =$$

$$|\mathbf{r}|^2 - (\hat{\mathbf{u}}\cdot\mathbf{r})^2 + \gamma^2\Big[\big(1 - (|\mathbf{v}|/c)^2\big)(\hat{\mathbf{u}}\cdot\mathbf{r})^2 - \big(1 - (|\mathbf{v}|/c)^2\big)(ct)^2\Big] = |\mathbf{r}|^2 - (ct)^2. \tag{22c}$$

Solution for and application of the electromagnetic wave-equation causal Green's function

Since $\delta^{(3)}(\mathbf{r} - \mathbf{r}_0) = (2\pi)^{-3}\int d^3\mathbf{k}\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)}$, we make for the $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$ of Eq. (22b) the Fourier *ansatz,*

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \int d^3\mathbf{k}\; h(\mathbf{k}, t; t_0)\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)}, \tag{23a}$$

which when inserted into Eq. (22b) yields that $h(\mathbf{k}, t; t_0)$ satisfies the time differential equation,

$$(1/c)^2(\partial^2 h(\mathbf{k}, t; t_0)/\partial t^2) + |\mathbf{k}|^2 h(\mathbf{k}, t; t_0) = (2\pi^2 c)^{-1}\delta(t - t_0). \tag{23b}$$

In order to be *causal,* $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$, and therefore also $h(\mathbf{k}, t; t_0)$, must vanish when $t < t_0$. Using the Heaviside step function $\theta(t - t_0)$, which vanishes when $t < t_0$ and is equal to unity when $t \geq t_0$, we construct the following form for $h(\mathbf{k}, t; t_0)$ that is a causally proper solution of Eq. (23b) in both of the time regions $t < t_0$ and $t > t_0$,

$$h(\mathbf{k}, t; t_0) = K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0) + \eta), \tag{23c}$$

where the constants K and η are to be chosen to produce the singular function $(2\pi^2 c)^{-1}\delta(t - t_0)$ on the right side of Eq. (23b). Since the second time derivative of $\theta(t - t_0)$ is $\delta'(t - t_0)$, *yet no such derivative of a delta function occurs on the right side of* Eq. (23b), $h(\mathbf{k}, t; t_0)$ must be *continuous at* $t = t_0$, so the correct value of the constant phase η in Eq. (23c) is zero. With η set to zero, we now prepare to insert the right side of Eq. (23c) into Eq. (23b) to determine the correct value of the constant K,

$$(1/c)(\partial(K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0)))/\partial t) =$$

$$(K/c)[\delta(t - t_0)\sin(|\mathbf{k}|c(t - t_0)) + |\mathbf{k}|c\theta(t - t_0)\cos(|\mathbf{k}|c(t - t_0))] =$$

$$K|\mathbf{k}|\theta(t - t_0)\cos(|\mathbf{k}|c(t - t_0)). \tag{23d}$$

Using the Eq. (23d) result to differentiate $K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0))$ *again* with respect to time yields,

$$(1/c)^2(\partial^2(K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0)))/\partial t^2) = (1/c)(\partial(K|\mathbf{k}|\theta(t - t_0)\cos(|\mathbf{k}|c(t - t_0)))/\partial t) =$$

$$(K|\mathbf{k}|/c)[\delta(t - t_0)\cos(|\mathbf{k}|c(t - t_0)) - |\mathbf{k}|c\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0))] =$$

$$(K|\mathbf{k}|/c)\delta(t - t_0) - |\mathbf{k}|^2 K\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0)). \tag{23e}$$

Comparison of Eq. (23e) with Eq. (23b) shows that the correct value of K is $(2\pi^2|\mathbf{k}|)^{-1}$. Putting this value of K into Eq. (23c) (with η set to zero) yields $h(\mathbf{k}, t; t_0) = (2\pi^2|\mathbf{k}|)^{-1}\theta(t - t_0)\sin(|\mathbf{k}|c(t - t_0))$ which is then inserted into Eq. (23a) to obtain for $G(\mathbf{r}, t; \mathbf{r}_0, t_0)$,

$$G(\mathbf{r}, t; \mathbf{r}_0, t_0) = \theta(t - t_0)(2\pi^2)^{-1}\int d^3\mathbf{k}\;(\sin(|\mathbf{k}|c(t - t_0))/|\mathbf{k}|)\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)} =$$

$$(\theta(t-t_0)/\pi)\int_0^\infty kdk\,\sin(kc(t-t_0))\int_{-1}^{1}d\alpha\,e^{ik|\mathbf{r}-\mathbf{r}_0|\alpha}=$$

$$\big(2\theta(t-t_0)/(\pi|\mathbf{r}-\mathbf{r}_0|)\big)\int_0^\infty dk\,\sin(kc(t-t_0))\sin(k|\mathbf{r}-\mathbf{r}_0|)=$$

