Spin and the gauge invariance

R I Khrapko[1]
Physics Department, Moscow Aviation Institute, Moscow 125993, Russia

The current state of the problem of the electrodynamics angular momentum, which has a long history, is described. Two concepts of classical electrodynamics spin, existing simultaneously but differing in relation to the spin tensor, are presented. By the example of radiation from a rotating dipole, a violation of the equivalence of vector potentials coupled by the gauge transformation is demonstrated when calculating the spin flux. It is shown that only the potentials obtained by integrating the electric field over the time gives the correct value of the emitted spin. So the inevitable recognition of the electrodynamics spin changes the foundations of classical electrodynamics and demands abandoning the fundamental gauge invariance principle.

Key words: classical spin; circular polarization; electrodynamics
PACS 75.10.Hk

1. Introduction

The conceptual foundations of electromagnetic angular momentum have been revised since the 19th century. This article presents some overview and discussion of the rather confused matter of classical electrodynamics spin. The problem of the classical spin is just the serious unsolved problem that demands modifications of the grammar of our scientific description of the physical world and demands abandoning the fundamental gauge invariance principle.

Electromagnetic radiation of circular or elliptical polarization, no doubt, contains spin. This is clear already because such radiation consists of correlated photons, and photons have spin $\hbar$. Therefore the spin density should be attributed, in particular, to a plane unbounded electromagnetic wave of circular polarization in the same way as the densities of photon's energy and momentum are attributed to such a wave. We discuss this circumstance below. Due to the principle of correspondence in the transition from the quantum description to the classical one, the presence of spin in an electromagnetic wave should be described by classical electrodynamics. And there are currently two concepts of the spin of electromagnetic waves within the framework of classical electrodynamics. But these concepts are mutually exclusive.

According to the common concept (for example [1-8]), the density of the spin angular momentum $\mathbf{S}$ is proportional to the *gradient* of the intensity u^2 of the electromagnetic wave [1,4]

$$S_z \sim \frac{r}{u^2}\frac{\partial(u^2)}{\partial r}\,. \tag{1}$$

This means, in particular, that a plane unbounded electromagnetic wave, which is usually considered as a carrier of energy and momentum density (for example, [9]), have no spin. To have spin, the wave must be *real* in the sense that it must have a boundary where the intensity drops to zero. There, at the boundary, according to formula (1), the spin of photons is manifested, whose energy and momentum are distributed over the entire region occupied by the wave. So according to this concept, the spin of electromagnetic radiation is separated from the energy and momentum by a macroscopic distance. Heitler writes about it [2]:

> "A plane wave travelling in *z*-direction and with infinite extension in the *xy*-directions can have no angular momentum about the *z*-axis. However, this is no longer the case for the wave with finite extension in the *xy*-plane. It can be shown that the wall of such a wave packet gives a finite contribution to the angular momentum about the *z*-axis".

An alternative concept of angular momentum of light originated in the 19th century. According to Sadowsky and Poynting [10, 11], there is a density of angular momentum in an

[1] Email: khrapko_ri@hotmail.com, khrapko_ri@mai.ru , http://khrapkori.wmsite.ru

electromagnetic wave of circular polarization, and this density is proportional to the energy density and not to the gradient of the energy density. In this case, the spatial boundary of the wave is not considered at all. Pointing writes:

> If we put E for the energy in unit volume and G for the torque per unit area, we have $G = E\lambda / 2\pi$”

According to Weyssenhoff [12], the Sadowsky-Poynting concept allows us to consider an electromagnetic wave of circular polarization as a “spin fluid”. Weyssenhoff writes:

> “By spin-fluid we mean a fluid each element of which possesses besides energy and linear momentum also a certain amount of angular momentum, proportional – just as energy and the linear momentum – to the volume of the element”.

Indeed, since the time of Emma Noether, such an angular momentum density, which is not a moment of linear momentum, is recognized as a spin density and is described by a spin tensor density. The Lagrange formalism using the Lagrangian $\mathsf{L} = -F_{\mu\nu}F^{\mu\nu}/4$ leads to the canonical expression for the spin tensor [13-15]

$$\Upsilon_c^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu} \quad . \tag{2}$$

Here A^{λ} and $F^{\mu\nu}$ are, respectively, the vector magnetic potential and the field-strength tensor.

