THE BINARY GOLDBACH CONJECTURE AND CIRCLES OF PARTITION

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Abstract. In this paper we use a new method to study problems in the additive number theory (see [1]). With the notion of circle of partition as a set of points whose weights are natural numbers of a particular subset under an additive condition we be able to prove that there exist infinitely many natural numbers ≥ 6 having at least one representation as the sum of two different prime numbers. Based on this by a consequent using of the method of circles of partition finally we can also prove the real binary Goldbach conjecture.

1. Introduction

The Goldbach conjecture dates from 1742 out of the correspondence between the Swiss mathematician Leonard Euler and the German mathematician Christian Goldbach. The problem has two folds, namely the binary case and the ternary case. The binary case ask if every even number ≥ 6 can be written as a sum of two primes, where as the ternary case ask if every odd number ≥ 7 can be written as a sum of three prime numbers. The ternary case has, however, been solved quite recently in the preprint [2] culminating several works. Though the binary problem remains unsolved as of now there has been substantive progress as well as on its variants. The first milestone in this direction can be found in (see [6]), where it is shown that every even number can be written as the sum of at most C primes, where C is an effectively computable constant. In the early twentieth century, G.H Hardy and J.E Littlewood assuming the Generalized Riemann hypothesis (see [9]), showed that the number of even numbers $\leq X$ and violating the binary Goldbach conjecture is much less than $X^{\frac{1}{2}+c}$, where c is a small positive constant. Jing-run Chen [4], using the methods of sieve theory, showed that every even number can either be written as a sum of two prime numbers or a prime number and a number which is a product of two primes. It also known that almost all even numbers can be written as the sum of two prime numbers, in the sense that the density of even numbers representable in this manner is one [8], [7]. It is also known that there exist a constant K such that every even number can be written as the sum of two prime numbers and at most K powers of two, where we can take $K = 13$ [5].
In [1] we have developed a method which we feel might be a valuable resource and a recipe for studying problems concerning partition of numbers in specified subsets of $\mathbb{N}$. The method is very elementary in nature and has parallels with configurations of points on the geometric circle.
Let us suppose that for any $n \in \mathbb{N}$ we can write $n = u + v$ where $u, v \in \mathbb{M} \subset \mathbb{N}$ then

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the new method associate each of this summands to points on the circle generated in a certain manner by $n > 2$ and a line joining any such associated points on the circle. This geometric correspondence turns out to useful in our development, as the results obtained in this setting are then transformed back to results concerning the partition of integers.

2. The Circle of Partition

Here we repeat the base results of the method of circles of partition developed in [1].

Definition 2.1. Let $n \in \mathbb{N}$ and $\mathbb{M} \subseteq \mathbb{N}$. We denote with

$$\mathcal{C}(n, \mathbb{M}) = \{[x] \mid x, y \in \mathbb{M}, n = x + y\}$$

the **Circle of Partition** generated by n with respect to the subset $\mathbb{M}$. We will abbreviate this in the further text as CoP. We call members of $\mathcal{C}(n, \mathbb{M})$ as points and denote them by $[x]$. For the special case $\mathbb{M} = \mathbb{N}$ we denote the CoP shortly as $\mathcal{C}(n)$. We denote with $\|[x]\| := x$ the **weight** of the point $[x]$ and correspondingly the weight set of points in the CoP $\mathcal{C}(n, \mathbb{M})$ as $\|\mathcal{C}(n, \mathbb{M})\|$. Obviously holds

$$\|\mathcal{C}(n)\| = \{1, 2, \ldots, n-1\}.$$

Definition 2.2. We denote the line $\mathbb{L}_{[x],[y]}$ joining the point $[x]$ and $[y]$ as an axis of the CoP $\mathcal{C}(n, \mathbb{M})$ if and only if $x + y = n$. We say the axis point $[y]$ is an axis partner of the axis point $[x]$ and vice versa. We do not distinguish between $\mathbb{L}_{[x],[y]}$ and $\mathbb{L}_{[y],[x]}$, since it is essentially the the same axis. The point $[x] \in \mathcal{C}(n, \mathbb{M})$ such that $2x = n$ is the **center** of the CoP. If it exists then we call it as a **degenerated axis** $\mathbb{L}_{[x]}$ in comparison to the **real axes** $\mathbb{L}_{[x],[y]}$. We denote the assignment of an axis $\mathbb{L}_{[x],[y]}$ to a CoP $\mathcal{C}(n, \mathbb{M})$ as

$$\mathbb{L}_{[x],[y]} \hat{\in} \mathcal{C}(n, \mathbb{M}) \text{ which means } [x], [y] \in \mathcal{C}(n, \mathbb{M}) \text{ with } x + y = n.$$

Remark 2.3. In the following we consider only real axes. Therefore we abstain from the attribute *real* in the sequel.

