Abridgment of Cycles in GCS

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Abstract

A shortened English version of new concepts appeared this year in three French papers about cycles in Generalized Collatz Sequences (GCS) and few new developments.

- Reduced and Compact Subseqs
- Shape Vector as Subseq Rank ρ
- Monoid of Transition Functions ω between elements of compact sequences
- Triplet Operator $\langle a, b, c \rangle$ as powerful tool to compose linear functions
- Diophantine Equation $p^m x - r^d y - q = 0$ related to each monoid element
- Shape Class as general solution of the diophantine equation
- Cyclic Solution Requirement
- Universal Rotation Function R on q parameters
- Parallel Rotation Function V on ρ ranks
- Cardinality of Equation Classes
- Cycle Layers
- Layer's Level
- Cycle Barycenter
- Explicit Role of δ in Cyclic Process
- Numerical Cycle Occurence Probability

Sequences and Subseqs

Definition 1. *Let F, G and H be functions. By convention, an FGH chain is assimilated to the compound $H \circ G \circ F$. Thus $FGH(x) = H(G(F(x)))$.*

Definition 2. *Let p and r be relatively prime integers where $p > r > 1$. The reduced form of a Generalized Collatz Sequence is a sequence of integers where the successor to x is*

$$\begin{aligned} D(x) &= x/r && \textit{if } x \bmod r = 0 \\ T(x) &= \lfloor px/r \rfloor && \textit{otherwise} \end{aligned} \tag{1}$$

and the full form is

$$\begin{aligned} D(x) &= x/r && \textit{if } x \bmod r = 0 \\ C(x) &= r\lfloor px/r \rfloor && \textit{otherwise} \end{aligned}$$

The C and T functions combine $r-1$ simple functions C_i and T_i where $0 < i < r$.

$$\begin{aligned} C_1(x) &= px - 1 & \qquad T_1(x) &= (px-1)/r \\ C_2(x) &= px - 2 & T_2(x) &= (px-2)/r \\ &\dots & &\dots \\ C_{r-1}(x) &= px - r + 1 & T_{r-1}(x) &= (px-r+1)/r \end{aligned}$$

Definition 3. *A Generalized Collatz subsequence, or "subseq", is a finite part of a Generalized Collatz Sequence that begins and ends with a non multiple of r. It comes in three varieties:*

- *full* subseq if extracted from a full sequence (C et D)
- *reduced* subseq if extracted from a reduced sequence (T et D)
- *compact* subseq if it does not include multiples of r.

Definition 4. *The shape vector of a subseq is the sequence of numbers that corresponds to the chain of functions between the first and the last term of the equivalent reduced subseq, as per*

$$\begin{aligned} D &\longrightarrow 0 \\ T_i &\longrightarrow i \quad (0 < i < r) \end{aligned} \tag{2}$$

Definition 5. *The shape class of a subseq is the set of all subseqs having the same shape vector.*

Functions T_i and D are followed by any function. So any number in a shape vector is allowed. This means that shape classes can be numbered by setting the rank ρ in the r base to the class shape vector.

Definition 6. *Let Ω be the set of transition functions ω between elements of compact subseq. This together with the composition law $\circ$ of functions and the identity I define the free monoid $(\Omega, \circ, I)$ whose basis is*

$$\{\beta_{qd}\} = \left\{T_q D^d\right\} \quad (0 < q < r) \quad (d \geq 0) \tag{3}$$

and the set of functions ω is

$$\Omega = \left\{\prod_{i=1}^{m} \beta_{q_i d_i}\right\} \;\; (m > 0) \tag{4}$$

The sequences $(q_i)_m$ and $(d_i)_m$ determine a distinct ω function:

$$\begin{aligned} \omega &= \beta_{q_1 d_1}\, \beta_{q_2 d_2} \ldots \beta_{q_m d_m} \\ &= T_{q_1} D^{d_1}\, T_{q_2} D^{d_2} \ldots T_{q_m} D^{d_m} \end{aligned} \tag{5}$$

Let $X = (x_1\; x_2 \ldots x_{m+1})$ be a compact subseq. A $q_i 0^{d_i}$ represents a base element shape vector where an isolated q_i (not followed by 0) is written as $q_i 0^0$.