$$\big(\theta(t-t_0)/(2\pi|\mathbf{r}-\mathbf{r}_0|)\big)\int_{-\infty}^{\infty}dk\big[\cos\big(k(|\mathbf{r}-\mathbf{r}_0|-c(t-t_0))\big)-\cos\big(k(|\mathbf{r}-\mathbf{r}_0|+c(t-t_0))\big)\big]=$$

$$\big(\theta(t-t_0)/|\mathbf{r}-\mathbf{r}_0|\big)\delta(|\mathbf{r}-\mathbf{r}_0|-c(t-t_0)). \tag{23f}$$

A second delta function of argument $|\mathbf{r}-\mathbf{r}_0|+c(t-t_0)$ *doesn't occur in* Eq. (23f) because of the presence of the causal Heaviside step function $\theta(t-t_0)$. A slightly more compact, neater form of the Eq. (23f) result for the electromagnetic wave-equation causal Green's function $G(\mathbf{r},t;\mathbf{r}_0,t_0)$ is,

$$G(\mathbf{r},t;\mathbf{r}_0,t_0)=2\theta(t-t_0)\,\delta(|\mathbf{r}-\mathbf{r}_0|^2-c^2(t-t_0)^2), \tag{23g}$$

which is nonzero *only on precisely the* Eq. (22a) lightspeed-expanding spherical-shell wavefront space-time locus $|\mathbf{r}-\mathbf{r}_0|^2=c^2(t-t_0)^2$ for $t\geq t_0$.

We now sketch an alternate calculation of the Eq. (19a) scalar potential ϕ for a point entity of charge q which moves at constant velocity $\mathbf{v}$ by propagating the effect of its source charge density $\rho=q\delta^{(3)}(\mathbf{r}-\mathbf{v}t)$ with the Eq. (23g) causal Green's function $G(\mathbf{r},t;\mathbf{r}_0,t_0)=2\theta(t-t_0)\,\delta(|\mathbf{r}-\mathbf{r}_0|^2-c^2(t-t_0)^2)$,

$$\phi(\mathbf{r},t)=\int G(\mathbf{r},t;\mathbf{r}_0,t_0)\,\rho(\mathbf{r}_0,t_0)\,d^3\mathbf{r}_0\,d(ct_0)=$$

$$2q\int\theta(t-t_0)\,\delta(|\mathbf{r}-\mathbf{r}_0|^2-c^2(t-t_0)^2)\,\delta^{(3)}(\mathbf{r}_0-\mathbf{v}t_0)\,d^3\mathbf{r}_0\,d(ct_0)=$$

$$2qc\int_{-\infty}^{t}\delta(|\mathbf{r}-\mathbf{v}t_0|^2-c^2(t-t_0)^2)\,dt_0=$$

$$2qc\int_{-\infty}^{t}\delta\big(|\mathbf{r}|^2-(ct)^2+2(ct_0)[(ct)-((\mathbf{v}/c)\cdot\mathbf{r})]-(ct_0)^2(1-|\mathbf{v}/c|^2)\big)dt_0=$$

$$2qc\int_{-\infty}^{t}\delta\big((ct_0)^2(1-|\mathbf{v}/c|^2)-2(ct_0)[(ct)-((\mathbf{v}/c)\cdot\mathbf{r})]+(ct)^2-|\mathbf{r}|^2\big)dt_0. \tag{24a}$$

where in the last step the sign of argument of the delta function was reversed, which is permitted since the delta function is an even function. The next step will be to change the variable of integration from from t_0 to $y=-(ct_0/\gamma)$, where, as usual, $\gamma=\big(1/\sqrt{1-|\mathbf{v}/c|^2}\,\big)$. Therefore $(ct_0)=-\gamma y$ and $dt_0=-(\gamma/c)dy$,

$$\phi(\mathbf{r},t)=2q\gamma\int_{-(ct/\gamma)}^{\infty}\delta\big(y^2+2y\gamma[(ct)-((\mathbf{v}/c)\cdot\mathbf{r})]+(ct)^2-|\mathbf{r}|^2\big)dy. \tag{24b}$$

The final change of the variable of integration is from y to $l=y+\gamma[(ct)-((\mathbf{v}/c)\cdot\mathbf{r})]$, so $dy=dl$. The lower limit of integration gets shifted from $-(ct/\gamma)$ to $-\gamma[((\mathbf{v}/c)\cdot\mathbf{r})-(1-(1/\gamma^2))(ct)]=-\gamma(\mathbf{v}/c)\cdot(\mathbf{r}-\mathbf{v}t)$,

$$\phi(\mathbf{r},t)=2q\gamma\int_{-\gamma(\mathbf{v}/c)\cdot(\mathbf{r}-\mathbf{v}t)}^{\infty}\delta\big(l^2-[|\mathbf{r}|^2+\gamma^2((ct)-((\mathbf{v}/c)\cdot\mathbf{r}))^2-(ct)^2]\big)dl \tag{24c}$$