Proposition 2.4. *Each axis is uniquely determined by points $[x] \in \mathcal{C}(n, \mathbb{M})$.*

Proof. Let $\mathbb{L}_{[x],[y]}$ be an axis of the CoP $\mathcal{C}(n, \mathbb{M})$. Suppose as well that $\mathbb{L}_{[x],[z]}$ is also an axis with $z \neq y$. Then it follows by Definition 2.2 that we must have $n = x + y = x + z$ and therefore $y = z$. This cannot be and the claim follows immediately. ◻

Corollary 2.5. *Each point of a CoP $\mathcal{C}(n, \mathbb{M})$ except its center has exactly one axis partner.*

Proof. Let $[x] \in \mathcal{C}(n, \mathbb{M})$ be a point without an axis partner being not the center of the CoP. Then holds for every point $[y] \neq [x]$ except the center

$$x + y \neq n.$$

This is a contradiction to the Definition 2.1. Due to Proposition 2.4 the case of more than one axis partners is impossible. This completes the proof. ◻

Notations. We denote by

$$\mathbb{N}_n = \{m \in \mathbb{N} \mid \ m \leq n\} \tag{2.1}$$

the **sequence** of the first n natural numbers.

3. The Fundamental Theorem and its Conclusions

Theorem 3.1 (Fundamental)**.** *Let $n, r \in \mathbb{N}$, $\mathbb{M} \subseteq \mathbb{N}$ and $\mathcal{C}(n, \mathbb{M})$ be a non-empty CoP with an axis $\mathbb{L}_{[x],[n-x]} \hat{\in} \mathcal{C}(n, \mathbb{M})$* [1] *. If holds $x + r \in \mathbb{M}$ then $\mathcal{C}(n + r, \mathbb{M})$ is a non-empty CoP too.*

Proof. Since $\mathbb{L}_{[x],[n-x]} \hat{\in} \mathcal{C}(n, \mathbb{M})$, x and $n - x$ are members of $\mathbb{M}$. And due to the premise also $x + r \in \mathbb{M}$. Then holds

$$n + r - (x + r) = n - x \in \mathbb{M}.$$

Ergo there is an axis $\mathbb{L}_{[x+r],[n+r-(x+r)]} \hat{\in} \mathcal{C}(n + r, \mathbb{M})$ and $\mathcal{C}(n + r, \mathbb{M})$ is non-empty. □

Corollary 3.2. *Let the requirements of Theorem 3.1 be fulfilled. If the base set $\mathbb{M}$ is an infinite set and there exists a non-empty CoP $\mathcal{C}(n_o, \mathbb{M})$ then there exist infinitely many positive integers $n > n_o$ with non-empty CoPs $\mathcal{C}(n, \mathbb{M})$.*

Proof. Let $\mathbb{L}_{[x],[n-x]}$ be an axis of $\mathcal{C}(n_o, \mathbb{M})$. Then is due to Theorem 3.1 also $\mathcal{C}(n_o + r_1, \mathbb{M})$ non-empty with $r_1 > 0$ and if $x + r_1 \in \mathbb{M}$. From this CoP we can continue this process with $r_2 > 0$ to the non-empty CoP $\mathcal{C}(n_o + r_1 + r_2, \mathbb{M})$. Since the base set is an infinite set this process can be repeated infinitely many. □

Lemma 3.3. *It is possible to construct all CoPs $\mathcal{C}(n, \mathbb{P})$ containing a certain member $[x_o]$ with $n \geq x_o + 3$.*