The spin tensor has been fruitfully used in recent decades. Calculations concerning plane waves and, in particular, spin transfer to a mirror are published [16-21]. Such calculations are impossible within the framework of the common concept. It is shown that the famous Beth experiment with a half-wave plate is inexplicable within the framework of the common concept and needs a spin tensor to explain [22-25]. The use of the spin tensor made it possible to detect the spin radiation of a rotating dipole [26-31].

The existence of the spin density, which is described by the gauge not invariant spin tensor, means that the potentials coupled by the gauge transformation are not equivalent to each other. The purpose of this article is to demonstrate, using a specific example of counting the spin radiation of a rotating dipole, how two different potentials, giving the same electromagnetic field, lead to different values of the spin flux.

Recall that the radiation from a rotating dipole is linearly polarized in the plane of rotation and circularly polarized in the direction of the axis of rotation. This is depicted in Figure 1, taken from the encyclopedia [32].

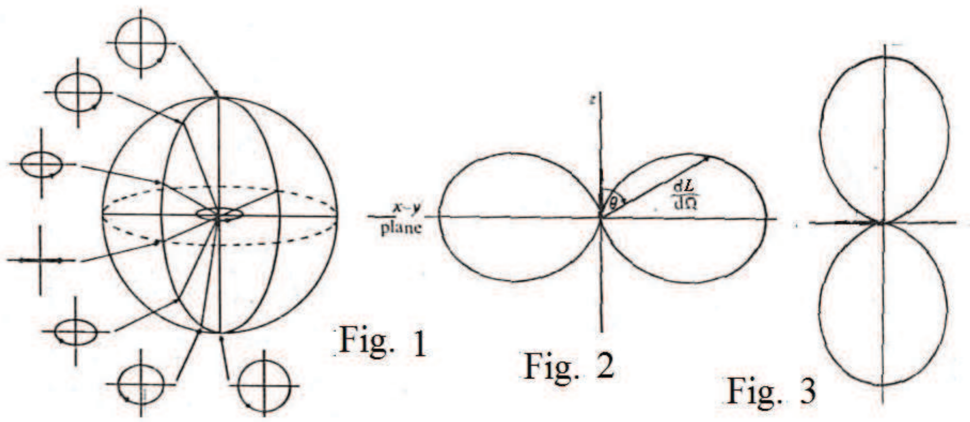

Fig. 1. Polarization of the radiation of a dipole rotating in xy-plane, observed from different directions.
Fig. 2. Angular distribution of the flux of the z-component of the orbital angular momentum $dL_z / d\Omega dt \propto \sin^2\theta$.
Fig. 3. Angular distribution of the flux of the z-component of the spin angular momentum $dS_z / d\Omega dt \propto \cos^2\theta$

This field of a rotating dipole contains a flow of moment of linear momentum, the angular distribution of which is shown in Figure 2. It is seen that this flow is contained mainly in the vicinity of the plane of rotation. The angular distribution of the flux is calculated by the formula for the moment of force $d\mathfrak{F}^j$, that acts on the area da_i, which is an element of the spherical surface [33-35]:

$$dL^{xy} / dt = r^{[x}T^{y]i} da_i, \quad d\mathfrak{F}^j = T^{ji} da_i \tag{3}$$

where T^{ji} is the Maxwell stress tensor. The spin radiation of the rotating dipole is directed mainly along the axis of rotation. Its angular distribution is shown in Figure 3 and is calculated in Section 2

Note that the emergence of the common concept is associated with the Belinfante-Rosenfeld procedure [36,37] and the Humblet transformation [38]. These documents was discussed in detail in the works [19,22,39]

2. Spin radiation by a rotating dipole

The physical meaning of the spin tensor $\Upsilon^{\lambda\mu\nu}$ is that the 4-spin of 4-volume element dV_ν is $dS^{\lambda\mu} = \Upsilon^{\lambda\mu\nu} dV_\nu$. This means, for example, that the component dS^{xy} of the spin passing through the area da_z in time dt is

$$dS^{xy} = \Upsilon^{xyz} da_z dt\,, \tag{4}$$

and the torque acting through this area is $d\tau^{xy} = dS^{xy} / dt = \Upsilon^{xyz} da_z$.