$$x_1 \xmapsto{q_1 0^{d_1}} x_2 \xmapsto{q_2 0^{d_2}} x_3 \cdots x_m \xmapsto{q_m 0^{d_m}} x_{m+1}$$

$$x_{m+1} = \omega(x_1)$$

The rank ρ in r base of an ω function is the shape vector produced by concatenation of successive $q_i 0^{d_i}$ but where 0^0 is not written.

$$\rho = q_1 0^{d_1}\, q_2 0^{d_2} \ldots q_m 0^{d_m} \tag{6}$$

Composition of Functions

Triplet Operator

To any function of the form $f(x) = (ax + c)/b$ we can associate the triplet $\langle a, b, c\rangle$ as an operator $\langle f\rangle$ equivalent to f:

$$\langle f\rangle\,(x) = \langle a, b, c\rangle\,(x) = f(x)$$

This implies the following composition law

$$\langle a_1, b_1, c_1\rangle\,\langle a_2, b_2, c_2\rangle = \langle a_1 a_2, b_1 b_2, b_1 c_2 + c_1 a_2\rangle \tag{7}$$

the neutral element $\langle 1, 1, 0\rangle$, the inverse $\langle a, b, c\rangle^{-1} = \langle b, a, -c\rangle$ and $\langle ka, kb, kc\rangle = \langle a, b, c\rangle$.

In the monoid there will be no inverse. Only descending functions are composed but all their inverse form the monoid of the rising functions. The triplets of the elementary functions are

$$\langle D\rangle = \langle 1, r, 0\rangle \quad \langle C_i\rangle = \langle p, 1, -i\rangle \quad \langle T_i\rangle = \langle p, r, -i\rangle$$

and those of the base elements are

$$\left\langle \beta_{q_i d_i}\right\rangle = \left\langle p, r^{d_i+1}, -q_i\right\rangle$$

To improve the notations in arrays let put

$$\left\langle \beta_{q_i d_i}\right\rangle = \langle p, r_i, -q_i\rangle \quad \text{where} \tag{8}$$

$$r_i = r^{d_i+1} \tag{9}$$

The only formula needed to compose the functions of a chain as a block is

$$\prod_{i=1}^{n} \langle a_i, b_i, c_i\rangle = \langle a, b, c\rangle$$

$$\begin{aligned}
a &= a_1\, a_2\, a_3\, a_4 \quad \cdots \quad a_n \\
b &= b_1\, b_2\, b_3\, b_4 \quad \cdots \quad b_n
\end{aligned}$$

$$c = \begin{array}{rcc}
c_1\, a_2\, a_3\, a_4 & \cdots & a_n \\
+\, b_1\, c_2\, a_3\, a_4 & \cdots & a_n \\
+\, b_1\, b_2\, c_3\, a_4 & \cdots & a_n \\
+\, b_1\, b_2\, b_3\, c_4 & \cdots & a_n \\
\vdots & \ddots & \vdots \\
+\, b_1\, b_2\, b_3\, b_4 & \cdots & c_n
\end{array} \tag{10}$$

Diophantine Equation

With $\langle p, r_i, -q_i\rangle$ the formula gives us $\langle \omega\rangle$ where the number of divisions $d = m + \sum d_i$

$$\langle \omega\rangle = \prod_{i=1}^{m} \langle p, r_i, -q_i\rangle = \left\langle p^m, r^d, -q\right\rangle$$

$$q = \begin{array}{rcc}
+\, q_1\, p\, p\, p & \cdots & p \\
+\, r_1\, q_2\, p\, p & \cdots & p \\
+\, r_1\, r_2\, q_3\, p & \cdots & p \\
+\, r_1\, r_2\, r_3\, q_4 & \cdots & p \\
\vdots & \ddots & \vdots \\
+\, r_1\, r_2\, r_3\, r_4 & \cdots & q_m
\end{array} \tag{11}$$

and the corresponding diophantine equation

$$p^m x - r^d y - q = 0 \tag{12}$$

whose q parameter can be written in a concise way

$$q = \sum_{i=1}^{m} q_i p^{m-i} r^{\sigma_i} \tag{13}$$

$$\text{by posing} \qquad \sigma_i = \sum_{k=1}^{i} (1 + d_{k-1}) \qquad \text{where} \qquad d_0 = -1 \tag{14}$$