Proof. We start with the least generator $n_o = x_o + 3$ of a CoP containing the axis $\mathbb{L}_{[3],[x_o]}$ and $y_o = n_o - x_o$. Now we consider the axis $\mathbb{L}_{[y_o],[n_o-y_o]} = \mathbb{L}_{[y_o],[x_o]} \hat{\in} \mathcal{C}(n_o, \mathbb{P})$. In virtue of Theorem 3.1 holds also

$$\mathbb{L}_{[y_1],[n_1-y_1]} = \mathbb{L}_{[y_o+d_o],[n_o-y_o]} \hat{\in} \mathcal{C}(n_o + d_o, \mathbb{P})$$

and therefore

$$\mathbb{L}_{[y_o+d_o],[n_o-y_o]} = \mathbb{L}_{[y_1],[x_o]} \hat{\in} \mathcal{C}(n_1, \mathbb{P})$$

with $y_1 = y_o + d_o$ and $n_1 = n_o + d_o$, if d_o is the distance to the immediately subsequent prime after y_o. Thus we have found with $\mathcal{C}(n_o, \mathbb{P})$ and $\mathcal{C}(n_1, \mathbb{P})$ two CoPs both containing $[x_o]$. Since y_1 is the immediately subsequent prime after y_o there is no CoP $\mathcal{C}(n, \mathbb{P})$ with $n_o < n < n_1$ containing $[x_o]$ because there is no axis $\mathbb{L}_{[x_o],[y]} \hat{\in} \mathcal{C}(n, \mathbb{P})$ between

$$\mathbb{L}_{[x_o],[y_o]} \hat{\in} \mathcal{C}(n_o, \mathbb{P}) \text{ and } \mathbb{L}_{[x_o],[y_1]} \hat{\in} \mathcal{C}(n_1, \mathbb{P}).$$

By virtue of Lemma 3.3 we can repeat this procedure with $y_1, y_2, \ldots, d_1, d_2, \ldots$ and $n_1, n_2, \ldots$ infinitely many often and obtain a chain of axes

$$\mathbb{L}_{[y_o],[x_o]}, \mathbb{L}_{[y_1],[x_o]}, \mathbb{L}_{[y_2],[x_o]}, \ldots \mathbb{L}_{[y_s],[x_o]}, \ldots$$

[1] The axis can also be a degenerated axis with $x = n - x = \frac{n}{2}$ if it exists.

of the chain of all CoPs

$$\mathcal{C}(n_o, \mathbb{P}), \mathcal{C}(n_1, \mathbb{P}), \mathcal{C}(n_2, \mathbb{P}), \ldots \mathcal{C}(n_s, \mathbb{P}), \ldots$$

containing all the fixed point $[x_o]$. □

Theorem 3.4 (Infinite Goldberg)**.** *There are infinitely many even numbers $n \geq 6$ having at least one representation as sum of two primes.*

Proof. As an equivalent to the claim we prove that there exist non-empty CoPs for infinitely many generators $n \geq 6$. We construct by virtue of Lemma 3.3 all chains of non-empty CoPs starting with generators $n_o = p + 3$ for all $p \geq 3$. These are infinitely many chains with infinite lengths of each one. □

Even if the statement of this theorem seems trivial, it illustrates in a beauty fashion the application of the method of circles of partition on issues in additive number theory.

Whether each even number ≥ 6 generates a CoP in this pool, remains still open, but it will be solved in the sequel.

By virtue of Lemma 3.3 let

$$\mathbb{G}_x := \{n \in 2\mathbb{N} \mid [x] \in \mathcal{C}(n, \mathbb{P})\}, x \in \mathbb{P} \tag{3.1}$$

be the set of the generators of all CoPs containing the point $[x]$ and

$$\mathbb{G}_x(n) := \{m \in \mathbb{G}_x \mid m \leq n\} \tag{3.2}$$

the set of such generators not greater than n. Further let be

$$\mathbb{G}(n) := \bigcup_{\substack{p \in \mathbb{P} \\ 3 \leq p \leq n-3}} \mathbb{G}_p(n). \tag{3.3}$$

Corollary 3.5. *From Proposition 3.6 and* (3.2) *follows immediately*

$$|\mathbb{G}_p(n)| = \pi(n) - \pi(p).$$

Proposition 3.6. *For all $p \in \mathbb{P} \mid p \geq 3$ holds*

$$\mathbb{G}_p = \mathbb{P} + p \text{ and } \mathbb{G}_p(n) = \mathbb{P}_n + p$$

where $\mathbb{P}_n = \mathbb{P} \cap \mathbb{N}_n$.