In this article, we use expression (4) with the component $\underset{c}{\Upsilon}^{xyr}$ of the canonical spin tensor (2) twice to visually demonstrate the difference that arises when the *temporal* vector potential is replaced by the standard potential obtained by the formula [40]

$$\phi = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{r} [\rho]_{\text{ret}} dV\,, \quad \mathbf{A} = \frac{1}{4\pi\varepsilon_0 c^2} \int \frac{1}{r} [\mathbf{j}]_{\text{ret}} dV\,. \tag{5}$$

We call the temporal potential the potential obtained by integrating the electric field over time, at a zero scalar potential. Since this gauge choice is the Weyl gauge, we denote temporal potential $\mathbf{W}$ so as not to be confused with $\mathbf{A}$. It turns out that the temporal potential gives the correct result, and the standard potential (5) gives the wrong result.

The starting point for constructing the canonical spin tensor with the temporal potential is the field of the dipole $\mathbf{p}\exp(-i\omega t)$ [8,41]

$$\mathbf{E} = \left[\frac{\omega^2 (\mathbf{p}r^2 - (\mathbf{pr})\mathbf{r})}{4\pi\varepsilon_0 c^2 r^3} + \frac{i\omega(\mathbf{p}r^2 - 3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0 c r^4} - \frac{(\mathbf{p}r^2 - 3(\mathbf{pr})\mathbf{r}}{4\pi\varepsilon_0 r^5} \right] \exp(ikr - i\omega t)\,, \tag{6}$$

$$\mathbf{H} = \left[\frac{\omega^2 \mathbf{r}\times\mathbf{p}}{4\pi c r^2} + \frac{i\omega \mathbf{r}\times\mathbf{p}}{4\pi r^3} \right] \exp(ikr - i\omega t)\,, \tag{7}$$

but we restrict ourselves only to the *radiation* field of rotating dipole.

$$\mathbf{E} = \frac{\omega^2 (\mathbf{p}r^2 - (\mathbf{pr})\mathbf{r})}{4\pi\varepsilon_0 c^2 r^3} \exp(ikr - i\omega t), \quad \mathbf{H} = \frac{\omega^2 \mathbf{r}\times\mathbf{p}}{4\pi c r^2} \exp(ikr - i\omega t)\,, \quad p_x = p, \quad p_y = ip\,. \tag{8}$$

This gives (no exponential factor)

$$E_x = F_{tx} = \frac{\omega^2 p(r^2 - x^2 - ixy)}{4\pi\varepsilon_o c^2 r^3}\,, \; E_y = F_{ty} = \frac{\omega^2 p(ir^2 - xy - iy^2)}{4\pi\varepsilon_o c^2 r^3}\,, \; E_z = F_{tz} = \frac{-\omega^2 p(zx + izy)}{4\pi\varepsilon_o c^2 r^3}\,, \tag{9}$$

$$H_x = F^{zy} = \frac{-i\omega^2 pz}{4\pi c r^2}\,, \quad H_y = F^{xz} = \frac{\omega^2 pz}{4\pi c r^2}\,, \quad H_z = F^{yx} = \frac{\omega^2 p(ix - y)}{4\pi c r^2}\,. \tag{10}$$

Since $\mathbf{E} = -\partial_t \mathbf{W} = i\omega \mathbf{W}$ or $F_{tx} = \partial_t W_x = -i\omega W_x$, we get

$$W^x = \frac{\omega p(-ir^2 + ix^2 - xy)}{4\pi\varepsilon_o c^2 r^3}, \quad W^y = \frac{\omega p(r^2 + ixy - y^2)}{4\pi\varepsilon_o c^2 r^3}, \quad W^z = \frac{\omega p(izx - zy)}{4\pi\varepsilon_o c^2 r^3} \tag{11}$$