We now express the entity $[|\mathbf{r}|^2+\gamma^2((ct)-((\mathbf{v}/c)\cdot\mathbf{r}))^2-(ct)^2]$ within the Eq. (24c) delta function in terms of the constant-velocity direction unit vector $\hat{\mathbf{u}}\equiv(\mathbf{v}/|\mathbf{v}|)$, *and we as well apply the identity* $\gamma^2-1=\gamma^2|\mathbf{v}/c|^2$ *twice in order to obtain the particular algebraic form of this entity we want,*

$$[|\mathbf{r}|^2+\gamma^2((ct)-((\mathbf{v}/c)\cdot\mathbf{r}))^2-(ct)^2]=[|\mathbf{r}|^2+\gamma^2((ct)-|\mathbf{v}/c|(\hat{\mathbf{u}}\cdot\mathbf{r}))^2-(ct)^2]=$$

$$|\mathbf{r}|^2+(\gamma^2-1)(ct)^2-2\gamma^2|\mathbf{v}|t(\hat{\mathbf{u}}\cdot\mathbf{r})+\gamma^2|\mathbf{v}/c|^2(\hat{\mathbf{u}}\cdot\mathbf{r})^2=$$

$$|\mathbf{r}|^2+\gamma^2|\mathbf{v}/c|^2c^2t^2-2\gamma^2|\mathbf{v}|t(\hat{\mathbf{u}}\cdot\mathbf{r})+(\gamma^2-1)(\hat{\mathbf{u}}\cdot\mathbf{r})^2=$$

$$|\mathbf{r}|^2-(\hat{\mathbf{u}}\cdot\mathbf{r})^2+\gamma^2[(\hat{\mathbf{u}}\cdot\mathbf{r})^2-2(\hat{\mathbf{u}}\cdot\mathbf{r})|\mathbf{v}|t+|\mathbf{v}|^2t^2]=$$

$$|\mathbf{r}|^2-(\hat{\mathbf{u}}\cdot\mathbf{r})^2+\gamma^2((\hat{\mathbf{u}}\cdot\mathbf{r})-|\mathbf{v}|t)^2=|\mathbf{r}|^2-(\hat{\mathbf{u}}\cdot\mathbf{r})^2+(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))^2=$$

$$|\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|^2. \tag{24d}$$

We insert the result of Eq. (24d) into Eq. (24c) to obtain,

$$\phi(\mathbf{r},t)=2q\gamma\int_{-|\mathbf{v}/c|(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))}^{\infty}\delta\big(l^2-|\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|^2\big)dl=$$

$$q\gamma/|\mathbf{r}-\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r})+\hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot(\gamma(\mathbf{r}-\mathbf{v}t)))|, \tag{24e}$$

the Eq. (19a) result, obtained there using a different approach. As one sees from Eq. (24d), however, *getting the algebraic forms of the identical results* of the two approaches *to align* is awkward and tedious.

Lorentz boosts as "hyperbolic rotations of space-time planes" in four dimensions

Eq. (22c) demonstrates that finite-c Lorentz boosts preserve $|\mathbf{r}|^2 - (ct)^2 = (x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2$, where we are taking the liberty of designating ct as x_4. That preservation of the quadratic form $(x_1)^2 + (x_2)^2 + (x_3)^2 - (x_4)^2$ by linear *Lorentz-boost transformations* of the coordinates (x_1, x_2, x_3, x_4) is *reminiscent* of the preservation of the quadratic form $(x_1)^2 + (x_2)^2 + (x_3)^2$ by linear *planar-rotation transformations* of the coordinates (x_1, x_2, x_3). *To sharpen the analogy of Lorentz boosts to planar rotations* we now reexpress Eqs. (20a) and (20e) in terms of $x_4 \equiv ct$ and $\beta \equiv |\mathbf{v}/c| = \sqrt{1 - (1/\gamma^2)}$,

$$\mathbf{r}' = \mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\gamma((\hat{\mathbf{u}}\cdot\mathbf{r}) - \beta x_4)) \;\text{ and }\; x_4' = \gamma(x_4 - \beta(\hat{\mathbf{u}}\cdot\mathbf{r})), \tag{25a}$$

which transforms *only the* $(\hat{\mathbf{u}}\cdot\mathbf{r})$*—*$x_4$ *"plane" subspace of the four-dimensional space* (x_1, x_2, x_3, x_4), leaving the four-dimensional vector $(\mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}), 0)$, which is *orthogonal* to that "plane", *unchanged.*