The $(d_i)_m$ sequence is thus replaced by the $(\sigma_i)_m$ sequence. For example:

$$(d_i)_5 = (2\ 3\ 0\ 2\ 1) \ \rightarrow \ (d_i + 1)_5 = (3\ 4\ 1\ 3\ 2) \ \rightarrow \ (\sigma_i)_5 = (0\ 3\ 7\ 8\ 11)$$

$$q = q_1 p^4 + q_2 p^3 r^3 + q_3 p^2 r^7 + q_4 p r^8 + q_5 r^{11}$$

General Solution of the equation

There's necessarily a solution as p and r are relatively prime. In the general solution $(r^d k + x_0,\ p^m k + y_0)$ where k is a relative integer we obtain $(p^m k + y_0) \bmod r = 0$ at the frequency of once on r. These k values will not be used because y cannot be multiple of r.

By solving Bezout's equation

$$p^m u + r^d v = 1$$

we have a minimal solution $y_0 = -qv \bmod p^m$ which can be a multiple of r. In this case, we should take the minimum value y_0 of opposite sign by decreasing or adding p^m. We then derive x_0 from the equation and calculate the k_0 value of the test $k \bmod r = k_0$ to eliminate the non-permitted k.

n	ρ	$\langle \omega \rangle$	$bk + x$	$ak + y$	*full*	*compact*
..						
54	2000	$\langle 4, 81, -2 \rangle$	$81k - 40,$	$4k - 2$	(-40 -162 -54 -18 -6 -2)	(-40 -2)
55	2001	$\langle 16, 81, -35 \rangle$	$81k + 68$	$16k + 13$	(68 270 90 30 10 39 13)	(68 10 13)
56	2002	$\langle 16, 81, -62 \rangle$	$81k - 67$	$16k - 14$	(-67 -270 -90 -30 -10 -42 -14)	(-67 -10 -14)
57	2010	$\langle 16, 81, -17 \rangle$	$81k - 4$	$16k - 1$	(-4 -18 -6 -2 -9 -3 -1)	(-4 -2 -1)
58	2011	$\langle 64, 81, -95 \rangle$	$81k + 23$	$64k + 17$	(23 90 30 10 39 13 51 17)	(23 10 13 17)
59	2012	$\langle 64, 81, -122 \rangle$	$81k + 50$	$64k + 38$	(50 198 66 22 87 29 114 38)	(50 22 29 38)
60	2020	$\langle 16, 81, -26 \rangle$	$81k - 49$	$16k - 10$	(-49 -198 -66 -22 -90 -30 -10)	(-49 -22 -10)
..						

Cyclic Solution

The cyclic solution $x = y$ is however quite simple:

$$x = q/(p^m - r^d) \tag{15}$$

and the requirement for a numerical cycle as well:

$$q \bmod \delta = 0 \quad \text{where} \quad \delta = p^m - r^d \tag{16}$$

In a numerical cycle the numbers rotate and so do the functions. The question is, can there be a function rotation cycle when the divisibility requirement by δ does not allow a numerical cycle? The answer is yes and the implications are substantial.

Universal Cycles and Derived Cycles

We need to reverse the idea that functional cyclicity derives from numerical cyclicity. In the algebraic elucidation all is ordered and numbered. All is regulated except the $\delta|q$ possibility. All q belong to a cycle $(q_i)_m$ resulting from rotation of i in the $\langle p, r_i, -q_i\rangle$ components. It also corresponds to rotation of i in the $T_i D^i$ of shape vectors and thus implies $(\rho_i)_m$ rank cycles.

Rotational cyclicity is a universal algebraic property. It is occasionally reflected in a derived numerical cycle when q/δ is an integer.

The mod* notation

It's a kind of modulo in multiplicative context. As well as $x = rk + q$ where $q < r$ implies $q = x \bmod r$ as the rest of the division, similarly $x = r^k q$ where q is not multiple of r implies $q = x \operatorname{mod^*} r$ as a non-multiple residue of r. Used to stylishly define functions like cyclical R, where $\operatorname{mod^*} r$ replace an inappropriate $r^{-v_r(pq-(pq \bmod r)\delta)}$ expression.