taking $W^x = -W_x$ into account because of the metric signature used. According to (2) $\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2W^{[\lambda}F^{\mu]\nu}$, we have

$$\underset{c}{\Upsilon}^{xyx} = -\frac{\Re}{2}\{\bar{W}^x F^{yx}\} = \frac{\omega^3 z^2 x p^2}{32\pi^2\varepsilon_0 c^3 r^5}, \quad \underset{c}{\Upsilon}^{xyy} = \frac{\Re}{2}\{\bar{W}^y F^{xy}\} = \frac{\omega^3 z^2 y p^2}{32\pi^2\varepsilon_0 c^3 r^5}$$

$$\underset{c}{\Upsilon}^{xyz} = -\frac{\Re}{2}\{\bar{W}^x F^{yz} - \bar{W}^y F^{xz}\} = \frac{\omega^3 (r^2 + z^2) z p^2}{32\pi^2\varepsilon_0 c^3 r^5} \tag{12}$$

To calculate the infinitesimal spin (4) $dS^{xy} = \underset{c}{\Upsilon}^{xyl} da_l dt$, we need the Cartesian coordinates of the elements da_l of a spherical surface $r = \text{Const}$, the spherical coordinates of which look like this

$$d\mathbf{a} = \{da_r = d\theta d\varphi, da_\theta = 0, da_\varphi = 0\}.$$

Conversion factors between Cartesian and spherical coordinates

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r} \quad \text{and} \quad \sqrt{g} = r^2 \sin\theta$$

allow us to get the desired Cartesian coordinates of the elements:

$$da_l = \{da_x = xr\sin\theta d\theta d\varphi, da_y = yr\sin\theta d\theta d\varphi, da_z = zr\sin\theta d\theta d\varphi\}$$

As a result, we obtain the flux of the spin component through the surface element da_l

$$dS^{xy}/dt = \underset{c}{\Upsilon}^{xyl} da_l = \underset{c}{\Upsilon}^{xyx} da_x + \underset{c}{\Upsilon}^{xyy} da_y + \underset{c}{\Upsilon}^{xyz} da_z =$$

$$= \frac{\omega^3 p^2 (z^2x^2 + z^2y^2 + r^2z^2 + z^4)}{32\pi^2\varepsilon_0 c^3 r^4} \sin\theta d\theta d\varphi = \frac{\omega^3 p^2}{16\pi^2\varepsilon_0 c^3} \cos^2\theta \sin\theta d\theta d\varphi. \tag{13}$$

This formula is presented in Fig. 3.

3. Standard field potential of a rotating dipole

Now we calculate the standard potential (5) $\{\phi, \mathbf{A}\}$ in the field of the rotating dipole to use it instead of the temporal potential $\mathbf{W}$ in formula (2) $\underset{c}{\Upsilon}^{\lambda\mu\nu} = -2A^{[\lambda}F^{\mu]\nu}$ to obtain the spin flux.

Consider first an oscillating (non-rotating) dipole directed along the x axis:

$$p^x = lq = p\exp(-i\omega t). \tag{14}$$

We consider it as "elementary vibrator" in the sense that the charges q are located at the ends, at a distance l, and the current

$$I = \partial_t q = -i\omega p^x / l \tag{15}$$

is the same at the points of the dipole. Using $jdV = Idx$ in equation (5) and integrating the constant current along the dipole, taking into account the retardation, we obtain

$$A^x = \frac{-i\omega p}{4\pi\varepsilon_0 c^2 r}\exp(ikr - i\omega t). \tag{16}$$

Representing a rotating dipole as a pair of dipoles with a quarter-period phase shift relative to each other, $\mathbf{p} = \{p, ip\}$, we obtain the standard vector potential of the rotating dipole field

$$\mathbf{A} = \{A^x = \frac{-i\omega p}{4\pi\varepsilon_0 c^2 r}, A^y = \frac{\omega p}{4\pi\varepsilon_0 c^2 r}, A^z = 0\}\exp(ikr - i\omega t). \tag{17}$$

It is easy to verify that such a potential, taking into account the corresponding φ-potential, gives the electromagnetic field (6), (7).