Furthermore, since, as is readily verified from Eq. (25a) and the definitions of β and γ in terms of $|\mathbf{v}/c|$,

$$(\hat{\mathbf{u}}\cdot\mathbf{r}') = \gamma((\hat{\mathbf{u}}\cdot\mathbf{r}) - \beta x_4), \;\; x_4' = \gamma(x_4 - \beta(\hat{\mathbf{u}}\cdot\mathbf{r})) \;\text{ and }\; \gamma^2 = (1-\beta^2)^{-1}, \tag{25b}$$

we readily work out that,

$$(\hat{\mathbf{u}}\cdot\mathbf{r}')^2 - (x_4')^2 = \gamma^2\big[((\hat{\mathbf{u}}\cdot\mathbf{r}) - \beta x_4)^2 - (x_4 - \beta(\hat{\mathbf{u}}\cdot\mathbf{r}))^2\big] = \gamma^2(1-\beta^2)\big[(\hat{\mathbf{u}}\cdot\mathbf{r})^2 - (x_4)^2\big] = (\hat{\mathbf{u}}\cdot\mathbf{r})^2 - (x_4)^2. \tag{25c}$$

Thus a direction-$\hat{\mathbf{u}}$ Lorentz boost *leaves the* $(\hat{\mathbf{u}}\cdot\mathbf{r})$*—*$x_4$ *"planar" quadratic form* $(\hat{\mathbf{u}}\cdot\mathbf{r})^2 - (x_4)^2$ *invariant.*

It is interesting and instructive to construct *the closest analogy to the Lorentz boost of constant velocity* $\mathbf{v}$ that can be achieved *using ordinary rotations in the three dimensions* $(x_1, x_2, x_3) = (\mathbf{r})$. Since Lorentz boosts transform *only those "planes" which contain the* x_4*-axis,* we shall *restrict our analogy* to rotations of *only those planes which contain the* $(\hat{\mathbf{j}}\cdot\mathbf{r})$*-axis* where $\hat{\mathbf{j}}$ *is a completely fixed (although arbitrary) three-dimensional unit vector.* Besides the $(\hat{\mathbf{j}}\cdot\mathbf{r})$-axis, the ingredients of our restricted-rotations analogy to the Lorentz boost include all three-dimensional vectors $\mathbf{s}$ *which are orthogonal to* $\hat{\mathbf{j}}$, namely all three-dimensional $\mathbf{s}$ which satisfy $(\hat{\mathbf{j}}\cdot\mathbf{s}) = 0$; all of these $\mathbf{s}$ can be expressed in the form $\mathbf{s} = \mathbf{r} - \hat{\mathbf{j}}(\hat{\mathbf{j}}\cdot\mathbf{r})$, where $\mathbf{r}$ is a three-dimensional vector. Among the three-dimensional vectors $\mathbf{s}$ *which are orthogonal* to $\hat{\mathbf{j}}$, *the unit vectors* $\hat{\mathbf{n}}$ *are of special interest;* they satisfy $|\hat{\mathbf{n}}|^2 = 1$ and $(\hat{\mathbf{j}}\cdot\hat{\mathbf{n}}) = 0$. Using these ingredients, our planar rotational *analogy* of the Lorentz boost is *the following ordinary rotation in the* $(\hat{\mathbf{n}}\cdot\mathbf{s})$*—*$(\hat{\mathbf{j}}\cdot\mathbf{r})$ *plane,*

$$\mathbf{s}' = \mathbf{s} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{s}) + \hat{\mathbf{n}}(\cos\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}) + \sin\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r})) = \mathbf{s} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{s}) + \hat{\mathbf{n}}(\cos\theta\,((\hat{\mathbf{n}}\cdot\mathbf{s}) + \tan\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r}))) \text{ and,}$$
$$(\hat{\mathbf{j}}\cdot\mathbf{r}') = \cos\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r}) - \sin\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}) = \cos\theta\,((\hat{\mathbf{j}}\cdot\mathbf{r}) - \tan\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s})). \tag{26a}$$