The N function to rotate $(x_i)_m$

The usual N function can be defined as $N(x) = T(x) \operatorname{mod^*} r$. The function N is not cyclic in itself, but may have a cyclic behavior $N^m(x) = x$ when x is an integer q/δ. Then the function N rotates the x_i in the numerical cycle $(x_i)_m$.

The R function to rotate $(q_i)_m$

The general rotation function to rotate q_i values contains the parameter δ which changes with count m of multiplications and count d of divisions. It's explicitly stated

$$R(q) = (pq - (pq \bmod r)\delta) \operatorname{mod^*} r \tag{17}$$

This can result in the following lines of Pascal code

$$\begin{array}{l} q := (p * q - ((p * q) \bmod r) * delta) \operatorname{div} r; \\ \textbf{while } (q \bmod r) = 0 \textbf{ do } q := q \operatorname{div} r; \end{array}$$

being used with the example $64x - 81y - q = 0$ where exist eight cycles $R^3(q) = q$.

$$\begin{array}{cccc} (\,37\;\;55\;\;79\,) & (\,46\;\;67\;\;95\,) & (\,58\;\;83\;\;122\,) & (\,74\;\;110\;\;158\,) \\ (\,49\;\;71\;\;106\,) & (\,53\;\;82\;\;115\,) & (\,62\;\;94\;\;131\,) & (\,65\;\;98\;\;142\,) \end{array}$$

The V function to rotate $(\rho_i)_m$

The shape vectors related to the q-values in the previous eight cycles are as follows:

$$\begin{array}{cccc} (\,1110\;\;1101\;\;1011\,) & (\,1120\;\;1201\;\;2011\,) & (\,1220\;\;2201\;\;2012\,) & (\,2220\;\;2202\;\;2022\,) \\ (\,1210\;\;2101\;\;1012\,) & (\,2110\;\;1102\;\;1021\,) & (\,2120\;\;1202\;\;2021\,) & (\,2210\;\;2102\;\;1022\,) \end{array}$$

The cyclic function V outlined by $V^m(\rho) = \rho$ is easily expressed in Pascal code by

```
repeat
  v := Concat (v, v[1]);
  Delete (v, 1, 1)
until v[1] <> '0';
```

Equation Classes Cardinality

There is no limit to the amount of equations having the same m number of multiplications because there is no upper limit to the number of divisions.

Definition 7. *Let's write $K_d[n]$ the class of equations having the same number n of divisions.*

Definition 8. *Let's write $K_{dm}[d,m]$ the subclasses of $K_d[d]$ having m multiplications.*

The geometrical representation of $K_{dm}[d,m]$ classes as a network enables cardinality calculation easy:

$$\operatorname{card} K_d[d] = (r-1)r^d \tag{18}$$

$$\operatorname{card} K_{dm}[d,m] = \binom{d-1}{m-1}(r-1)^m \tag{19}$$

In the following figure each circle represents a $K_{dm}[d,m]$ class at abcissa d and ordinate m. Each circle contains card $K_{dm}[d,m]$ with a number $n = r-1$. Diagonal lines represent T_i and horizontal lines represent D. How to get these amounts? The problem is as simple as how many possible numbers in base r with m non-zero digits and $d-m$ zeros?