Using expressions (17) instead of temporal potential (11) to calculate the canonical spin tensor (2) gives, instead of (12):

$$\underset{c}{\tilde{\Upsilon}}^{xyx} = -\frac{\Re}{2}\{\bar{A}^x F^{yx}\} = \frac{\omega^3 x p^2}{32\pi^2\varepsilon_0 c^3 r^3}, \quad \underset{c}{\tilde{\Upsilon}}^{xyy} = \frac{\Re}{2}\{\bar{A}^y F^{xy}\} = \frac{\omega^3 y p^2}{32\pi^2\varepsilon_0 c^3 r^3}$$

$$\underset{c}{\tilde{\Upsilon}}^{xyz} = -\frac{\Re}{2}\{\bar{A}^x F^{yz} - \bar{A}^y F^{xz}\} = \frac{\omega^3 z p^2}{16\pi^2\varepsilon_0 c^3 r^3}. \quad (18)$$

These expressions for the spin tensor lead to the correct result and coincide with (12) on the axis of rotation of the dipole, where $z = r$. However, they are not equal to zero on the plane of rotation, where the radiation polarization is linear and where there is no spin. This means they are not correct.

4. Conclusion

It is shown that the calculation of the dipole spin radiation by the formula (4) depends on which of the gauge equivalent vector potentials is used to form the spin tensor (2). Gauge transformations of electromagnetic potentials do not change the electromagnetic field determined by these potentials, and, accordingly, do not change the calculated values of energy, momentum and moment of momentum of the field. However, they change the calculated value of the spin of the electromagnetic field. The correct spin is given only by the temporal potentials obtained by integrating the electric field over time. So the inevitable recognition of the electrodynamics spin entails the denial of the gauge equivalence of vector potentials. This changes the foundations of classical electrodynamics.

I am eternally grateful to Robert Romer for courageously publication the question, "Does a plane wave really carry no spin?" [42].

References

1. Allen L and Padgett M J 2002 *Am. J. Phys*. **70** 567
2. Heitler W 1954 *The Quantum Theory of Radiation* (Oxford: Clarendon) p 401
3. Ohanian H C 1986 *Amer. J. Phys*. **54** 500-505
4. Allen L, Padgett M J and Babiker M 1999 *Progress in Optics XXXIX* (Elsevier, Amsterdam) p 291
5. Allen L, Barnett S M and Padgett M J 2003 *Optical Angular Momentum* (Institute of Physics Publishing, Bristol and Philadelphia)
6. Gotte J B and Barnett S M 2013 *The Angular Momentum of Light* (Cambrige) p 7
7. Simmonds J W and Guttmann M J 1970 *States, Waves and Photons* (Addison-Wesley Reading MA)
8. Jackson J D 1999 *Classical Electrodynamics* (John Wiley) p 350
9. Einstein A. Zur Elektrodynamik bewegter Korper *Annalen der Physik* **17**, 891, 1905
10. Sadowsky A. *Acta et Comm. Imp. Universitatis Jurievensis* **7**, No. 1-3 (1899)
11. Poynting J. H., "The wave motion of a revolving shaft, and a suggestion as to the angular momentum in a beam of circularly polarised light". *Proc. R. Soc. Lond. A* **82**, 560-567 (1909)
12 Weyssenhoff J. and Raabe A. Relativistic Dynamics of Spin-Fluids and Spin-Particles. *Acta Phys. Polon.* V. 9 7-19, 26, 46 (1947
13. Corson E M *Introduction to tensors, spinors, and reativistic wave-equation* NY, Hafner, 1953 p.71
14. Soper D. E., *Classical Field Theory* (N.Y.: Dover, 2008), p. 114
15. Barut A. O. *Electrodynamics and Classical Theory of Particles and Fields* (Macmillan, New York, 1964), p. 102
16. Khrapko R.I. "Reflection of light from a moving mirror" *Optik* **136** (2017) 503–506 http://khrapkori.wmsite.ru/ftpgetfile.php?id=153&module=files
17. Khrapko R.I. Absorption of spin from an electromagnetic wave *Optik* **154** (2018) 806–810 http://khrapkori.wmsite.ru/ftpgetfile.php?id=161&module=files
18. Khrapko R.I. Absorption of spin of a plane circularly polarized wave *Optik* **210** May 2020 Article 164527. http://khrapkori.wmsite.ru/ftpgetfile.php?id=187&module=files