This rotation in the $(\hat{\mathbf{n}}\cdot\mathbf{s})$—$(\hat{\mathbf{j}}\cdot\mathbf{r})$ plane leaves the vector $\mathbf{s} - \hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{s})$, which is *orthogonal* to that plane, *unchanged.* Furthermore, since Eq. (26a) implies that,

$$(\hat{\mathbf{n}}\cdot\mathbf{s}') = \cos\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}) + \sin\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r}) \;\text{ and }\; (\hat{\mathbf{j}}\cdot\mathbf{r}') = \cos\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r}) - \sin\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}), \tag{26b}$$

it follows that,

$$(\hat{\mathbf{n}}\cdot\mathbf{s}')^2 + (\hat{\mathbf{j}}\cdot\mathbf{r}')^2 = \big(\cos\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s}) + \sin\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r})\big)^2 + \big(\cos\theta\,(\hat{\mathbf{j}}\cdot\mathbf{r}) - \sin\theta\,(\hat{\mathbf{n}}\cdot\mathbf{s})\big)^2 = (\hat{\mathbf{n}}\cdot\mathbf{s})^2 + \big((\hat{\mathbf{j}}\cdot\mathbf{r})\big)^2, \tag{26c}$$

so this rotation in the $(\hat{\mathbf{n}}\cdot\mathbf{s})$—$(\hat{\mathbf{j}}\cdot\mathbf{r})$ plane *leaves the planar quadratic form* $(\hat{\mathbf{n}}\cdot\mathbf{s})^2 + ((\hat{\mathbf{j}}\cdot\mathbf{r}))^2$ *invariant.*

The *most obvious inadequacy* of this rotational analogy to Lorentz boosts is *the sign difference* in the associated planar quadratic forms, which are $(\hat{\mathbf{u}}\cdot\mathbf{r})^2 - (x_4)^2$ for the Lorentz boosts, but $(\hat{\mathbf{n}}\cdot\mathbf{s})^2 + (\hat{\mathbf{j}}\cdot\mathbf{r})^2$ for its rotational analogy. Therefore using the comparison of Eq. (25a) with Eq. (26a) to try to draw an analogy between γ and $\cos\theta$ and between β and $\tan\theta$ isn't useful. However, inserting *an imaginary angle* $\theta = i\upsilon$ *into the rotational analogy* is more enlightening; taking $\gamma = \cosh\upsilon$ and $\beta = \tanh\upsilon$ *is indeed viable in light of the fact that* $\gamma = (1/\sqrt{1-\beta^2})$. Thus Lorentz boosts *aren't ordinary circular (trigonometric) rotations, but "hyperbolic" ones*, which makes it useful to express Eq. (25a) in the form,

$$\mathbf{r}' = \mathbf{r} - \hat{\mathbf{u}}(\hat{\mathbf{u}}\cdot\mathbf{r}) + \hat{\mathbf{u}}(\cosh\upsilon\,(\hat{\mathbf{u}}\cdot\mathbf{r}) - \sinh\upsilon\,x_4) \;\text{ and }\; x_4' = \cosh\upsilon\,x_4 - \sinh\upsilon\,(\hat{\mathbf{u}}\cdot\mathbf{r}), \tag{27a}$$

where the unit vector $\hat{\mathbf{u}}$ and the rapidity parameter υ are related to the boost velocity $\mathbf{v}$ as follows,

$$\hat{\mathbf{u}} = (\mathbf{v}/|\mathbf{v}|), \;\; \cosh\upsilon = 1/\sqrt{1-|\mathbf{v}/c|^2} = \gamma, \;\; \tanh\upsilon = |\mathbf{v}/c| = \beta, \;\; \sinh\upsilon = |\mathbf{v}/c|/\sqrt{1-|\mathbf{v}/c|^2} = \gamma\beta,$$
$$\exp(\upsilon) = \cosh\upsilon + \sinh\upsilon = \sqrt{(1+|\mathbf{v}/c|)/(1-|\mathbf{v}/c|)} \;\text{ and }\; \upsilon = (\ln(1+|\mathbf{v}/c|) - \ln(1-|\mathbf{v}/c|))/2, \tag{27b}$$

so when $|\mathbf{v}/c| \ll 1$, υ is very close to $|\mathbf{v}/c| = \beta$, but as $|\mathbf{v}/c| \to 1$, $\upsilon \to \infty$.

Although Lorentz boosts *differ* from ordinary rotations in having *hyperbolic* instead of circular (i.e., trigonometric) character, they *share* the characteristic of rotations of affecting *only a planar, two-dimensional subspace of the space which they transform.* That notwithstanding, ordinary rotations *are free to transform any plane whatsoever of the three-dimensional space*, whereas Lorentz boosts *are rigidly constrained to transform only those planes of the four-dimensional space which include the x_4-axis.* An *issue* that arises with Lorentz boosts' *rigid limitation to planes which include the x_4 axis* is that *the net effect of two successive Lorentz boosts* may be to produce transformation of a plane in the four dimensional space *which doesn't include the initial x_4 axis* (of course Lorentz boosts, *unlike* Galilean boosts, *change* the x_4 axis). As a matter of fact, two successive Lorentz boosts in nonparallel directions *always* have *the net effect* of transforming a plane in the four-dimensional space *which doesn't include the initial x_4 axis,* so *two successive nonparallel Lorentz boosts don't comprise a Lorentz boost of the original space-time coordinates.* Such a pair of boosts, however, turns out to transform those original space-time coordinates *by a boost plus an ordinary space rotation.* An entity *which is accelerating in a direction not parallel to its velocity is continuously subjected to successive nonparallel boosts,* and therefore continuously undergoes ordinary space rotations, a condition known as *Thomas precession.*