Proof. Since $[p] \in \mathcal{C}(n, \mathbb{P})$ also holds $[n-p] \in \mathcal{C}(n, \mathbb{P})$ and hence $n - p \in \mathbb{P}$ and for $p \geq 3$

$$\begin{aligned} \mathbb{G}_p &= \{n \in 2\mathbb{N} \mid n - p \in \mathbb{P},\ n - p \geq 3\} \\ &= \{q + p \mid q \in \mathbb{P},\ q \geq 3\} \\ &= \mathbb{P} + p. \end{aligned}$$

It follows obviously that $\mathbb{G}_p(n) = \mathbb{P}_n + p$. □

Lemma 3.7 (The Little Lemma)**.** *Let* $\mathbb{G}_x(n)$ *be defined as in* (3.2)*. If for an even* n_o *holds that* $\mathbb{G}(n_o)$ *as defined in* (3.3) *contains all even numbers* $6 \le n < n_o$ *then there exist some even* $2 \le d_o \le n_o - 6$ *and some prime* p_o *such that*

$$[p_o - d_o] \in \mathcal{C}(n_o - d_o, \mathbb{P}).$$

Proof. Since $\mathbb{G}(n_o)$ contains all even numbers $6 \le n < n_o$ then for all these even numbers as CoP generators holds $\mathcal{C}(n, \mathbb{P}) \ne \emptyset$. Let $n = n_o - d$ be for all even d with $2 \le d \le n_o - 6$ so that we can choose a point

$$[x] \in \mathcal{C}(n_o - d_o, \mathbb{P}) \tag{3.4}$$

for some choice of even d_o and a prime $p_o \in \mathbb{P}_{n_o}$ with $p_o > \frac{n_o}{2}$ such that $[x]$ is the lower or center point of a real or degenerated axis with

$$x \le \frac{n_o - d_o}{2} \text{ and } p_o = x + d_o \in \mathbb{P}_{n_o}.$$

This representation of the prime p_o exists because

$$\frac{n_o - d_o}{2} < \frac{n_o}{2} < n_o$$

and in virtue of *Bertrand's postulate* there exists a prime number insight of the open interval $\left(\frac{n_o}{2}, n_o\right)$. Hence by replacing of x by $p_o - d_o$ in (3.4) we obtain

$$[p_o - d_o] \in \mathcal{C}(n_o - d_o, \mathbb{P}).$$

□

Lemma 3.8 (Main Lemma)**.** *Let* $\mathbb{G}_x(2n)$ *by virtue of* (3.2) *be the generator set of CoPs containing the point* $[x]$ *such that their generators are not greater than* $2n$ *by* $n \in \mathbb{N} \mid n \ge 3$*. Then contains* $\mathbb{G}(2n)$ *as defined in* (3.3) *all even numbers between* 6 *and* $2n$ *inclusively.*

Proof. At first we prove that the following statement is equivalent to the claim

$$\forall n \in 2\mathbb{N} \mid 6 \le n \le 2n \text{ holds } \mathcal{C}(n, \mathbb{P}) \ne \emptyset. \tag{3.5}$$

Let be

$$\omega(n, p) = \begin{cases} 0 & \text{for } p > n - 3 \vee n - p \notin \mathbb{P} \\ 1 & \text{for } n - p \in \mathbb{P}. \end{cases} \tag{3.6}$$

Then is obviously

$$\|\mathcal{C}(n, \mathbb{P})\| = \{p \in \mathbb{P} \mid 3 \le p \le n - 3,\ \omega(n, p) > 0\}$$
$$\text{and for } n \in \mathbb{N}$$
$$\mathbb{G}_p(2n) = \{m \in 2\mathbb{N} \mid 6 \le m \le 2n,\ \omega(m, p) > 0\}.$$

It follows that if $\mathcal{C}(2n, \mathbb{P}) = \emptyset$ then holds

$$\omega(2n, p) = 0 \text{ for } 3 \le p \le 2n - 3$$

and reversely. And this means that the sets $\mathbb{G}_p(2n)$ contain for no p the generator $2n$ and reversely that if $\mathcal{C}(2n, \mathbb{P}) \ne \emptyset$ then $2n$ belongs to at least one set $\mathbb{G}_p$. The equivalence between (3.5) and the claim of this lemma is demonstrated.