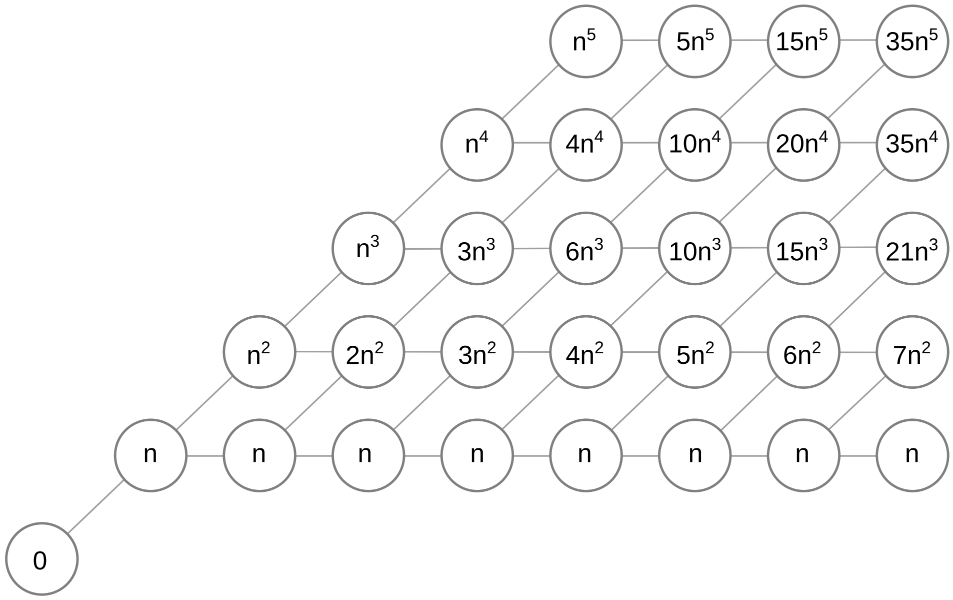

Figure 1: Amount of equations in K_{dm} classes with $n = r - 1$.

Subseqs of equal length

How many full subseqs of length L having distinct shape vector? Simply add the amounts in the diagonals where $d + m = L - 1$.

$$n,\ n,\ n(n+1),\ n(2n+1),\ n(n^2+3n+1)...$$
$$r-1,\ r-1,\ r(r-1),\ (2r-1)(r-1),\ (r^2+r-1)(r-1)...$$

It seems complex but dividing by $n = r - 1$ we get the following (u_k) sequences

$$\begin{aligned} &u_0 = 0,\ u_1 = 1 \\ &u_k = u_{k-1} + nu_{k-2} \end{aligned} \tag{20}$$

These are Lucas sequences $U_k(P, Q)$ where $P = 1$ and $Q = 1 - r$ including sequences of Fibonacci for $r = 2$ and Jacobsthal for $r = 3$.

Linking of Cycles

Let $q = Q(\rho)$ be the q parameter in the equation of rank ρ. Adding D at the end of a chain or '0' at the end of a shape vector ρ does not change q.

$$Q(r^k \times \rho) = Q(\rho) \tag{21}$$

Thus all q-values in a K_{dm} class are found in those with same m but higher d. In addition by applying the R rotation function with new δ to all q-values of the previous class we get all values of the new class. The monoid Ω is structured in layers starting from an initial q-value in $K_{dm}[m, m]$. Some of the following topics have not yet been addressed in French.

Cycle Layers

Definition 9. *A cycle layer is the set of all algebraic cycles reachable using only rotation functions starting from a q-value in* $K_{dm}[m, m]$.

The amount of cycles in a layer has no limit.

Definition 10. *The seed of a cycle layer is the smallest q-value of the layer.*

For example, the cycle $(94\ 185\ 122)$ is obtained by $R_\delta[-179]$, the cycle $(94\ 131\ 62)$ by $R_\delta[-17]$, finally $(62\ 58\ 65)$ by $R_\delta[37]$. The seed is $q_0 = 58$. The same, more simply, in terms of shape vectors ρ obtained by V is

$$(12020\ 20201\ 20120) \longrightarrow (1202\ 2021\ 2120) \longrightarrow (212\ 122\ 221)$$

The seed is $\rho_0 = 122$. With $r = 3$ and $m = 3$ there are four layers, the ρ-seeds being

$$\{111,\ 112,\ 122,\ 222\}$$

Definition 11. *The layer's level* λ *is the mean value of* q_i *in the shape vector* ρ

$$\lambda = m^{-1} \sum_{i=1}^{m} (q_i) \tag{22}$$

With $r > 5$ and $m = 4$ these five ρ-seeds have the same level $\lambda = 2$

$$\{1115,\ 1124,\ 1133,\ 1223,\ 2222\}$$

Cycle Barycenter

The μ function and the balance center μ_0 of a cycle were proposed in the first paper. The definitions are slightly modified to suited the new context of Generalized Collatz Sequences.