In Newtonian theory, a frictionless gyroscope will always point in the same direction (toward the North Star, for example), regardless of any acceleration of the vehicle carrying the gyroscope, which is the basis for inertial guidance systems. But Lorentz boosts violate the precepts of inertial guidance to a very slight extent to produce the Thomas precession phenomenon. Another way to see that inertial guidance is very slightly at odds with the Lorentz-boost consequences of acceleration not parallel to velocity is to note that those inside a moving vehicle observe Fitzgerald contraction of the universe in the direction of their velocity, so acceleration of the vehicle in a direction not parallel to its velocity produces rotation of the direction in which the universe appears to be contracted to those inside the vehicle. A gyroscope which is supposed to faithfully point toward a given star thus faces the shifting of the apparent positions of a great many stars that is produced by acceleration not parallel to the vehicle's velocity.

We next work out the Thomas-precession vector angular velocity of an entity whose acceleration isn't parallel to its velocity. We do so by applying two successive Lorentz boosts in arbitrary nonparallel directions, the second of which is infinitesimal, since acceleration changes velocity smoothly. We verify that that pair of nonparallel boosts fails to produce solely a Lorentz boost of the original space-time coordinates, but instead effects a boost plus a rotation of those coordinates.

Nonparallel Lorentz-boost pairs and the vector angular velocity of Thomas precession

Before plunging into the algebra of a pair of nonparallel Lorentz boosts, *it is extremely useful* to note from Eq. (27a) that two particular partial derivatives of respectively the space part $\mathbf{r}'(\mathbf{r}, x_4)$ and the time part $x'_4(\mathbf{r}, x_4)$ of a *bona fide* Lorentz boost yield *identical complete information about the boost velocity,*

$$-(\partial\,\mathbf{r}'(\mathbf{r}, x_4)/\partial x_4) = -\nabla_{\mathbf{r}}\, x'_4(\mathbf{r}, x_4) = \sinh\upsilon\,\hat{\mathbf{u}}. \tag{28a}$$

The first of our nonparallel pair of Lorentz boosts is in direction $\hat{\mathbf{u}}_1$ with rapidity υ_1 (see Eq. (27a)),

$$\mathbf{r}' = \mathbf{r} - \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1\cdot\mathbf{r}) + \hat{\mathbf{u}}_1(\cosh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \sinh\upsilon_1\,x_4) \;\text{ and }\; x'_4 = \cosh\upsilon_1\,x_4 - \sinh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}), \tag{28b}$$

while our second boost is in direction $\hat{\mathbf{u}}_2$ with infinitesimal rapidity $\delta\upsilon_2$ that is kept only to first order,

$$\mathbf{r}'' = \mathbf{r}' - \hat{\mathbf{u}}_2\,\delta\upsilon_2\,x'_4 \;\text{ and }\; x''_4 = x'_4 - \delta\upsilon_2\,(\hat{\mathbf{u}}_2\cdot\mathbf{r}'), \tag{28c}$$

Inserting Eq. (28b) into Eq. (28c) to eliminate $\mathbf{r}'$ and x'_4 from the latter produces an unwieldy result,

$$\mathbf{r}'' = \mathbf{r} - \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1\cdot\mathbf{r}) + \hat{\mathbf{u}}_1(\cosh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \sinh\upsilon_1\,x_4) - \hat{\mathbf{u}}_2\,\delta\upsilon_2(\cosh\upsilon_1\,x_4 - \sinh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r})) \;\text{ and}$$

$$x''_4 = \cosh\upsilon_1\,x_4 - \sinh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \delta\upsilon_2\,\hat{\mathbf{u}}_2\cdot(\mathbf{r} - \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1\cdot\mathbf{r}) + \hat{\mathbf{u}}_1(\cosh\upsilon_1\,(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \sinh\upsilon_1\,x_4)). \tag{28d}$$

which, after regrouping terms reads,

$$\mathbf{r}'' = \mathbf{r} - \hat{\mathbf{u}}_1(\hat{\mathbf{u}}_1\cdot\mathbf{r}) + (\hat{\mathbf{u}}_1\cosh\upsilon_1 + \hat{\mathbf{u}}_2\,\delta\upsilon_2\sinh\upsilon_1)(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - (\hat{\mathbf{u}}_1\sinh\upsilon_1 + \hat{\mathbf{u}}_2\,\delta\upsilon_2\cosh\upsilon_1)x_4 \;\text{ and}$$

$$x''_4 = (\cosh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\sinh\upsilon_1)x_4 - (\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)(\cosh\upsilon_1 - 1))(\hat{\mathbf{u}}_1\cdot\mathbf{r}) - \delta\upsilon_2(\hat{\mathbf{u}}_2\cdot\mathbf{r}). \tag{28e}$$