19. Khrapko R.I. Absorption of Spin by a Conducting Medium *AASCIT Journal of Physics* Vol. 4, No. 2, Page: 59-63 (2018) http://www.aascit.org/journal/archive2?journalId=977&paperId=6355
20. Khrapko R.I. Spin transferred to a mirror reflecting light "International Conference ICECEIC" January 2019 Kanchipuram, Tamil Nadu, India. http://khrapkori.wmsite.ru/ftpgetfile.php?id=188&module=files
21. Khrapko R.I. Spin transmitted to a mirror when light is reflected http://trudymai.ru/published.php?ID=34126
22. Khrapko R.I. True energy-momentum tensors are unique. Electrodynamics spin tensor is not zero https://arxiv.org/abs/physics/0102084
23. Khrapko R.I. Origin of Spin: Paradox of the classical Beth experiment. In *Unfolding the Labyrinth: Open Problems in Mathematics, Physics, Astrophysics, and other areas of science* (Hexis - Phoenix 2006), pp. 57-71 https://arxiv.org/abs/math/0609238
24. Khrapko R.I. Experimental verification of Maxwellian electrodynamics, *Measurement Techniques*, **46**, No. 4, 317-321 (2003) http://khrapkori.wmsite.ru/ftpgetfile.php?id=60&module=files
25. Khrapko R.I. Spin of the Classical Electrodynamics Bulletin of Peoples' Friendship University of Russia. Series Physic. 2002, #10(1), p. 40
26. Khrapko R.I. Spin of dipole radiation (2001) http://trudymai.ru/published.php?ID=34635
27. Khrapko R.I. Radiation of spin by a rotator (2003) https://web.ma.utexas.edu/cgi-bin/mps?key=03-315
28. Khrapko R.I. Spin is not a moment of momentum (2012) http://trudymai.ru/published.php?ID=28834
29. Khrapko R.I. Unknown spin radiation. *J. Phys.: Conf. Ser.* 1172 012055 (2019) http://khrapkori.wmsite.ru/ftpgetfile.php?id=193&module=files
30. Khrapko R.I. Spin radiation from a rotating dipole. *Optik* **181** (2019) 1080-1084 http://khrapkori.wmsite.ru/ftpgetfile.php?id=172&module=files
31. Khrapko R.I. Radiation damping of a rotating dipole *Optik*. 203 (2020) Article 164021 , http://khrapkori.wmsite.ru/ftpgetfile.php?id=189&module=files
32 Meyers, R. A. *Encyclopedie of Physics Science and Technology*, v. 2 (N.Y., AP, 1987) p. 266
33. Sommerfeld A. Atombau und Spektrallinien 1 Band FR1EDR. V1EWEG & SOHN BRAUNSCHWEIO 1951
34. Vul'fson K S "Angular momentum of electromagnetic waves" *Soviet Physics Uspekhi* (1987), **30**(8): 724-728
35. Khrapko R.I. Spin density of electromagnetic waves (2001). http://trudymai.ru/published.php?ID=34691
36. F. J. Belinfante, *Physica* **6**, 887 (1939).
37. L. Rosenfeld, "Sur le Tenseur D'Impulsion-Energie." *Memoires de l'Academie Royale des Sciences de Belgiques* **8** No 6 (1940).
38. J Humblet, Physica, **10**, 585 (1943)
39. Khrapko R.I. Mechanical stresses produced by a light beam *J. Modern Optics*, 55, 1487-1500 (2008) http://khrapkori.wmsite.ru/ftpgetfile.php?id=9&module=files
40. Landau L. D., Lifshitz E. M. *The Classical Theory of Fields* (Pergamon, N. Y. 1975).
41. Corney A. *Atomic and Laser Spectroscopy* (Oxford University Press, 1977).
42. Khrapko R.I. "Does plane wave not carry a spin?" *Amer. J. Phys.* **69**, 405 (2001)