Definition 12. *The* μ *function and the balance center* μ_0 *are defined by*

$$\mu(x) = (p - r^x)^{-1} \tag{23}$$

$$\mu_0 = \mu(d/m) \tag{24}$$

Here is the source of these definitions showed first in a restricted context. In some cases the q_i values of the shape vector $\rho = (q_i 0^{d_i})_m$ are a constant value c. Further on we will introduce the mean value $\bar{q}$ as a constant λ level. But for now we just suppose that the parameter r is shifted to x such that $d = m$. Then

$$\begin{aligned}
\langle \omega \rangle &= \left\langle p^m, r^d, -q \right\rangle = \langle p, x, -c \rangle^m && (25)\\
&= \langle p^m, x^m, -c(p^m - x^m)/(p-x) \rangle && \text{from } \langle f \rangle^m \text{ formula}\\
&= \left\langle p^m, r^d, -c(p^m - r^d)/(p - r^{d/m}) \right\rangle && \text{since } x^m = r^d\\
&= \left\langle p^m, r^d, -c\delta\mu_0 \right\rangle && (26)
\end{aligned}$$

Here is a known example to grasp the meaning of μ_0. Let (p, r) be $(3, 2)$. These are the negative Collatz-Kakutani sequences with $\delta = 139$ and $c = 1$. The $K_{dm}[11, 7]$ contains the well known cycle $(x_k)_7 = (q_k/139)_7 = (17\ \ 25\ \ 37\ \ 55\ \ 41\ \ 61\ \ 91)$. Now suppose that d is changed to d_k, or m is changed to m_k, so that

$$x_k = \mu(d_k/7) = \mu(11/m_k)$$

Consistent with $\mu_0 = \mu(11/7) \approx 35.6997$ indicated by $\bullet$ we have

$$\begin{aligned}
(d_k)_7 &\approx (10.8948\ \ 10.9590\ \ \bullet\ \ 11.0033\ \ 11.0335\ \ 11.0123\ \ 11.0394\ \ 11.0577)\\
(m_k)_7 &\approx (7.0676\ \ 7.0261\ \ \bullet\ \ 6.9979\ \ 6.9789\ \ 6.9922\ \ 6.9750\ \ 6.9635)
\end{aligned}$$

The following equations being verified this shows that μ_0 is the equilibrium point.

$$\sum_{k=1}^{m} 11 - d_k = \sum_{k=1}^{m} 7 - m_k = 0$$

The μ_0 value is the cycle barycenter of the minimum layer where $c = 1$. For a layer with $c = 2$ the barycenter is $2\mu_0$ as this example illustrates.

$$\begin{aligned}
V^{(k)}(\rho) &= (1110\ \ 1101\ \ 1011) \longrightarrow R^{(k)}(q) = (37\ \ 55\ \ 79)\\
V^{(k)}(\rho) &= (2220\ \ 2202\ \ 2022) \longrightarrow R^{(k)}(q) = (74\ \ 110\ \ 158)
\end{aligned}$$

Definition 13. *The barycenter q_0 of an algebraic cycle in any layer is the minimum barycenter multiply by the layer's level. Ditto for the related x_0 which is the barycenter of $(x_i)_m$ when the cycle exists.*

$$\begin{aligned}
q_0 &= \lambda\delta\mu_0\\
x_0 &= q_0/\delta = \lambda\mu_0
\end{aligned}$$

Now we need to show how is derived this general definition for non-constant q_i. We have

$$q = \sum_{i=1}^{m} q_i p^{m-i} r^{\sigma_i} \tag{27}$$

$$= \sum_{i=1}^{m} q_i \hat{q}_i \qquad \text{where } \hat{q}_i = p^{m-i} r^{\sigma_i} \tag{28}$$

$$= [\, q_1 \; q_2 \; .. \; q_m \,] \, [\, \hat{q_1} \; \hat{q_2} \; .. \; \hat{q_m} \,]^\top \tag{29}$$

The second matrix changes with division amount d while the first matrix is constant throughout the layer. If we add all q-values in a cycle the rotation of i generates a matrix with constant values.