Putting the Eq. (28e) space-time transformation *to the* Eq. (28a) *test for a bona fide Lorentz boost* yields,

$$-(\partial\,\mathbf{r}''(\mathbf{r},x_4)/\partial x_4) = \sinh\upsilon_1\,\hat{\mathbf{u}}_1 + \delta\upsilon_2\,\cosh\upsilon_1\,\hat{\mathbf{u}}_2 \quad \text{versus}$$
$$-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4) = (\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)(\cosh\upsilon_1 - 1))\hat{\mathbf{u}}_1 + \delta\upsilon_2\,\hat{\mathbf{u}}_2, \tag{28f}$$

so the Eq. (28e) *pair of nonparallel Lorentz boosts isn't a Lorentz boost.* However, we note *the norm equality,*

$$|\partial\,\mathbf{r}''(\mathbf{r},x_4)/\partial x_4)|^2 = |\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4)|^2 = \sinh^2\upsilon_1 + 2\delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\sinh\upsilon_1\cosh\upsilon_1 =$$
$$(\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\cosh\upsilon_1)^2 = (\sinh(\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)))^2, \tag{28g}$$

so $-(\partial\,\mathbf{r}''(\mathbf{r},x_4)/\partial x_4)$ *is a rotated version of* $-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4)$; it needs to be emphasized that $-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4)$ *isn't rotated because* $x_4''(\mathbf{r},x_4)$ *is the coordinate time component, which is rotationally invariant.*

Thus *the two* Eq. (28a) *equalities,* i.e., $-(\partial\,\mathbf{r}'(\mathbf{r},x_4)/\partial x_4) = -\nabla_{\mathbf{r}}\,x_4'(\mathbf{r},x_4) = \sinh\upsilon\,\hat{\mathbf{u}}$, *for bona fide Lorentz boosts* are *changed* for the Eq. (28e) non-Lorentz-boost pair of nonparallel Lorentz boosts as follows: the equality $-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4) = \sinh\upsilon\,\hat{\mathbf{u}}$ *still holds for the rotationally-invariant* $-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4)$*, and determines the boost part* of the Eq. (28e) non-Lorentz-boost pair of nonparallel Lorentz boosts, but $-(\partial\,\mathbf{r}''(\mathbf{r},x_4)/\partial x_4)$ *is a rotated version of the rotationally-invariant* $-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4)$*, and determines the rotation part* of the Eq. (28e) non-Lorentz-boost pair of nonparallel Lorentz boosts.

From $-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4) = \sinh\upsilon\,\hat{\mathbf{u}}$, together with the last equality of Eq. (28g) it follows that,

$$\upsilon = \upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2), \tag{28h}$$

for the boost part of the Eq. (28e) non-Lorentz-boost pair of nonparallel Lorentz boosts. We furthermore obtain from $-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4) = \sinh\upsilon\,\hat{\mathbf{u}}$ together with the results of Eqs. (28f) and (28g) that,

$$\hat{\mathbf{u}} = ((\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)(\cosh\upsilon_1 - 1))\hat{\mathbf{u}}_1 + \delta\upsilon_2\,\hat{\mathbf{u}}_2)/(\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\cosh\upsilon_1) =$$
$$(1 - (\delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)/\sinh\upsilon_1))\hat{\mathbf{u}}_1 + (\delta\upsilon_2/\sinh\upsilon_1)\hat{\mathbf{u}}_2. \tag{28i}$$

The Eq. (28h) rapidity parameter υ and the Eq. (28i) velocity-direction unit vector $\hat{\mathbf{u}}$ *fully specify the boost part* of the Eq. (28e) non-Lorentz-boost pair of nonparallel Lorentz boosts. The remaining *non-boost part* of the Eq. (28e) non-Lorentz-boost pair of nonparallel Lorentz boosts is *the rotation angle* it produces between $-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4)$ and $-(\partial\,\mathbf{r}''(\mathbf{r},x_4)/\partial x_4)$. Therefore *as a complement* to the velocity-direction unit vector $\hat{\mathbf{u}} = ((-\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4))/|\nabla_{\mathbf{r}}\,x_4''(\mathbf{r},x_4)|)$ obtained in Eq. (28i), we now obtain *its rotated counterpart* $\hat{\mathbf{u}}^{\mathrm{rot}} = ((-\partial\,\mathbf{r}''(\mathbf{r},x_4)/\partial x_4)/|\partial\,\mathbf{r}''(\mathbf{r},x_4)/\partial x_4|)$ from Eqs. (28f) and (28g),