Now we assume that for the even number $2n_o$ holds that $\mathbb{G}(2n_o)$ contains all even numbers between 6 and $2n_o - 2$ except $2n_o$. This would mean that holds $\mathcal{C}(2n_o, \mathbb{P}) = \emptyset$ and $\mathcal{C}(n, \mathbb{P}) \ne \emptyset$ for $6 \le n \le 2n_o - 2$.

Because of Lemma 3.7 there is always a prime p_o and an even number d_o such that $p_o - d_o$ is also prime and it holds [2]

$$[p_o - d_o] \in \mathcal{C}(2n_o - d_o, \mathbb{P}).$$

Then holds in virtue of Theorem 3.1 that there is an axis

$$\mathbb{L}_{[p_o],[2n_o - p_o]} \hat{\in} \mathcal{C}(2n_o, \mathbb{P}).$$

But this contradicts the assumption that $\mathcal{C}(2n_o, \mathbb{P}) = \emptyset$. Hence $2n_o$ is member of

$$\mathbb{G}(2n_o) = \{6, 8, \ldots, 2n_o - 2, 2n_o\} = 2\mathbb{N}_{n_o} \setminus \{2, 4\}$$

and the CoP $\mathcal{C}(2n_o, \mathbb{P})$ is non-empty. □

Since we did not make put any restriction on the even number $2n_o$ this statement is valid for all even numbers. Empirical calculations by the authors are resulted in confirmation of this statement for even generators until more than $2 \cdot 10^8$.

Corollary 3.9. *From Lemma 3.8 follows by $n \to \infty$*

$$\mathbb{G}(n) \longrightarrow 2\mathbb{N} \setminus \{2, 4\}.$$

This means that there are no empty CoPs with the base set $\mathbb{P}$ for all even generators ≥ 6 and proves the following Theorem.

Theorem 3.10 (Binary Goldbach Conjecture)**.** *For all even numbers ≥ 6 there exists at least one representation as sum of two primes.*

References

1. Agama, Theophilus and Gensel, Berndt *Studies in Additive Number Theory by Circles of Partition*, arXiv:2012.01329, 2020.
2. Helfgott, Harald A *The ternary Goldbach conjecture is true*, arXiv preprint arXiv:1312.7748, 2013.
3. Green, Ben and Tao, Terence *The primes contain arbitrarily long arithmetic progressions*, Annals of Mathematics, JSTOR, 2008, pp. 481–547.
4. Chen, Jing-run *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, The Goldbach Conjecture, World Scientific, 2002, pp. 275–294.
5. Heath-Brown, D Roger and Puchta, J-C *Integers represented as a sum of primes and powers of two*, Asian J. Math, vol: 6(3), 2002, 535–566.
6. Shnirel'man, Lev Genrikhovich, *On the additive properties of numbers*, Uspekhi Matematicheskikh Nauk, vol. 2:6, Russian Academy of Sciences, Steklov Mathematical Institute of Russian , 1939, pp. 9–25.
7. Estermann, Theodor *On Goldbach's problem: Proof that almost all even positive integers are sums of two primes*, Proceedings of the London Mathematical Society, vol. 2:1, Wiley Online Library, 1938, pp. 307–314.
8. Chudakov, Nikolai Grigor'evich, *The Goldbach's problem*, Uspekhi Matematicheskikh Nauk, vol. 4, Russian Academy of Sciences, Steklov Mathematical Institute of Russian , 1938, 14–33.
9. Hardy, Godfrey H and Littlewood, John E *Some problems of Partitio Numerorum(V): A further contribution to the study of Goldbach's problem*, Proceedings of the London Mathematical Society, vol. 2(1), Wiley Online Library, 1924, pp. 46–56.

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[2]Empirical calculations of the authors have shown that such solutions can be found always with $d_o = 6$.

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