$$\sum_{i=1}^{m} R^i(q) = [\, m\lambda \; m\lambda \; .. \; m\lambda \,] \, [\, \hat{q_1} \; \hat{q_2} \; .. \; \hat{q_m} \,]^\top \tag{30}$$

$$q_0 = m^{-1} \sum_{i=1}^{m} R^i(q) = \lambda[\, 1 \; 1 \; .. \; 1 \,] \, [\, \hat{q_1} \; \hat{q_2} \; .. \; \hat{q_m} \,]^\top \tag{31}$$

$$q_0 = \lambda\delta\mu_0 \qquad \text{always} \tag{32}$$

$$x_0 = q_0/\delta = \lambda\mu_0 \qquad \text{conditional} \tag{33}$$

How can the barycenter $\lambda\mu_0$ be useful? In the context $(p, r) = (3, 2)$ for example, we can say that no cycle $(x_i)_m$ can exist if $d > 2m$, since $|\lambda\mu_0| < 1$.

Explicit role of δ in cyclic process

There is no inner connection between δ and q but, with the rotation of q in cycles, δ plays an external role plain to see with the following matrices shifts yet providing a more accurate grasp of the R function implying q_1.

$$q = \left[p^{m-1} \; p^{m-2} \; .. \; p \; 1\right] [q_1 \; q_2 r^{\sigma_2} \; .. \; q_m r^{\sigma_m}]^\top \tag{34}$$

$$pq = \left[p^{m} \; p^{m-1} \; .. \; p^2 \; p\right] [q_1 \; q_2 r^{\sigma_2} \; .. \; q_m r^{\sigma_m}]^\top \tag{35}$$

$$pq + q_1 r^d = \left[p^{m} \; p^{m-1} \; .. \; p^2 \; p \; 1\right] \left[q_1 \; q_2 r^{\sigma_2} \; .. \; q_m r^{\sigma_m} \; q_1 r^d\right]^\top \tag{36}$$

$$pq + q_1 r^d - q_1 p^m = \left[p^{m-1} \; .. \; p^2 \; p \; 1\right] \left[q_2 r^{\sigma_2} \; .. \; q_m r^{\sigma_m} \; q_1 r^d\right]^\top \tag{37}$$

$$(pq - q_1\delta) \bmod^* r = \left[p^{m-1} \; p^{m-2} \; .. \; p \; 1\right] \left[q_2 \; q_3 r^{\sigma_3-\sigma_2} \; .. \; q_1 r^{d-\sigma_2}\right]^\top \tag{38}$$

$$R(q) = \left[p^{m-1} \; p^{m-2} \; .. \; p \; 1\right] [q_2 \; q_3 r^{s_2} \; .. \; q_1 r^{s_m}]^\top \tag{39}$$

The rotation of $(d_i)_m$ is less evident than that of $(q_i)_m$ but not less true. I leave it to check. Instead, here is the inverse Collatz-Kakutani example in $(p,r) = (3,2)$ about the cycle $(x_i)_7 = (17\ 25\ 37\ 55\ 41\ 61\ 91)$ with $(1+d_i)_7 = (1\ 1\ 1\ 2\ 1\ 1\ 4)$ thus $(\sigma_i)_7 = (0\ 1\ 2\ 3\ 5\ 6\ 7)$.

$$
\begin{aligned}
q &= \left[3^6\ 3^5\ 3^4\ 3^3\ 3^2\ 3^1\ 3^0\right] \left[2^0\ 2^1\ 2^2\ 2^3\ 2^5\ 2^6\ 2^7\right]^\top = 2363 \\
3q &= \left[3^7\ 3^6\ 3^5\ 3^4\ 3^3\ 3^2\ 3^1\right] \left[2^0\ 2^1\ 2^2\ 2^3\ 2^5\ 2^6\ 2^7\right]^\top = 7089 \\
3q + 2^{11} &= \left[3^7\ 3^6\ 3^5\ 3^4\ 3^3\ 3^2\ 3^1\ 3^0\right] \left[2^0\ 2^1\ 2^2\ 2^3\ 2^5\ 2^6\ 2^7\ 2^{11}\right]^\top = 9137 \\
3q + 2^{11} - 3^7 &= \left[3^6\ 3^5\ 3^4\ 3^3\ 3^2\ 3^1\ 3^0\right] \left[2^1\ 2^2\ 2^3\ 2^5\ 2^6\ 2^7\ 2^{11}\right]^\top = 6950 \\
R(q) = (3q - \delta) \bmod^* 2 &= \left[3^6\ 3^5\ 3^4\ 3^3\ 3^2\ 3^1\ 3^0\right] \left[2^0\ 2^1\ 2^2\ 2^4\ 2^5\ 2^6\ 2^{10}\right]^\top = 3475
\end{aligned}
$$