$$\hat{\mathbf{u}}^{\mathrm{rot}} = (\sinh\upsilon_1\,\hat{\mathbf{u}}_1 + \delta\upsilon_2\,\cosh\upsilon_1\,\hat{\mathbf{u}}_2)/(\sinh\upsilon_1 + \delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\cosh\upsilon_1) =$$
$$(1 - (\delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\cosh\upsilon_1/\sinh\upsilon_1))\hat{\mathbf{u}}_1 + (\delta\upsilon_2\cosh\upsilon_1/\sinh\upsilon_1)\hat{\mathbf{u}}_2. \tag{28j}$$

The unit vectors $\hat{\mathbf{u}}$ and $\hat{\mathbf{u}}^{\mathrm{rot}}$ differ only by an infinitesimal amount. Therefore their cross product gives the infinitesimal angle between them multiplied by the unit normal vector to the plane in which they lie, which is *an infinitesimal vector angle* that we denote as $\delta\mathbf{\Theta}$,

$$\delta\mathbf{\Theta} = \hat{\mathbf{u}}\times\hat{\mathbf{u}}^{\mathrm{rot}} = [(1 - (\delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)/\sinh\upsilon_1))\hat{\mathbf{u}}_1 + (\delta\upsilon_2/\sinh\upsilon_1)\hat{\mathbf{u}}_2]\times$$
$$[(1 - (\delta\upsilon_2(\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_2)\cosh\upsilon_1/\sinh\upsilon_1))\hat{\mathbf{u}}_1 + (\delta\upsilon_2\cosh\upsilon_1/\sinh\upsilon_1)\hat{\mathbf{u}}_2] =$$
$$\delta\upsilon_2((\cosh\upsilon_1 - 1)/\sinh\upsilon_1)(\hat{\mathbf{u}}_1\times\hat{\mathbf{u}}_2). \tag{28k}$$

Eq. (28k) is written in terms of direction unit vectors and rapidities, but we need it expressed in terms of velocities. For the infinitesimal rapidity $\delta\upsilon_2$ that is straightforward because $\delta\upsilon_2 = |\delta\mathbf{v}_2/c|$, so $\delta\upsilon_2\hat{\mathbf{u}}_2 = (\delta\mathbf{v}_2/c)$. For $\hat{\mathbf{u}}_1$ we proceed from its definition: $\hat{\mathbf{u}}_1 = (\mathbf{v}_1/|\mathbf{v}_1|) = (\mathbf{v}_1/c)(\cosh\upsilon_1/\sinh\upsilon_1)$. Putting these results, namely $\delta\upsilon_2\hat{\mathbf{u}}_2 = (\delta\mathbf{v}_2/c)$ and $\hat{\mathbf{u}}_1 = (\mathbf{v}_1/c)(\cosh\upsilon_1/\sinh\upsilon_1)$, into Eq. (28k) yields,

$$\delta\mathbf{\Theta} = (\cosh\upsilon_1(\cosh\upsilon_1 - 1)/\sinh^2\upsilon_1)(\mathbf{v}_1\times\delta\mathbf{v}_2)/c^2 =$$
$$(\cosh\upsilon_1/(\cosh\upsilon_1 + 1))(\mathbf{v}_1\times\delta\mathbf{v}_2)/c^2 = \left(1\Big/\left(1 + \sqrt{1 - |\mathbf{v}_1/c|^2}\right)\right)(\mathbf{v}_1\times\delta\mathbf{v}_2)/c^2. \tag{28l}$$

If an entity has trajectory $\mathbf{r}(t)$, then in the infinitesimal time interval $[t,\, t+\delta t]$, $\mathbf{v}_1 = \dot{\mathbf{r}}(t)$, $\delta\mathbf{v}_2 = (\ddot{\mathbf{r}}(t))\delta t$ and $\delta\mathbf{\Theta} = (d\mathbf{\Theta}(t)/dt)\delta t$, so,

$$d\mathbf{\Theta}(t)/dt = \left(1\Big/\left(1 + \sqrt{1 - |\dot{\mathbf{r}}(t)/c|^2}\right)\right)(\dot{\mathbf{r}}(t)\times\ddot{\mathbf{r}}(t))/c^2, \tag{28l}$$

the Thomas-precession vector angular velocity of an entity whose trajectory is $\mathbf{r}(t)$.