The new $(\sigma_i)_7 = (0\ 1\ 2\ 4\ 5\ 6\ 10)$ matches $(1+d_i)_7 = (1\ 1\ 2\ 1\ 1\ 4\ 1)$ hence the rotated previous $(1+d_i)_7$.

Incidental divisibility

Beside the constant external role of δ in q-parameter cycles there is an independant link between δ and q in the previous example: $\delta = 139$ divide 2363 giving 17, fulfilling so the requirement to get a numerical cycle, either $q \bmod \delta = 0$. Then, forcely, $25 \times \delta = 3475$.

Such numerical cycle occurs very rarely, randomly and presumably never with large integers. Often, in addition, there is no way to predict that a potential $q \bmod \delta$ value will be more frequent than another. Is it therefore justified to assume the equiprobability of the potential values, especially since this would allow us to account what is observed: rarely, randomly and never seen with large numbers ? The issue requires investigation to go beyond the conditional calculus of probabilities in the first French paper.

Equiprobability violation

With a given m there exist a class K_{dm} for which

$$
|\delta| = \left|p^m - r^d\right| \quad \text{and} \quad |\mu_0|^{-1} = \left|p - r^{d/m}\right|
$$

are minima. For K_{dm} classes with greater d the μ_0 values are decreasing and eventualy $|\lambda\mu_0| < 1$. This implies that there is no possibility of a numerical cycle and therefore 0 can't be a potential value of $q \bmod \delta$. There are a few other indirect ways of knowing that 0 is excluded or required as a potential value, but nothing can be said about $q \bmod \delta$ values between 0 and δ. On the other hand, if δ divide a q_i in a cycle $(q_i)_m$ it can be shown that δ divide all the others q_i of that cycle.

Numerical Cycle Occurence Probability

If all possible values of q mod δ are allowed in a class K_{dm} and there is no way to predict the relative frequency of the values, we can assume that they are equal and calculate, under this condition, what is the occurence probability of a cycle in the class.

Let $P_{one} = |\delta|^{-1}$ be the probability that a $(q_i)_m$ cycle results in a numerical $(x_i)_m$ cycle and e be the number of cycles of length m (which excludes spirals when d and m are not relatively prime). The probability P_{no} that there is no numerical cycle in this class K_{dm} is

$$P_{no} = (1 - P_{one})^e \tag{40}$$

The number of cycles when d an m are relatively prime is

$$e = m^{-1}\,\text{card}\,K_{dm} \tag{41}$$

If there are spirals we should subtract from card K_{dm} the elements in the spirals but this is only a minor correction that can be neglected. So we have

$$P_{no} = \left(1 - \frac{1}{|p^m - r^d|}\right)^{\frac{1}{d}\binom{d}{m}(r-1)^m} \tag{42}$$

Except when d and m are small we can use the following approximation where $1/N$ means a chance on N of at least one cycle.

$$\begin{aligned} P_{no} &= \left(\frac{|\delta| - 1}{|\delta|}\right)^e \\ &= 1 - \binom{e}{1}|\delta|^{-1} + \binom{e}{2}|\delta|^{-2} - \cdots \\ 1/N = P_{yes} &\cong e\,|\delta|^{-1} \\ N &\cong \frac{|p^m - r^d|\,d}{\binom{d}{m}(r-1)^m} \end{aligned} \tag{43}$$

References

1. Pierre Lamothe, 2020, *Approche algébrique du problème* $3n+1$ *généralisé*, https://vixra.org/abs/2002.0571

2. Pierre Lamothe, 2020, *Cycles universels et cycles dérivés en* $3x+1$ *généralisé*, https://vixra.org/abs/2002.0594

3. Pierre Lamothe, 2020, *Les cycles dans les suites de Collatz généralisées*, https://vixra.org/abs/2002.0204